ON THE SUM OF TWO KRONECKER SETS

BY

SADAHIRO SAEKI

T. W. Körner asks in his very brilliant paper [6; p. 111 and p. 223] if the sum of two Kronecker sets is of synthesis. In this note we construct a counter-example. This result implies in particular that the following theorem of S. W. Drury [1] (see also [2]) is best possible in some sense: every finite sum of closed subsets of a totally disconnected Kronecker set is of synthesis.

LEMMA (cf. [6; Lemma 6.11]). *Let G be a metrizable LCA I-group with dual $\hat{G}$, and E a strongly independent Cantor set in G.*

(a) *Then there exist two Kronecker sets E_1 and $E_2 \subset G$, both homeomorphic to E, such that $E \subset E_1 + E_2$ and*

$$Gp(E_1) \cap Gp(E_2) = Gp(E) \cap Gp(E_j) = \{0\} \quad (j = 1, 2).$$

(b) *Suppose, in addition, that E is a set of type M and that E_1 and E_2 are as in part* (a). *Then there exist an H_1-set $K \subset E$ of type M and a compact set $K_1 \subset E_1$ such that $K \subset K_1 + E_2$ and $K_1 \cup E_2$ is an H_1-set.*

Proof. Part (a) is an easy consequence of Kaufman's method in [5] (see also [4]) and so we give only a sketch of the proof (a detailed proof will appear in [9]).

Let $C(E; G)$ be the additive group of all continuous mappings from E into G. It is easy to see that $C(E; G)$ forms a complete metric, topological group under the topology of uniform convergence, and that, for quasi-all $f \in C(E; G)$, $f(E)$ is a Kronecker set in G which is homeomorphic to E by Kaufman's argument in [5]. Moreover, we can show that the set B of all $f \in C(E; G)$ such that

$$Gp(f(E)) \cap Gp(E) \neq \{0\}$$

is of the first Baire category in $C(E; G)$, since E is independent. Therefore quasi-all $f \in C(E; G)\backslash B$ have the property that the sets $E_1 = f(E)$ and $E_2 = (f_0 - f)(E)$ are Kronecker sets homeomorphic to E. Here f_0 denotes the identity mapping on E. Since E is independent over $\mathbf{Z}$, we conclude that E_1 and E_2 have the desired properties.

To prove part (b), we must repeat some arguments in [8]. Since E_2 is a Kronecker set, we can choose the characters $\chi_1, \ldots, \chi_N \in \hat{G}$ in Lemma 2 of [8] so that

$$|1 - \chi_j| < \varepsilon \quad \text{on } E_2 \quad (j = 1, 2, \ldots, N), \tag{1}$$

where ε is an arbitrary, but preassigned, positive real number. It follows from Theorem 2 and its proof in [8] that E contains an H_1-set K of type M such that

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for each $\varepsilon > 0$ and $g \in C(K)$ and $|g| = 1$, there exist finitely many characters $\chi_1, \ldots, \chi_N \in \hat{G}$ which satisfy (1) and

$$\left| g - N^{-1} \sum_{j=1}^{N} \chi_j \right| < \varepsilon \quad \text{on } K. \tag{2}$$

These characters are the same as in (1), and this requires some variations in the construction in [8].

Notice that every element x of $E_1 + E_2$ can be uniquely written as $x = x_1 + x_2$ with $x_j \in E_j$ $(j = 1, 2)$, because $E_1 \cap E_2 = \emptyset$ and $E_1 \cup E_2$ is independent by part (a). Thus we may identify the sum $E_1 + E_2$ with the product $E_1 \times E_2$. Let $\pi\colon E_1 + E_2 \to E_1$ be the natural projection, and let $K_1 = \pi(K)$; hence $K \subset K_1 + E_2$. It only remains to show that $K_1 \cup E_2$ is an H_1-set.

Let $\gamma \in \hat{G}$ and $\varepsilon > 0$ be given. Choose finitely many characters $\chi_1, \ldots, \chi_N \in \hat{G}$ so that they satisfy (1) and (2) with $g = \gamma \circ \pi$. Then

$$\left| 1 - N^{-1} \sum_{j=1}^{N} \chi_j \right| < \varepsilon \quad \text{on } E_2. \tag{3}$$

Moreover, we have

$$\left| \gamma - N^{-1} \sum_{j=1}^{N} \chi_j \right| < 2\varepsilon \quad \text{on } K_1. \tag{4}$$

Indeed, given $x_1 \in K_1$, choose $x_2 \in E_2$ such that $x_1 + x_2 \in K$. It follows from (1) and (2) that

$$\begin{aligned}
&\left| \gamma(x_1) - N^{-1} \sum_{j=1}^{N} \chi_j(x_1) \right| \\
&\quad \leq \left| g(x_1 + x_2) - N^{-1} \sum_{j=1}^{N} \chi_j(x_1 + x_2) \right| + N^{-1} \sum_{j=1}^{N} |1 - \chi_j(x_2)| \\
&\quad < \varepsilon + \varepsilon \\
&\quad = 2\varepsilon.
\end{aligned}$$

This establishes (4). Since K_1 and E_2 are Kronecker sets, (3) and (4) imply that $K_1 \cup E_2$ is an H_1-set, which completes the proof.

