CONVOLUTION PRODUCTS WITH SMALL FOURIER-STIELTJES TRANSFORMS

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Let G be a LCA group with dual Γ. Denote by $M(G)$ the convolution algebra of Borel measures on G and $\wedge$ the Fourier-Stieltjes transformation. Let $M_a(G)$ be those $\mu \in M(G)$ which are absolutely continuous with respect to Haar measure on G. I. Glicksberg [3] has proved:

THEOREM 1. *Let S be a measurable subset of Γ such that $(S - \gamma) \cap S$ has finite Haar measure for a dense set of γ. If $\mu_i \in M(G)$ $(i = 1, 2)$ and* supp $\hat{\mu}_i \subset S$ *then $\mu_1 * \mu_2 \in M_a(G)$. Furthermore, if G is metrizable, $|\mu_1| * |\mu_2| \in M_a(G)$.*

Glicksberg's proof is quite deep and uses disintegration of measures. Recently, Colin Graham [4] has given a simple proof of Theorem 1 in which the metrizability hypothesis is removed.

Let $\mathscr{R} \subset \Gamma$ be a Riesz set, i.e., whenever $\mu \in M(G)$ and supp $\hat{\mu} \subset \mathscr{R}$ then $\mu \in M_a(G)$. Our next theorem gives when $\mathscr{R}$ is empty the above mentioned results of Glicksberg and Graham:

THEOREM 2. *Let S be a measurable subset of Γ such that $\{(S - \gamma) \cap S\}$ is a Riesz set for a dense subset $\mathscr{D}$ of $\gamma \notin \mathscr{R}$. If $\mu_i \in M(G)$ $(i = 1, 2)$ and* supp $\hat{\mu}_i \subset S$ *then $|\mu_1| * |\mu_2| \in M_a(G)$.*

Proof. Assume μ satisfies the hypothesis of the present theorem with $\mathscr{D} = \mathscr{R}$. We shall prove $\mu^2 \in M_a(G)$. Let $\gamma \notin \mathscr{R}$. Then

$$\mu * \bar{\gamma}\mu \in M_a(G) \quad (\gamma \notin \mathscr{R}) \tag{1}$$

since $(S - \gamma) \cap S$ is a Riesz set. Now (1) implies inasmuch as $M_a(G)$ is an ideal that

$$\mu_s * \bar{\gamma}\mu_s \in M_a(G) \quad (\gamma \notin \mathscr{R}) \tag{2}$$

where μ_s is the singular part of μ.

From (2) it follows that

$$\mu_s * p\mu_s \in M_a(G) \tag{3}$$

for all trigonometric polynomials $p(x) = \sum_\gamma c_\gamma \gamma(-x)$, $\gamma \notin \mathscr{R}$.

We claim there exists a sequence p_n of trigonometric polynomials with characters not in $\mathscr{R}$ such that

$$p_n\mu_s \to \mu_s \quad \text{in } M(G). \tag{4}$$

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Since $M_a(G)$ is closed, (3) and (4) give

$$\mu_s^2 \in M_a(G) \tag{5}$$

and so $\mu^2 \in M_a(G)$.

To establish our claim we prove that the set of all trigonometric polynomials on G with characters not in $\mathscr{R}$ is dense in $L^1(d\mu_s)$. Suppose not. Then by the Hahn-Banach Theorem there is a $\phi \in L^\infty(d\mu_s)$ with $\phi \neq 0 \bmod \mu_s$ such that

$$\int_G p(x)\phi(x)\, d\mu_s(x) = 0 \tag{6}$$

for all trigonometric polynomials p with characters not in $\mathscr{R}$. In particular (6) gives

$$(\phi\mu_s)^\wedge(\gamma) = 0 \quad (\gamma \notin \mathscr{R}). \tag{7}$$

Since $\phi\mu_s$ is a singular measure it is immediate from (7) that $\phi = 0 \bmod \mu_s$. This contradiction establishes our claim. Simple modifications (see also [4]) of the above argument now give the full theorem. We omit the details.

COROLLARY (Wallen [7]). *Let* $\mathbf{A} = \{n_k\}$ *be a sequence of positive integers satisfying the Faber gap condition* $\lim_{k\to\infty} (n_{k+1} - n_k) = \infty$. *If* supp $\hat{\mu} \subset \mathbf{Z}^- \cup \mathbf{A}$ *then* $\mu^2 \in M_a(\mathbf{T})$.

Proof. Take $\mathscr{R} = \mathbf{Z}^-$ in Theorem 2 and $S = \mathbf{Z}^- \cup \mathbf{A}$.

Comments. (i) There are many possible variants on Theorem 2. In this connection see [4] and [7].

(ii) For an extension of Theorem 1 see [6, Theorem 2].

(iii) For related work the reader is referred to [1] and [2].

(iv) Interesting examples of Riesz sets are given in [5].

REFERENCES

1. RAOUF DOSS, *On measures with small transforms*, Pacific J. Math., vol. 26 (1968), pp. 257–263.
2. CARL-GUSTAV ESSEN, *A note on Fourier-Stieltjes transforms and absolutely continuous functions*, Math. Scand., vol. 2 (1954), pp. 153–157.
3. IRVING GLICKSBERG, *Fourier-Stieltjes transforms with small supports*, Illinois J. Math., vol. 9 (1965), pp. 418–427.
4. COLIN C. GRAHAM, *Fourier-Stieltjes transforms with small supports*, Illinois J. Math., vol. 18 (1974), pp. 532–534.
5. YVES MEYER, *Spectres des mesures et mesures absolument continues*, Studia Math., vol. 30 (1968), pp. 87–99.
6. LOUIS PIGNO, *Approximations to the norm of the singular part of a measure*, Kansas State University Technical Report 44, 1974.
7. LAWRENCE J. WALLEN, *Fourier-Stieltjes transforms tending to zero*, Proc. Amer. Math. Soc., vol. 24 (1970), pp. 651–652.

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