A DIOPHANTINE PROBLEM ON GROUPS IV

BY

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1. Introduction

In this paper we give some improvements of results in earlier papers in this series [1], [2], [3] and add some related results. G will denote a locally compact abelian group with dual group $\hat{G}$. $\Lambda = (\lambda_n)_{n=1}^{\infty}$ is a sequence in $\hat{G}$, $\lambda_n \to \infty$. Except in Theorems 3.2 and 3.3 we assume that for some neighbourhood V of 0 in $\hat{G}$ and some constant $C > 0$,

(1.1) every translate $V + \gamma$ of V contains at most C terms of the sequence $(\lambda_r)_{r=1}^{\infty}$.

We write $E(\Lambda)$ for the set of x in G for which the sequence $((x, \lambda_j))_{j=1}^{\infty}$ is not uniformly distributed on the circle

$$T = \{e^{2\pi i\theta} : \theta \in [0, 1)\}$$

We identify T with the interval $[0, 1)$ where convenient. In case $G = T$, $\hat{G} = Z$ [14, §1.2.7] where Z denotes the additive group of integers. Thus λ_j can be thought of as integers, where $(x, \lambda_j) = e^{2\pi i\lambda_j x}$ $(x \in T)$. Thus $E(\Lambda)$ is the set of x in $[0, 1)$ for which the fractional parts $\{\lambda_1 x\}$, $\{\lambda_2 x\}$, $\cdots$ are not uniformly distributed in $[0, 1)$. Similar remarks apply when $G = \hat{G} = R$, the real line. In both cases Weyl showed $E(\Lambda)$ has Lebesgue measure zero [15]. In [3] we generalised his theorem as follows.

THEOREM 1.1. *If G has an open subgroup $R^a \times H$ where H is compact and $\hat{H}$ is almost torsion free (i.e. has finitely many elements of order m, each $m \geq 1$), $E(\Lambda)$ has local Haar measure zero if* (1.1) *holds.*

The hypothesis about the open subgroup is 'best possible' [3]. Actually (1.1) is relaxed very considerably in [15], [3]. We call a group G having an open subgroup as described above a *Weyl group*.

Note. 'E has local Haar measure zero' means $m(E \cap K) = 0$ for every compact K in G where $dm(x)$ (or dx) is Haar measure in G. We use 'locally a.e' to mean 'except for a set of local Haar measure zero in G'. 'Locally' can be dropped when G is σ-compact.

We now briefly describe the main results of this paper. In §2 we sharpen Theorem 1.1 for some groups by considering the rate of vanishing of the *discrepancy*

$$D(n; x) = \sup_A | N_n(A, x)/n - | A | /2\pi |$$

locally a.e. Here the supremum is over all arcs A of T, of length $| A |$, $N_n(A)$

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being the number of terms $(x, \lambda_1), \cdots, (x, \lambda_n)$ that fall in A. When G is connected the results are like the classical theorems [5], [7] for the case $G = R$ or T.

In §3 we show, when $G = T$, that a finite measure $d\mu(x) \geq 0$ concentrated on $E(\Lambda)$ (that is, $|\mu|(T\backslash E(\Lambda)) = 0$) cannot satisfy

$$\hat{\mu}(n) = O((\log n)^{-1-\varepsilon}) \quad \text{as} \quad n \to \infty \quad (\varepsilon > 0).$$

This improves a result in [3]. Here $\hat{\mu}$ is the Fourier Stieltjes transform defined in the general group case by

$$\hat{\mu}(\gamma) = \int_G (x, \gamma)^- \, d\mu(x) \quad \text{for} \quad \mu \in M(G), \gamma \in \hat{G}.$$

(We write $M(G)$ for the space of complex regular Borel measures on G with total variation norm and $M_0(G)$ for the set of μ in $M(G)$ such that $\hat{\mu}(\gamma) \to 0$ as $\gamma \to \infty$ in $\hat{G}$.)

We also show in §3 when G is a quotient group of a torus $T^{\mathbf{b}}$ ($\mathbf{b}$ any cardinal) that if $\lambda_n \to \infty$ in $\hat{G}$ and $F(\Lambda)$ is the subset of $E(\Lambda)$ defined by

$$(1.2) \qquad F(\Lambda) = \{x \in G : ((x, \lambda_n))_{n=1}^{\infty} \text{ is not dense in } T\}$$

then $F(\Lambda)$ is a $U^* -$ *set*—that is, no nonzero μ in $M_0(G)$ is concentrated on $F(\Lambda)$.