THEOREM. *Every LCA I-group G contains two disjoint perfect Kronecker sets K_1 and K_2 such that*

(i) *$K_1 \cup K_2$ is an independent H_1-set;*

(ii) *$K_1 + K_2$ contains a strongly independent H_1-set K of nonsynthesis such that $Gp(K) \cap Gp(K_j) = \{0\}$ for $j = 1, 2$;*

(iii) *$K_1 + K_2$ is of nonsynthesis.*

Proof. Without loss of generality, we may assume that G is metrizable. Then G contains a strongly independent Cantor set E which is an H_1-set of type M by Theorem 2 of [8]. Let K, K_1, and $K_2 = E_2$ be as in part (b) of the lemma. Obviously (i) and (ii) hold. To get a contradiction, we suppose that $K_1 + K_2$ is of synthesis. Under this hypothesis, we want to prove that the set K satisfies Condition $(\mathscr{H}_b)$ in [7] for some positive constant b. This will yield the desired contradiction, since K is of nonsynthesis.

First notice that the natural projections $\pi_j: K_1 + K_2 \to K_j$ are injective on K $(j = 1, 2)$ by (i) and (ii). On the other hand, Kaijser's theorem [3] and (i) assure that the Fourier restriction algebra $A(K_1 + K_2)$ of $A(G)$ is topologically isomorphic to the tensor algebra $V(K_1, K_2) = C(K_1) \hat{\otimes} C(K_2)$ under the map $f \to \tilde{f}$, where

$$\tilde{f}(x_1, x_2) = f(x_1 + x_2) \quad (x_j \in K_j; j = 1, 2).$$

Indeed, it is an easy consequence of our construction of K_1 and K_2 that these algebras are isometrically isomorphic. Let A and B be two disjoint compact subsets of K. Then $\pi_1(A) \cap \pi_1(B) = \emptyset$, and so there exists a clopen subset C of K_1 such that $\pi_1(A) \subset C$ and $\pi_1(B) \cap C = \emptyset$. Let $f \in A(K_1 + K_2)$ be the characteristic function of $C + K_2$. Then $\|f\|_A < b$ for some constant $b < \infty$, because $\|\tilde{f}\|_V = 1$ (b depends only on K_1 and K_2, and is independent of A and B). Since $C + K_2$ is a clopen subset of $K_1 + K_2$, and since $K_1 + K_2$ is assumed to be of synthesis, it follows that there exists a $g \in A(G)$ such that $g = 1$ on some neighborhood of $C + K_2$, $g = 0$ on some neighborhood of $(K_1 \backslash C) + K_2$ and $\|g\|_{A(G)} < b$. But $A \subset C + K_2$ and $B \subset (K_1 \backslash C) + K_2$. Hence K satisfies Condition $(\mathscr{H}_b)$ in [7], and is therefore of synthesis by Theorem 1 of [7]. This contradiction establishes (iii).

Remarks. (a) For tensor algebras, we refer the reader to [11].

(b) Suppose that K is a compact subset of a nondiscrete LCA group G and that X is any compact infinite space. If there exists an isometric algebra homomorphism $\psi: C(X) \to A(K)$ with $\psi(1) = 1$, and if K carries a pseudomeasure P such that

$$\limsup_{\gamma \to \infty} |P(\hat{\gamma})| < \|P\|_{PM}, \tag{i}$$

then K is of nonsynthesis. Notice that such a ψ exists if $A(K) = C(X) \hat{\otimes} C(Y)$ isometrically for some Y.

A rough proof of this fact is as follows. We may assume that K contains $0 \in G$. Suppose that K is of synthesis; hence $P \in A'(K)$. Then every unimodular function f in $A(K)$ such that $\|f\|_A = f(0) = 1$ is the restriction of some character in $\hat{G}$ by (i) and Lemma 4.2 of [10]. The existence of ψ as above then forces K to be a Dirichlet set. But this is impossible by (i), because P is carried by K.

(c) Our theorem holds for every nondiscrete LCA group G if we substitute the words "K_q-set" and "weak K_q-set" for "Kronecker set" and "H_1-set", respectively. To prove this, we must replace E and E_2 by appropriate subsets thereof in the proof of the lemma, and avoid appealing to Kaijser's theorem in the last proof. We omit the details.

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Kansas State University
Manhattan, Kansas

Tokyo Metropolitan University
Tokyo