ALGEBRAIC AUTOMORPHISM GROUPS

BY

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1. Introduction

Let F be an algebraically closed field of characteristic 0. It is known that there is a large class of affine algebraic F-groups G for which the group $\mathscr{W}(G)$ of all affine algebraic group automorphisms is again an affine algebraic F-group in a natural way. In fact, [2] gives an intrinsic structural characterization of the connected groups G for which this is the case. It is our present purpose to apply the underlying principle and technique of [2] quite generally to an exploration of $\mathscr{W}(G)$ when G is an arbitrary pro-affine algebraic group over an arbitrary algebraically closed field.

Although $\mathscr{W}(G)$ is usually not an algebraic group, there is a natural notion of *algebraic subgroup* of $\mathscr{W}(G)$. The guiding principle for this is the following. For a subgroup P of $\mathscr{W}(G)$ to qualify as an *algebraic* subgroup, it should be possible to endow P with the structure of a pro-affine algebraic group in such a way that the canonical map $P \times G \to G$, sending each (α, x) onto $\alpha(x)$, becomes a morphism of pro-affine algebraic varieties. We shall see in Section 2 that if P qualifies in this sense then there is a unique minimum pro-affine structure on P. If each such P is given this minimum structure then the resulting family of algebraic subgroups of $\mathscr{W}(G)$ has all the desired naturality properties.

In Section 3, we obtain the appropriate extensions of the results of [2] for affine algebraic groups over fields of characteristic 0. The main result here is that the subgroup of $\mathscr{W}(G)$ that is generated by the family of all connected algebraic subgroups is still an algebraic subgroup, the *maximum connected algebraic subgroup* of $\mathscr{W}(G)$. This illustrates the strength of the structure theory of affine algebraic groups in characteristic 0, as compared with the case of characteristic $p \neq 0$. Indeed, we exhibit a simple and nonpathological example showing that the result does not extend to the case of nonzero characteristic. Another simple example shows that, independently of the characteristic, an extension to the pro-affine (rather than affine) case is also not possible.

In Section 4, we deal with connected affine algebraic groups in characteristic 0. First, we verify that the algebraic subgroups of $\mathscr{W}(G)$ are precisely those subgroups whose natural images in the automorphism group of the Lie algebra are algebraic, in the usual sense (essentially, the proof of this is already in [2]). Next, we show how one can describe the image in the Lie algebra automorphism group of the maximum connected algebraic subgroup of $\mathscr{W}(G)$ in purely Lie algebra theoretic terms. Loosely speaking, this shows that the maximum

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connected algebraic subgroup is far more *computable* than is the whole group $\mathscr{W}(G)$. Finally, we point out the reassuring fact that when the base field is the field of complex numbers, in which case $\mathscr{W}(G)$ carries the structure of a complex Lie group, the maximum connected algebraic subgroup of $\mathscr{W}(G)$ coincides with the *topological* identity component of $\mathscr{W}(G)$.

Added in proof. I have just learned that, long before [2] was written, the automorphism groups of affine algebraic groups over fields of characteristic 0 were analysed by A. Borel and J-P. Serre (*Théorèmes de finitude en cohomologie galoisienne*, Comment. Math. Helv., vol. 39, Fasc. 2 (1964), pp. 111–163). They are shown to be groups of a special type (ALA) in a suitably defined category of *locally algebraic* schemes. Although Borel and Serre work in the general algebraic geometric setting, where the Hopf algebras of polynomial functions do not appear explicitly as such, it is not difficult to see that the identity component of the group Q of Theorem 3.2 coincides with the identity component (in the sense of Borel-Serre) of the automorphism group of G, and that Theorems 3.1 and 3.2 are covered by the results of the paper cited above.

2. Algebraic automorphism groups

Let F be an algebraically closed field, and let G be a pro-affine algebraic F-group, with Hopf algebra of polynomial functions $A = \mathscr{A}(G)$, in the sense of [1]. Let P be a subgroup of the group $\mathscr{W}(G)$ of all pro-affine algebraic group automorphisms of G, and suppose that P can be equipped with a Hopf algebra B of F-valued functions such that (P, B) is the structure of a pro-affine algebraic F-group, with the property that the map $P \times G \to G$ sending each (α, x) onto $\alpha(x)$ is a morphism of pro-affine algebraic varieties. Then, for every element f of A, the F-valued function on $P \times G$ sending each (α, x) onto $f(\alpha(x))$ is a polynomial function on $P \times G$. This means that there are elements $g_1, \ldots, g_n$ of A and $h_1, \ldots, h_n$ of B such that $f(\alpha(x)) = \sum_{i=1}^n h_i(\alpha)g_i(x)$, so that

$$f \circ \alpha = \sum_{i=1}^{n} h_i(\alpha)g_i.$$

Thus, P and B must satisfy the following conditions.

(1) For every f in A, the composites $f \circ \alpha$, with α ranging over P, all lie in some finite-dimensional F-subspace of A, i.e., *A is locally finite as a (right) P-module.*

(2) For every ρ in $A^\circ = \mathrm{Hom}_F(A, F)$ and every f in A, the F-valued function ρ/f on P defined by

$$(\rho/f)(\alpha) = \rho(f \circ \alpha)$$

belongs to the algebra B of polynomial functions on P.

Recall, from [1], that a family of functions f on a group is called *fully stable* if it is stable with respect to the translation actions $f \mapsto x \cdot f$ and $f \mapsto f \cdot x$, where $(x \cdot f)(y) = f(yx)$ and $(f \cdot x)(y) = f(xy)$, as well as with respect to the map $f \mapsto f'$, where $f'(x) = f(x^{-1})$. For any subgroup P of $\mathscr{W}(G)$, let us denote by $A(P)$ the smallest fully stable F-algebra of F-valued functions on P that contains all the functions ρ/f, with ρ in A° and f in A.