In §4 we exhibit a class of Weyl groups, the $\mathbf{a}$-adic groups $\Omega_{\mathbf{a}}$ studied by von Neumann and van Dantzig; see [11] for references. We prove some simple algebraic results (for example, that $\Omega_{\mathbf{a}}$ is almost torsion free, which appears to be new). §4 ends with some supplementary remarks on the earlier papers in the series.

2. Character sums and discrepancy locally a.e.

In this section we prove the following theorems.

THEOREM 2.1. *Let G be any locally compact abelian group. The hypothesis (1.1) implies*

$$(2.1) \qquad \textstyle\sum_{k=1}^{n} (x, \lambda_k) = O(n^{1/2} (\log n)^b)$$

for a suitable constant $b = b(G)$, locally a.e. in G.

Note. We shall use constants C like Zygmund [16], that is they need not be the same at each occurrence, they are independent of variables such as x and n, and all other dependences are indicated where desirable.

THEOREM 2.2. *Let G be a compact abelian group and suppose $Q(h)$ elements λ in $\hat{G}$ belong to the group*

$$F_h(\hat{G}) = \{\lambda \in \hat{G} : h\lambda = 0\} \quad (h \geq 1).$$

Then if

$$(2.2) \qquad Q(h) = O(h^A) \quad \textit{as} \quad h \to \infty \quad (A > 0)$$

the hypothesis (1.1) implies

$$D(n, x) = o(n^{-1/(A+2)} (\log n)^{\varepsilon+1/2}) \tag{2.3}$$

for every $\varepsilon > 0$, *a.e in* G. *If*

$$Q(h) = O(\log h)^{B}) \quad as \quad h \to \infty, \tag{2.4}$$

the hypothesis (1.1) implies that, locally a.e.,

$$D(n; x) = o(n^{-1/2}(\log n)^{(B+5+\varepsilon)/2}) \quad (\varepsilon > 0 \textit{ arbitrary}). \tag{2.5}$$

THEOREM 2.3. *Let* G *be a locally compact abelian group and suppose the component of* 0 *is open in* G. *Then (1.1) implies*

$$D(n; x) = O(n^{-1/2}(\log n)^{b}) \tag{2.6}$$

for a suitable constant $b = b(G)$, *locally a.e in* G.

We prove Theorem 2.3 first. We require two lemmas.

LEMMA 2.1. *We have for every integer* $M \geq 1$,

$$nD(n; x) \leq 300 \left(\frac{n}{M+1} + \sum_{h=1}^{M} \frac{1}{h} \left| \sum_{j=1}^{n} (x, h\lambda_j) \right| \right). \tag{2.7}$$

Proof. See [8, Theorem III].

LEMMA 2.2. Let $z^{(p)} = (z_j^{(p)})_{j=1}^{a}$ *be a vector in* R^a *for* $p = 1, \cdots, n$. *Suppose there are at most* C *of these vectors in each box* $[0, 1)^a + m$, *where* m *has integer coordinates. Then for* $A > 1$, $h > 0$, $n \geq 2$,

$$S = \sum_{p=1}^{n} \prod_{j=1}^{a} \min\,(1/\pi h\,|\,z_j^{(p)}\,|, 2A) \leq C_1 \sum_{r=0}^{a} (\log n)^r/h^r \tag{2.8}$$

where C_1 *depends only on* C, A *and* a.

We use the convention that min $(1/0, 2A)$ is $2A$.

Proof. The contribution to S of those $z^{(p)}$ with $z_j^{(p)} \geqq n + 1$ or $z_j^{(p)} < -n - 1$ for some j is at most

$$n(2A)^{a-1}/(n + 1) \leqq (2A)^{a-1}.$$

The contribution of those $z^{(p)}$ with $-1 \leqq z_j^{(p)} < 1$ for all j is at most $2^a \cdot C \cdot (2A)^a$.

We partition the remaining $z^{(p)}$ as follows. Consider those $z^{(p)}$ for which

$$-1 \leqq z_j^{(p)} < 1 \quad \text{for} \quad j \notin K,$$

$$1 \leq z_j^{(p)} < n + 1 \quad \text{or} \quad -n - 1 \leqq z_j^{(p)} < -1 \quad \text{for} \quad j \in K,$$

where K is a subset of $\{1, 2, \cdots, a\}$ with r members $(1 \leqq r \leqq a)$. These $z^{(p)}$ satisfy

$$-m_j - 1 \leqq z_j^{(P)} < -m_j \quad \text{or} \quad m_j \leqq z_j^{(P)} < m_j + 1$$

for some integers m_j, $1 \leqq m_j \leqq n$, for all j in K.

For a fixed set K and fixed m_j $(j \in K)$ the contribution of these $z^{(p)}$ to S is at most

$$2^a C \cdot \frac{(2A)^{a-r}}{(\pi h)^r} \frac{1}{m_1 \cdot \cdots \cdot m_r}$$

If now K is fixed but the m_j vary $(1 \leqq m_j \leqq n)$ the total is clearly

$$2^a C \cdot \frac{(2A)^{a-r}}{(\pi h)^r} \left(\frac{1}{1} + \cdots + \frac{1}{n}\right)^r.$$

If K then varies over all subsets of $\{1, 2, \cdots, a\}$ with r members the total is

$$\frac{a!}{(a-r)!r!} 2^a \cdot C \cdot \frac{(2A)^{a-r}}{(\pi h)^r} \left(\frac{1}{1} + \cdots + \frac{1}{h}\right)^r \leqq \frac{(4A)^a a! C(1 + \log n)^r}{h^r}.$$

Summing from $r = 1$ to a and adding the two contributions computed first,

$$S \leqq C(2A)^{a-1} + (4A)^a a! C \textstyle\sum_{r=0}^{a} (1/h^r)(1 + \log n)^r$$

which yields the desired result.

Proof of Theorem 2.3. Let $dm_1(x)$ be Haar measure on $R^a \times H$, the component of 0 in G; H is connected and $\hat{H}$ is torsion free [11, (24.35)]. Now

$$dm_1(x) = dx_1 \cdots dx_a \, dz.$$

(dx_j = Lebesgue measure on R, $1 \leq j \leq a$; dz = Haar measure on H. We always take compact groups to have total Haar measure = unity.) Suppose we can prove (2.6) for the sequence $((x - x_0, \lambda_j))_{j=1}^{\infty}$ for any x_0 in G, a.e w.r.t. $dm_1(x)$. Then it is easy to deduce (2.6) holds locally a.e in G; see the reasoning in [3, §2.8]. Now $(\lambda_j)_{j=1}^{\infty}$ considered as a sequence in $(R^a \times H)^\wedge$ also satisfies (1.1) with adjusted C and V; this is proved exactly like [3, Lemma 2.6]. We can thus assume throughout that $G = R^a \times H$ (as taking $x_0 \neq 0$ has no significant effect on the calculation).

By Theorem 3 of [9], (2.6) will follow, with exponent $b = (C_1 + 3)/2$, if we can show

$$I = \int_{[-A,A]^a \times H} n^2 D^2(m, n; x) \, dm(x) \leq Cn (\log n)^{b_1}, \tag{2.9}$$

for any $A > 0$, where $D(m, n; x)$ is the discrepancy of the sequence $(\lambda_{m+n})_{n=1}^{\infty}$ $(m \geq 0)$. We take $m = 0$ to simplify the writing. By Lemma 2.1, and Minkowski's inequality,

$$\begin{aligned} I^{1/2} &\leq C(n^{1/2} + \textstyle\sum_{1 \leq h^2 \leq n} (1/h) \\ &\quad \cdot \left(\displaystyle\int_{[-A,A]^a \times H} | \textstyle\sum_{k=1}^{n} (x, h\lambda_k) |^2 \, dm_1(x)\right)^{1/2} \end{aligned} \tag{2.10}$$

(we have taken $M = [n^{1/2}]$).
Now

$$\begin{aligned} J &= \int_{[-A,A]^a \times H} | \textstyle\sum_{k=1}^{n} (x, h\lambda_k) |^2 \, dm_1(x) \\ &= \textstyle\sum_{p,q=1}^{n} \left\{\prod_{j=1}^{a} \displaystyle\int_{-A}^{A} \exp\,(2\pi i h x_j(y_j^{(p)} - y_j^{(q)})) \, dx_j\right\} \int_H (z, h\gamma_p - h\gamma_q) \, dz \end{aligned}$$

where

$$\lambda_k = (y_1^{(k)}, \cdots, y_a^{(k)}, \gamma_k) = (y^{(k)}, \gamma_k)$$

and

$$(x, \lambda_k) = \exp(2\pi i \textstyle\sum_{j=1}^{a} x_j y_j^{(k)}) \cdot (z, \gamma_k).$$