If (P, B) is as above then, by the definition of pro-affine algebraic group, B is a fully stable F-algebra of *representative* functions on P (which is equivalent to saying that B is a Hopf algebra of functions on P). By virtue of condition (2) above, $A(P)$ is a sub Hopf algebra of B, and it is clear that $A(P)$ still separates the elements of P. By assumption, P may be identified with the group $\mathscr{G}(B)$ of all F-algebra homomorphisms $B \to F$, and our last statement says that the restriction map $\mathscr{G}(B) \to \mathscr{G}(A(P))$ is injective. On the other hand, it is a general fact of the theory of pro-affine algebraic groups that such a restriction map is always surjective (cf. [1, Theorem 2.1]). Therefore $(P, A(P))$ is the structure of a pro-affine algebraic F-group, and the canonical map $P \times G \to G$ is clearly a morphism of pro-affine algebraic varieties also with respect to this *minimum admissible* algebra $A(P)$ of polynomial functions on P. This provides us with a justification for referring to P as an *algebraic subgroup* of $\mathscr{W}(G)$, making the agreement that *the Hopf algebra of polynomial functions on P is to be $A(P)$*. We observe that, if the base field F is of characteristic 0, then the injectiveness of the map $\mathscr{G}(B) \to \mathscr{G}(A(P))$ implies that $A(P) = B$.

The following result is a straightforward extension of [2, Theorem 2.1].

THEOREM 2.1. *Let (G, A) be the structure of a pro-affine F-group, where F is an algebraically closed field. Let P be a subgroup of $\mathscr{W}(G)$. Then P is contained in an algebraic subgroup of $\mathscr{W}(G)$ if and only if A is locally finite as a P-module. If this condition is satisfied, then $A(P)$ is a fully stable F-algebra of representative functions on P, and the associated group $\mathscr{G}(A(P))$ may be identified with an algebraic subgroup of $\mathscr{W}(G)$, coinciding with the intersection of the family of all algebraic subgroups that contain P.*

Proof. The necessity of the condition being already clear (condition (1) above), we assume that A is locally finite as a P-module. Let f be an element of A, and let $(f_1, \ldots, f_n)$ be an F-basis for the F-linear span in A of the family of composites $f \circ \alpha$, with α in P. Let ρ be an element of A°. Evidently, for every α in P, we have $(\rho/f) \cdot \alpha = \rho/(f \circ \alpha)$. Now $f \circ \alpha$ is an F-linear combination $\sum_{i=1}^{n} h_i(\alpha) f_i$, whence $(\rho/f) \cdot \alpha = \sum_{i=1}^{n} h_i(\alpha) \rho/f_i$. This shows that the F-linear span in $A(P)$ of the family of translates $(\rho/f) \cdot \alpha$, where α ranges over P, is finite-dimensional, so that each ρ/f is a *representative* function on P. Since $A(P)$ is the *smallest* fully stable algebra of F-valued functions on P containing the functions ρ/f, it follows that $A(P)$ is a sub Hopf algebra of the Hopf algebra of all representative functions on P.

Now let σ be any element of $\mathscr{G}(A(P))$, and let x be an element of G. Viewing x as an element of A°, we consider the effect of σ on the elements x/f of $A(P)$, where f ranges over A. Thus we define an F-valued function σ_x on A by setting $\sigma_x(f) = \sigma(x/f)$. From the fact that x is an F-algebra homomorphism $A \to F$

we have $x/(fg) = (x/f)(x/g)$ for all elements f and g of A. It follows that σ_x is actually an F-algebra homomorphism $A \to F$, i.e., that σ_x is an element of G.

Let $\gamma: A \to A \otimes A$ denote the comultiplication of A, so that, for x and y in G, we have $xy = (x \otimes y) \circ \gamma$. For f in A, let us write $\gamma(f) = \sum_{i=1}^n f_{1i} \otimes f_{2i}$. Then we have

$$\begin{aligned}((xy)/f)(\alpha) &= (xy)(f \circ \alpha) = f(\alpha(x)\alpha(y)) \\ &= (\alpha(x) \otimes \alpha(y))(\gamma(f)) \\ &= \sum_{i=1}^n f_{1i}(\alpha(x))f_{2i}(\alpha(y)),\end{aligned}$$

which shows that

$$(xy)/f = \sum_{i=1}^n (x/f_{1i})(y/f_{2i})$$

This gives

$$\sigma_{xy}(f) = \sum_{i=1}^n \sigma_x(f_{1i})\sigma_y(f_{2i}),$$

whence

$$\sigma_{xy} = (\sigma_x \otimes \sigma_y) \circ \gamma = \sigma_x\sigma_y.$$

Thus, the map sending each element x of G onto σ_x is a group homomorphism $G \to G$. Let us denote this group homomorphism by σ^*.

For a fixed f in A, the functions x/f, with x ranging over G, all lie in some finite-dimensional F-subspace of $A(P)$. In fact, if $f \circ \alpha = \sum_{i=1}^n h_i(\alpha)f_i$ (as above), then

$$x/f = \sum_{i=1}^n f_i(x)h_i.$$

Therefore, the restriction of σ to the set of these functions x/f coincides with a finite F-linear combination of evaluations at elements of P, i.e., there are elements $c_1, \ldots, c_n$ of F, and elements $\alpha_1, \ldots, \alpha_n$ of P, such that $\sigma(x/f) = \sum_{i=1}^n c_i x(f \circ \alpha_i)$ for every x in G. By the definition of σ^*, this means that $f \circ \sigma^* = \sum_{i=1}^n c_i f \circ \alpha_i$. In particular, this shows that $f \circ \sigma^*$ belongs to A, and we have thus shown that σ^* is a morphism of pro-affine algebraic groups $G \to G$.

Next, let us observe that if the element σ of $\mathcal{G}(A(P))$ is the canonical image of an element α of P then $\sigma^* = \alpha$. Indeed, for every f in A and every x in G, we have

$$f(\sigma^*(x)) = \sigma(x/f) = (x/f)(\alpha) = f(\alpha(x)).$$

Now let $\delta: A(P) \to A(P) \otimes A(P)$ be the comultiplication of $A(P)$. Let σ and τ be elements of $\mathcal{G}(A(P))$, and let h be an element of $A(P)$. Then we have $(\sigma\tau)(h) = \tau((\sigma \otimes i)(\delta(h)))$, where i stands for the identity map on $A(P)$. If σ

is the canonical image, α' say, of an element α of P then $(\sigma \otimes i)(\delta(h))$ is simply the translate $h \cdot \alpha$, and the above reads $(\alpha'\tau)(h) = \tau(h \cdot \alpha)$. On the other hand,

$$(\alpha'\tau)(h) = \alpha'((i \otimes \tau)(\delta(h))) = (i \otimes \tau)(\delta(h))(\alpha).$$

Thus, we have $(i \otimes \tau)(\delta(h))(\alpha) = \tau(h \cdot \alpha)$. In particular, for $h = x/f$, this gives

$$(i \otimes \tau)(\delta(x/f))(\alpha) = \tau(x/(f \circ \alpha)) = (f \circ \alpha)(\tau^*(x)) = (\tau^*(x)/f)(\alpha).$$

Hence $(i \otimes \tau)(\delta(x/f)) = \tau^*(x)/f$. Now, if σ is an arbitrary element of $\mathscr{G}(A(P))$, this gives

$$(\sigma \otimes \tau)(\delta(x/f)) = f(\sigma^*(\tau^*(x))).$$

The expression on the left is equal to $(\sigma\tau)(x/f) = f((\sigma\tau)^*(x))$. Letting f range over A, we conclude from this that $(\sigma\tau)^*(x) = \sigma^*(\tau^*(x))$ for every x in G, whence $(\sigma\tau)^* = \sigma^* \circ \tau^*$.

Clearly, if 1 denotes the neutral element of $\mathscr{G}(A(P))$, then 1^* is the identity map on G. Therefore, the last result above shows that every σ^* is in fact an element of $\mathscr{W}(G)$, its inverse $(\sigma^*)^{-1}$ in $\mathscr{W}(G)$ being $(\sigma^{-1})^*$. The map sending each σ onto σ^* is therefore an injective group homomorphism $\mathscr{G}(A(P)) \to \mathscr{W}(G)$, by means of which we identify $\mathscr{G}(A(P))$ with a subgroup of $\mathscr{W}(G)$. This subgroup contains P, because $(\alpha')^* = \alpha$ for every element α of P. It is now evident that, by this identification, $\mathscr{G}(A(P))$ becomes an algebraic subgroup of $\mathscr{W}(G)$, that P is algebraically dense in $\mathscr{G}(A(P))$, and that the restriction map is a Hopf algebra isomorphism $A(\mathscr{G}(A(P))) \to A(P)$. Finally, the initial discussion in this section shows that every algebraic subgroup of $\mathscr{W}(P)$ that contains P must also contain $\mathscr{G}(A(P))$. Our proof of Theorem 2.1 is now complete.

PROPOSITION 2.2. *Let (G, A) be the structure of a pro-affine algebraic F-group, where F is an algebraically closed field. Let P be an algebraic subgroup of $\mathscr{W}(G)$, and let H be a P-stable algebraic subgroup of G. Then the image P_H of P in $\mathscr{W}(H)$ is an algebraic subgroup of $\mathscr{W}(H)$, and the canonical map $P \to P_H$ is a morphism of pro-affine algebraic groups. If H is normal in G, then the same facts hold with G/H in the place of H.*

Proof. Let $\pi\colon A \to A_H$ denote the restriction homomorphism of A onto the Hopf algebra A_H of polynomial functions on H. Let $\tau\colon P \to P_H$ denote the canonical map of automorphism groups. For f in A and α in P, we have $\pi(f \circ \alpha) = \pi(f) \circ \tau(\alpha)$. This shows that A_H is locally finite as a P_H-module, and that τ is a morphism of pro-affine algebraic groups from $P = \mathscr{G}(A(P))$ to $\mathscr{G}(A_H(P_H))$. This implies (cf. [1, Section 2]) that the image P_H of P is an algebraic subgroup of $\mathscr{G}(A_H(P_H))$, whence, actually, $P_H = \mathscr{G}(A_H(P_H))$. This proves the first part of the proposition.

Now suppose that H is normal in G. Then G/H is a pro-affine algebraic F-group, its Hopf algebra of polynomial functions being the H-fixed part A^H of A (cf. [1, Section 2]). Let $\tau\colon P \to P_{G/H}$ be the canonical map of P onto the group $P_{G/H}$ of automorphisms of G/H induced by the elements of P. The $P_{G/H}$-module structure of A^H, when lifted to a P-module structure by means of τ, becomes the restriction to A^H of the P-module structure of A. Therefore, it is clear that A^H, viewed as the algebra of polynomial functions on G/H, is locally finite as a $P_{G/H}$-module, and that τ is a morphism of pro-affine algebraic groups from $P = \mathcal{G}(A(P))$ to $\mathcal{G}(A^H(P_{G/H}))$. As in the first part of this proof, it follows that $P_{G/H}$ is an algebraic subgroup of $\mathcal{W}(G/H)$, and that $\tau\colon P \to P_{G/H}$ is a morphism of pro-affine algebraic groups. This proves Proposition 2.2.

Somewhat more generally than in the second part of Proposition 2.2, let us consider a P-stable sub Hopf algebra B of A. The restriction map $\mathcal{G}(A) \to \mathcal{G}(B)$ is a surjective morphism of pro-affine algebraic groups. Now P acts on B by Hopf algebra automorphisms, whence we have a canonical map $P \to \mathcal{W}(\mathcal{G}(B))$. If P_B denotes the image of P in $\mathcal{W}(\mathcal{G}(B))$, then $B(P_B)$ may evidently be identified with the sub Hopf algebra $B(P)$ of $A(P)$ constructed in the same way as was $A(P)$, starting from the functions ρ/f, with ρ in B° (or A°) and f in B. Now the canonical map $P \to \mathcal{W}(\mathcal{G}(B))$ becomes simply the restriction map $\mathcal{G}(A(P)) \to \mathcal{G}(B(P))$, and we conclude as above that it is a surjective morphism of pro-affine algebraic groups from P to an algebraic subgroup of $\mathcal{W}(\mathcal{G}(B))$.

PROPOSITION 2.3. *Let (G, A) and P be as in Proposition 2.2. Every finitely generated sub Hopf algebra of A is contained in a finitely generated P-stable sub Hopf algebra of A. If B is a finitely generated P-stable sub Hopf algebra of A, then $B(P)$ is finitely generated, so that the canonical image of P in $\mathcal{W}(\mathcal{G}(B))$ is an affine algebraic group.*

Proof. Let $f \in A$, $\alpha \in P$, $x \in G$. Then we have

$$(f \circ \alpha)\cdot x = (f\cdot \alpha(x)) \circ \alpha, \quad x\cdot(f\circ\alpha) = (\alpha(x)\cdot f)\circ\alpha \quad \text{and} \quad (f\circ\alpha)' = f'\circ\alpha.$$

This shows that, if C is a sub Hopf algebra of A, then the F-subalgebra of A that is generated by the elements $f \circ \alpha$, with f in C and α in P, is still a sub Hopf algebra of A. If C is finitely generated as an F-algebra, it follows from the fact that A is locally finite as a P-module that the subalgebra generated by the elements $f \circ \alpha$ is still finitely generated. This proves the first statement of the proposition.

Now suppose that B is a finitely generated P-stable sub Hopf algebra of A. Since B is locally finite as a P-module, there is a finite-dimensional P-stable subspace S of B that generates B as an F-algebra. Let T denote the F-subspace of $B(P)$ that is spanned by the functions ρ/f, with ρ in B° and f in S. The argument we made in the beginning of our proof of Theorem 2.1 (showing that each ρ/f is a representative function on P) shows that T is stable under the translation action of P from the right, and also that T is still finite-dimensional. For α in P, let α° denote the linear endomorphism of B defined by $\alpha^\circ(f) = f\circ\alpha$. Then we have $\alpha\cdot(\rho/f) = (\rho\circ\alpha^\circ)/f$, whence we see that T is also stable under the translation action of P from the left.

Now let f and g be elements of B, and let ρ be an element of B°. The restriction of ρ to the P-orbit $(fg) \circ P$ is an F-linear combination of evaluations at elements of G. Since the evaluations are F-algebra homomorphisms, it follows that the F-subalgebra of $B(P)$ that is generated by T contains all the functions ρ/f, with f in the F-algebra generated by S, i.e., with f in B. Therefore, $B(P)$ is generated as an F-algebra by the elements t of T and the corresponding functions t', where $t'(\alpha) = t(\alpha^{-1})$. Our proof of Proposition 2.3 is now complete.

As an immediate corollary, we have that *if G is an affine algebraic group then every algebraic subgroup of $\mathscr{W}(G)$ is an affine algebraic group.*

PROPOSITION 2.4. *Let (G, A) be as above, and let P and Q be algebraic subgroups of $\mathscr{W}(G)$ such that P normalizes Q. Then QP is an algebraic subgroup of $\mathscr{W}(G)$.*

Proof. Evidently, the assumptions imply that A is locally finite as a QP-module. Let ρ be an element of A°, and let f be an element of A. Consider the element ρ/f of $A(QP)$. Clearly, its restriction to P is the element ρ/f of $A(P)$. Since the restriction map from $A(QP)$ to the F-algebra of all representative functions on P is a morphism of Hopf algebras, it follows that the restriction image of $A(QP)$ is contained in $A(P)$. Moreover, it is a sub Hopf algebra containing the functions ρ/f, and therefore coincides with $A(P)$ (by the definition of $A(P)$). The same holds for Q in the place of P. Therefore, as pro-affine algebraic groups, P and Q may be identified with algebraic subgroups of $\mathscr{G}(A(QP))$, via the above restriction maps. This implies that the conjugation action of P on Q is an action by automorphisms of pro-affine algebraic groups, that $A(Q)$ thereby becomes a locally finite P-module, and that the associated representative functions on P belong to $A(P)$. Therefore, the semidirect product $Q \cdot P$, where $(q_1, p_1)(q_2, p_2)$ is defined to be $(q_1(p_1 q_2 p_1^{-1}), p_1 p_2)$, can be endowed with the structure of a pro-affine algebraic F-group, such that the algebra of polynomial functions is canonically isomorphic with $A(Q) \otimes A(P)$. Clearly, the multiplication of $\mathscr{W}(G)$ yields a morphism of pro-affine algebraic groups $Q \cdot P \to \mathscr{G}(A(QP))$. Therefore, the image QP is an algebraic subgroup of $\mathscr{G}(A(QP))$, and so coincides with it. In particular, QP is an algebraic subgroup of $\mathscr{W}(G)$, so that Proposition 2.4 is established.

3. Affine groups in characteristic 0

If G is a pro-affine algebraic group, we denote by G_u the *unipotent radical* of G, i.e., the maximum unipotent normal algebraic subgroup of G. The identity component of G will be denoted by G_1, and the center of G by $\mathscr{C}(G)$. We shall usually write $\mathscr{C}_1(G)$ for $\mathscr{C}(G)_1$. The algebra of polynomial functions on G will be denoted by $\mathscr{A}(G)$. The following result is implicit in [2], and the proof given below is confined to making a safe connection with the reasoning carried out in [2].

THEOREM 3.1. *Let F be an algebraically closed field of characteristic* 0, *and let G be an affine algebraic F-group. Let P be a subgroup of $\mathcal{W}(G)$. Then $\mathcal{A}(G)$ is locally finite as a P-module if and only if the canonical image of P in $\mathcal{W}(\mathcal{C}_1(G_1/G_u))$ is finite.*

Proof. First, we show that it will suffice to prove the theorem in the case where G is connected. Suppose that $\mathcal{A}(G)$ is locally finite as a P-module. Let P' denote the canonical image of P in $\mathcal{W}(G_1)$. Clearly, $\mathcal{A}(G_1)$ is locally finite as a P'-module. Therefore, assuming that the theorem holds for G_1, the canonical image of P' in $\mathcal{W}(\mathcal{C}_1(G_1/G_u))$ is finite. But this coincides with the canonical image of P. Conversely, if the canonical image of P in $\mathcal{W}(\mathcal{C}_1(G_1/G_u))$ is finite, and if the theorem holds for G_1, then $\mathcal{A}(G_1)$ is locally finite as a P'-module. By a simple elementary argument, given in the proof of Proposition 2.3 in [2], this implies that $\mathcal{A}(G)$ is locally finite as a P-module.

It remains to prove the theorem in the case where G is connected. Assume that G is connected, and that $\mathcal{A}(G)$ is locally finite as a P-module. We know from Section 2 that the local finite-ness is preserved in passing to factor groups and subgroups. Therefore, $\mathcal{A}(C_1(G/G_u))$ is locally finite as a P-module. Now $\mathcal{C}_1(G/G_u)$ is an algebraic toroid, and the elementary argument given at the beginning of the proof of Lemma 3.1 in [2] shows that the canonical image of P in $\mathcal{W}(\mathcal{C}_1(G/G_u))$ must therefore be finite.

Now suppose that the canonical image of P in $\mathcal{W}(\mathcal{C}_1(G/G_u))$ is finite (and $G = G_1$). Let G' denote the group of inner automorphisms of G. Since the canonical image of $G'P$ in $\mathcal{W}(\mathcal{C}_1(G/G_u))$ coincides with that of P, we may replace P with $G'P$, and so we now suppose that $G' \subset P$. We may write G as a semidirect product $G_u \cdot K$, where K is a maximal reductive subgroup of G. Let X denote the stabilizer of K in P. Since all the maximal reductive subgroups of G are conjugate by inner automorphisms, we have $G'X = P$.

Put $T = \mathcal{C}_1(K)$, and let Z denote the element-wise fixer of T in P. Suppose that γ is an element of X whose canonical image in $\mathcal{W}(\mathcal{C}_1(G/G_u))$ is trivial. Let t be an element of T. Then the coset tG_u is an element of $\mathcal{C}_1(G/G_u)$, so that $\gamma(t)G_u = tG_u$, which gives $t^{-1}\gamma(t) \in G_u \cap T = (1)$, so that $\gamma \in Z$. Since the canonical image of X in $\mathcal{W}(\mathcal{C}_1(G/G_u))$ is finite, we see that $Z \cap X$ is therefore of finite index in X.

From here on, the second half of the proof of Lemma 3.1 in [2] can be copied and yields the conclusion that $\mathcal{A}(G)$ is locally finite as a P-module. This establishes Theorem 3.1.

THEOREM 3.2. *Let F be an algebraically closed field of characteristic* 0, *and let G be an affine algebraic F-group. Let Q be the kernel of the canonical homomorphism $\mathcal{W}(G) \to \mathcal{W}(\mathcal{C}_1(G_1/G_u))$. Then Q is an algebraic subgroup of $\mathcal{W}(G)$, and every connected algebraic subgroup of $\mathcal{W}(G)$ is contained in Q.*

Proof. By Theorem 3.1, $\mathcal{A}(G)$ is locally finite as a Q-module. Let us write A for $\mathcal{A}(G)$, and let us consider the algebraic hull $\mathcal{G}(A(Q))$ of Q in $\mathcal{W}(G)$. We can apply Proposition 2.2, first passing to the factor group G/G_u, and then to the subgroup $\mathscr{C}_1(G_1/G_u)$. The conclusion is that the image of $\mathscr{G}(A(Q))$ in $\mathscr{W}(\mathscr{C}_1(G_1/G_u))$ is an algebraic automorphism group, P say, and that the canonical map $\mathscr{G}(A(Q)) \to P$ is a morphism of affine algebraic groups. Since Q is the kernel of this morphism, it is an algebraic subgroup of $\mathscr{G}(A(Q))$, and therefore $Q = \mathscr{G}(A(Q))$, so that Q is an algebraic subgroup of $\mathscr{W}(G)$.

Now let R be any connected algebraic subgroup of $\mathscr{W}(G)$. Again, the canonical image of R in $\mathscr{W}(\mathscr{C}_1(G_1/G_u))$ is an algebraic group of automorphisms, S say, and the canonical map $R \to S$ is a morphism of affine algebraic groups. Therefore, S is connected. On the other hand, by Theorem 3.1, S is finite. Therefore, S is trivial, which means that $R \subset Q$, so that Theorem 3.2 is proved.

Evidently, Theorem 3.2 implies that if R and S are connected algebraic subgroups of $\mathscr{W}(G)$ then $\mathscr{A}(G)$ is locally finite as a module for the group generated by R and S. The following example shows that *this fails in characteristic* $p \neq 0$.

Let F be an algebraically closed field of characteristic $p \neq 0$. Let G be the direct product of two copies of the additive group of F. More precisely, G is the (unipotent) algebraic vector group whose elements are the pairs (u, v) of elements of F, and $\mathscr{A}(G)$ is the polynomial algebra $F[x, y]$, where x and y are the usual coordinate functions on G.

For each element a of F, define the automorphism ρ_a of G by $\rho_a(u, v) = (u, v + au^p)$. It is easy to see that these automorphisms constitute an algebraic subgroup R of $\mathscr{W}(G)$, and that R is isomorphic, as an affine algebraic group, with the additive group of F. Similarly, define the automorphisms σ_a by $\sigma_a(u, v) = (u + av^p, v)$. Clearly, these constitute an algebraic subgroup S of $\mathscr{W}(G)$, and in fact S is isomorphic with R. We have

$$x \circ \rho_a = x, \quad y \circ \rho_a = y + ax^p, \quad x \circ \sigma_a = x + ay^p, \quad y \circ \sigma_a = y.$$

Now let γ be the automorphism $\rho_1 \circ \sigma_1$ of G. We have

$$x \circ \gamma = x + y^p, \quad y \circ \gamma = y + y^{p^2} + x^p.$$

It follows that, for every positive integer k, the function $x \circ \gamma^k$ is a sum of terms x^r or y^s, where r and s are powers of p. the term of highest degree being $y^{p^{2k-1}}$. In particular, $\mathscr{A}(G)$ is *not* locally finite as a module for the group generated by R and S.

The same kind of failure occurs, independently of the characteristic, with pro-affine (rather than affine) algebraic groups. The simplest example for this is as follows.

Let F be an algebraically closed field, and let G be the group of infinite sequences $(a_0, a_1, \ldots)$ of elements of F, with entry-wise addition. We can endow G with the structure of a (unipotent) pro-affine algebraic F-group, whose algebra of polynomial functions is the algebra $F[x_0, x_1, \ldots]$ of polynomials in the usual coordinate functions x_i. For every t in F, we define automorphisms ρ_t and σ_t of G as follows. Writing a for $(a_0, a_1, \ldots)$, etc., put $\rho_t(a) = b$, where $b_{2i} = a_{2i} + ta_{2i+1}$ and $b_{2i+1} = a_{2i+1}$ $(i = 0, 1, \ldots)$. On the other hand, put $\sigma_t(a) = c$, where $c_{2i} = a_{2i}$ and $c_{2i+1} = a_{2i+1} + ta_{2i+2}$. It is easy to see that the automorphisms ρ_t constitute an algebraic subgroup R of $\mathscr{W}(G)$, and the automorphisms σ_t constitute an algebraic subgroup S of $\mathscr{W}(G)$, and that both R and S are isomorphic, as affine algebraic groups, with the additive group of F. We have

$$x_{2i} \circ \rho_t = x_{2i} + tx_{2i+1}, \quad x_{2i+1} \circ \rho_t = x_{2i+1},$$

$$x_{2i} \circ \sigma_t = x_{2i}, \quad x_{2i+1} \circ \sigma_t = x_{2i+1} + tx_{2i+2}.$$

If $\gamma = \rho_1 \circ \sigma_1$ we have

$$x_{2i} \circ \gamma = x_{2i} + x_{2i+1} + x_{2i+2}, \quad x_{2i+1} \circ \gamma = x_{2i+1} + x_{2i+2}.$$

Clearly, $\mathscr{A}(G)$ is not locally finite as a module for the group generated by R and S.

Finally, let us return for a moment to the case of an affine group G over an algebraically closed field of characteristic 0. If R and S are algebraic subgroups of $\mathscr{W}(G)$, and if R is connected, it is still true that $\mathscr{A}(G)$ is locally finite as a module for the group generated by R and S. Indeed, the canonical image of R in $\mathscr{W}(\mathscr{C}_1(G_1/G_u))$ is trivial (by Theorem 3.2), and the canonical image of S is finite (by Theorem 3.1). Therefore, the canonical image in $\mathscr{W}(\mathscr{C}_1(G_1/G_u))$ of the group generated by R and S is finite, and the conclusion follows from Theorem 3.1. However, this can fail if neither R nor S is connected. The simplest and typical example for this is as follows.

Let G be the direct product of two copies of the multiplicative group of F. Then, as is well known, $\mathscr{W}(G)$ is isomorphic with the multiplicative group of the 2 by 2 integral matrices of determinant 1 or -1. Let σ be the automorphism corresponding to the matrix

$$\begin{pmatrix} -2 & -3 \\ 1 & 1 \end{pmatrix},$$

and let τ be the automorphism corresponding to the matrix

$$\begin{pmatrix} -2 & 1 \\ -3 & 1 \end{pmatrix}.$$

Then each σ and τ is of order 3, while $\tau \circ \sigma$ corresponds to the matrix

$$\begin{pmatrix} 5 & 7 \\ 7 & 10 \end{pmatrix},$$

which is evidently of infinite order.

4. Lie algebra automorphisms

Let F be an algebraically closed field, and let G be an affine algebraic F-group. Let L denote the Lie algebra of G, and let $\mathscr{W}(L)$ denote the group of all Lie algebra automorphisms of L, with its natural structure of an affine algebraic

F-group (an algebraic subgroup of the full linear group on the F-space L). The proof of [2, Proposition 2.2] shows that, *if P is an algebraic subgroup of $\mathcal{W}(G)$ then the natural map $P \to \mathcal{W}(L)$ is a morphism of affine algebraic groups, so that the image of P in $\mathcal{W}(L)$ is an algebraic subgroup of $\mathcal{W}(L)$.*

THEOREM 4.1. *Let F be an algebraically closed field of characteristic 0, and let G be a connected affine algebraic F-group, with Lie algebra L. Let P be a subgroup of $\mathcal{W}(G)$. Then P is algebraic if and only if its image in $\mathcal{W}(L)$ is an algebraic subgroup of $\mathcal{W}(L)$.*

Proof. We have already seen that the condition is necessary. Now suppose that the image of P in $\mathcal{W}(L)$ is an algebraic subgroup of $\mathcal{W}(L)$. Then the proof of [2, Theorem 3.3] shows that the image of P in $\mathcal{W}(\mathcal{C}_1(G/G_u))$ is finite. (In order to adapt this proof to the present situation, it suffices to observe, as in the proof of Theorem 3.1 above, that we may assume P to contain the group of inner automorphisms of G). By Theorem 3.1, this implies that $\mathcal{A}(G)$ is locally finite as a P-module. From the above, we know that the natural map from $\mathcal{G}(A(P))$ (where $A = \mathcal{A}(G)$) to $\mathcal{W}(L)$ is a morphism of affine algebraic groups. Since G is connected and F is of characteristic 0, this map is injective. Therefore, P is the full inverse image of an algebraic subgroup of $\mathcal{W}(L)$, so that P is an algebraic subgroup of $\mathcal{G}(A(P))$. Hence $P = \mathcal{G}(A(P))$ and is an algebraic subgroup of $\mathcal{W}(G)$. Theorem 4.1 is therefore proved.

Let G be as in Theorem 4.1. It will be convenient to have a notation for the maximum connected algebraic subgroup of $\mathcal{W}(G)$, whose existence has been established in Theorem 3.2. Let us denote it by $\mathcal{W}_1(G)$.

Let τ denote the natural homomorphism $\mathcal{W}(G) \to \mathcal{W}(L)$. Since G is connected and F is of characteristic 0, we know that τ is injective, and that its restriction to $\mathcal{W}_1(G)$ is an isomorphism of affine algebraic groups from $\mathcal{W}_1(G)$ to the connected algebraic subgroup $\tau(\mathcal{W}_1(G))$ of $\mathcal{W}(L)$. We shall obtain a description of $\tau(\mathcal{W}_1(G))$ in purely Lie algebra theoretic terms.

From a semidirect product decomposition $G = G_u \cdot K$, with K a maximal reductive subgroup of G, we have a semidirect sum decomposition $L = L_u + R$, where L_u is the Lie algebra of G_u, and so an ad-nilpotent ideal of the Lie algebra L of G, and where R is the Lie algebra of K. If α is an element of $\tau(\mathcal{W}(G))$, then $\alpha(L_u) = L_u$, while $\alpha(R)$ is the Lie algebra of some maximal reductive subgroup of G, which is a conjugate xKx^{-1}, where x is some element of G_u. Therefore, there is an element t in L_u such that $\alpha(R) = \text{Exp}\,(D_t)(R)$, where D_t is the *nilpotent* inner derivation effected by t on L ($D_t(s) = [t, s]$). The automorphisms $\text{Exp}\,(D_t)$, with t ranging over L_u, constitute a unipotent algebraic subgroup of $\mathcal{W}(L)$, which we denote by E_u. Thus, G determines the ad-nilpotent ideal L_u of L, as well as an E_u-orbit $\Sigma = E_u(R)$ of reductive sub Lie algebras of L. Let $\mathcal{W}_G(L)$ denote the subgroup of $\mathcal{W}(L)$ consisting of those automorphisms which stabilize L_u and permute the members of Σ among themselves. Note that, although R is not determined by G, the E_u-orbit Σ of R is determined by G

alone, i.e., is independent of the choice of R. Therefore, $\mathscr{W}_G(L)$ is determined by G alone. In proving the theorem below, we shall obtain a computationally effective (noninvariant) description of $\mathscr{W}_G(L)$, which will show that it is actually an *algebraic* subgroup of $\mathscr{W}(L)$.

Next, let us consider the reductive Lie algebra L/L_u. Denote the *group of inner automorphisms* of L/L_u by $\mathscr{W}_i(L/L_u)$. This may be defined as the connected algebraic subgroup of $\mathscr{W}(L/L_u)$ whose Lie algebra is the Lie algebra of inner derivations of L/L_u, but it coincides with the natural image of the group G' of inner automorphisms. It is known (from the elementary structure theory of semisimple Lie algebras) that $\mathscr{W}_i(L/L_u)$ is generated by the automorphisms Exp (D_t), with t in L/L_u and such that D_t is nilpotent.

THEOREM 4.2. *Let F be an algebraically closed field of characteristic* 0, *and let G be a connected affine algebraic F-group, with Lie algebra L. In the notation introduced above, $\mathscr{W}_G(L)$ is an algebraic subgroup of $\mathscr{W}(L)$, and the natural image in $\mathscr{W}(L)$ of the maximum connected algebraic subgroup $\mathscr{W}_1(G)$ of $\mathscr{W}(G)$ coincides with $\rho^{-1}(\mathscr{W}_i(L/L_u))_1$, where ρ is the canonical map $\mathscr{W}_G(L) \to \mathscr{W}(L/L_u)$.*

Proof. Write $L = L_u + R$, as above, and let V denote the subgroup of $\mathscr{W}(L)$ consisting of the automorphisms that stabilize R, as well as L_u. Clearly, V is an algebraic subgroup of $\mathscr{W}(L)$. Let α be an element of $\mathscr{W}_G(L)$. From our above discussion, we know that there is an element Exp (D_t) of E_u such that Exp $(D_t) \circ \alpha$ stabilizes R, and therefore belongs to V. It follows that $\mathscr{W}_G(L) \subset E_u V$. From the definition of E_u and the fact that V stabilizes L_u, it is clear that V normalizes E_u. It follows that $V \subset \mathscr{W}_G(L)$, so that $\mathscr{W}_G(L) = E_u V$ (evidently, $E_u \subset \mathscr{W}_G(L)$). Since E_u is a connected algebraic subgroup of $\mathscr{W}(L)$, and is normalized by the algebraic subgroup V, this shows that $\mathscr{W}_G(L)$ is an algebraic subgroup of $\mathscr{W}(L)$ (owing to the fact that E_u is connected, and because all the groups involved are affine, the argument needed here is standard, and simpler than the one we used at the end of our proof of Proposition 2.4).

Let Q be the algebraic subgroup of $\mathscr{W}(G)$ defined in Theorem 3.2, so that $Q_1 = \mathscr{W}_1(G)$. Clearly, the group G' of inner automorphisms of G is a connected algebraic subgroup of Q_1. Referring to the semidirect product decomposition $G = G_u \cdot K$, let H denote the element-wise fixer of K in $\mathscr{W}(G)$. It is clear from the definition of Q that H is an algebraic subgroup of Q. Since G' is a connected normal algebraic subgroup of Q, it follows that $G'H$ is an algebraic subgroup of Q, whence $(G'H)_1$ is a connected algebraic subgroup of Q_1. Clearly, $(G'H)_1 = G'H_1$.

Let α be any element of Q. There is an element t in G_u such that, if t' is the corresponding element of G', the automorphism $t'\alpha$ stabilizes K. We have $K = TS$, where $T = \mathscr{C}_1(K)$ and S is the semisimple commutator subgroup $[K, K]$. Since $t'\alpha$ belongs to Q, we know from the proof of Theorem 3.1 that $t'\alpha$ leaves the elements of T fixed. Clearly, $t'\alpha$ stabilizes S. Since the image of G_u in G' is a connected normal algebraic subgroup of Q_1, it is clear that $t'\alpha$ belongs to Q_1 whenever α belongs to Q_1.

The restriction to S of the canonical morphism

$$[G, G]G_u \to ([G, G]G_u)/G_u$$

is an isomorphism of affine algebraic groups $S \to ([G, G]G_u)/G_u$, from which we have an isomorphism $\mathscr{W}(([G, G]G_u)/G_u) \to \mathscr{W}(S)$. Composing this with the canonical homomorphism $\mathscr{W}(G) \to \mathscr{W}(([G, G])/G_u)$, we obtain a homomorphism $\mathscr{W}(G) \to \mathscr{W}(S)$. With the help of Proposition 2.2, it is easy to see that the restriction of this homomorphism to Q is a morphism of affine algebraic groups $\eta\colon Q \to \mathscr{W}(S)$ (it is evident from Theorem 3.1 that $\mathscr{W}(S)$ is algebraic). Moreover, if α is an element of Q_1, and if $t'\alpha$ is a corresponding element of Q_1 as obtained above, stabilizing S, then the automorphism of S induced by $t'\alpha$ clearly coincides with $\eta(t'\alpha)$. Therefore, the automorphism of S induced by $t'\alpha$ actually belongs to $\mathscr{W}(S)_1$, whence it is the inner automorphism effected by an element s of S. Now we have $(s')^{-1}t'\alpha \in H$. Hence $Q_1 \subset G'H$, and therefore even $Q_1 \subset (G'H)_1 = G'H_1$. Since $G'H_1 \subset Q_1$ (as we have seen above), our conclusion is that $Q_1 = G'H_1$.

Let τ denote the natural homomorphism $\mathscr{W}(G) \to \mathscr{W}(L)$. Clearly, $\tau(\mathscr{W}(G)) \subset \mathscr{W}_G(L)$, and the restriction of τ to Q_1 is an injective morphism of affine algebraic groups $Q_1 \to \mathscr{W}_G(L)$. The kernel of

$$\rho\colon \mathscr{W}_G(L) \to \mathscr{W}(L/L_u)$$

evidently contains $\tau(H_1)$, whence $(\rho \circ \tau)(Q_1) = (\rho \circ \tau)(G') \subset \mathscr{W}_i(L/L_u)$. Since $\tau(Q_1)$ is connected, this gives

$$\tau(Q_1) \subset \rho^{-1}(\mathscr{W}_i(L/L_u))_1.$$

Conversely, let α be any element of $\rho^{-1}(\mathscr{W}_i(L/L_u))$. There is an element x in G such that $\tau(x')\alpha$ lies in the kernel of ρ. Since this is still an element of $\mathscr{W}_G(L)$, there is an element t in G_u such that $\tau(t')\tau(x')\alpha$ stabilizes R. Since this automorphism is still in the kernel of ρ, it follows that $\tau(y')\alpha$ leaves the elements of R fixed, where we have written y for tx. Therefore, $\tau(y')\alpha$ induces an automorphism of L_u that commutes with every derivation D_r, with r in R. Using that G_u is unipotent, we see that this automorphism of L_u is the differential of an affine algebraic group automorphism μ of G_u that commutes with every conjugation effected by an element of K. Clearly, μ can therefore be extended to yield an element μ^* of $\mathscr{W}(G)$ that leaves the elements of K fixed. Now μ^* belongs to Q, and $\tau(y')\alpha = \tau(\mu^*)$, which shows that α belongs to $\tau(Q)$. Thus we have $\rho^{-1}(\mathscr{W}_i(L/L_u)) \subset \tau(Q)$, whence

$$\rho^{-1}(\mathscr{W}_i(L/L_u))_1 \subset \tau(Q)_1 = \tau(Q_1).$$

Since the reversed inclusion has already been established above, this completes the proof of Theorem 4.2.

Finally, it is interesting to consider the above in the case where F is the field of complex numbers, so that G *is a connected complex algebraic group.* In this case, $\mathcal{W}(G)$ carries the structure of a complex Lie group, being a closed complex Lie subgroup of the group of all complex analytic group automorphisms of G. It is known [3, Theorem 3] that the *topological* identity component of $\mathcal{W}(G)$ is algebraic, in the sense that its natural image in $\mathcal{W}(L)$ is an algebraic subgroup of $\mathcal{W}(L)$. By Theorem 4.1, this means that it is an algebraic subgroup of $\mathcal{W}(G)$. Since it is topologically connected, it is also connected as an algebraic group. Therefore, it is contained in the maximum connected algebraic subgroup $\mathcal{W}_1(G)$ of $\mathcal{W}(G)$. Since $\mathcal{W}_1(G)$ is a connected complex algebraic group, it is also topologically connected. Therefore, the *maximum connected algebraic subgroup $\mathcal{W}_1(G)$ coincides with the topological identity component of the complex Lie group $\mathcal{W}(G)$.*

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