Let $S(p, q)$ be 1 if $\gamma_p = \gamma_q$, 0 otherwise. Since $\hat{H}$ is torsion free,

$$J \leq \textstyle\sum_{p,q=1}^{n} \prod_{j=1}^{A} \min(1/\pi h \mid y_j^{(p)} - y_j^{(q)} \mid, 2A) \cdot S(p, q).$$

We now use (1.1). Consider a fixed value of the index q. If $S(p, q) = 1$, the vector

$$z^{(p)} = y^{(p)} - y^{(q)}$$

can only fall in any cube $[0, 1)^a + m(m \in R^a)$ for at most C values of p. So Lemma 2.2 applies. Summing over p, and then q,

$$J < Cn\left(1 + \frac{\log n}{h} + \cdots + \frac{(\log n)^a}{h^a}\right).$$

Thus the sum on the right of (2.10) is at most

$$\begin{array}{ll} Cn^{1/2} \log n & \text{if } a = 1 \\ Cn^{1/2} (\log n)^{a/2} & \text{if } a \geq 2. \end{array}$$

Here we use the fact that $\sum_h 1/h^{3/2}$, $\sum_h 1/h^{5/2}$, $\cdots$ are convergent series. This completes the proof of (2.9) and thus of Theorem 2.3.

Theorem 2.1. can obviously be proved by the same technique. The value of b can be taken to be $\frac{1}{2}(a + 3 + \varepsilon)$ in (2.1), where $R^a \times H$ is open in G, H compact; [14, §2.4].

The proof of Theorem 2.2 is similar. We have to obtain the following inequalities:

$$\int_G n^2 D^2(m, n; x)\, dx \leq Cn^{2-2/(A+2)} \tag{2.11}$$

if (2.2) is assumed;

$$\int_G n^2 D^2(m, n; x)\, dx < Cn(\log n)^{B+2} \tag{2.12}$$

if (2.4) is assumed. We can then apply Theorems 3 and 5 of [9]. We only prove (2.11). We have

$$\int_G |\textstyle\sum_{k=1}^{n} (x, h\lambda_k)|^2\, dx = \sum_{p,q=1}^{n} \int_G (x, h\lambda_p - h\lambda_q)\, dx = \sum_{q=1}^{n} N_q$$

where N_q is the number of solutions p of $h\lambda_p = h\lambda_q$. The number of distinct values of λ_p that could solve this equation is clearly at most $Q(h)$, and by (1.1) the number of solutions p is at most $CQ(h)$ for fixed q; that is, $N_q \leq CQ(h)$, and

$$\sum_{h=1}^{M} \frac{1}{h}\left(\int_G \left|\sum_{k=1}^{n} (x, hg_k)\right|^2 dx\right)^{1/2} \leq C \sum_{k=1}^{M} \frac{Q(h)^{1/2}}{h} \cdot n^{1/2} \leq CM^{A/2} n^{1/2}.$$

We now apply lemma 2.1 with $M = [n^{1/(2+A)}]$ to obtain (2.11).

3. Measures concentrated on $E(\Lambda)$ and $F(\Lambda)$

We begin with the following theorem.

THEOREM 3.1. *Let $\lambda_1, \lambda_2, \cdots$ be a sequence in R or Z satisfying (1.1). Let μ be a positive finite measure concentrated on $E(\Lambda)$. Then for every $\varepsilon > 0$,*

$$\limsup_{\lambda \to +\infty} |\hat{\mu}(\lambda)| (\log \lambda)^{1+\varepsilon} = +\infty. \tag{3.1}$$

If $\lambda_1 > 0$, $\lambda_{n+1}/\lambda_n \geq c > 1$ $(n > 1)$, then we have in fact

$$\limsup_{\lambda \to +\infty} |\hat{\mu}(\lambda)| (\log\log \lambda)^{1+\varepsilon} = +\infty. \tag{3.2}$$

Proof. Assume for definiteness that $(\lambda_n)_{n=1}^{\infty}$ is a sequence in Z. Suppose if possible that

$$|\hat{\mu}(-n)| = |\hat{\mu}(n)| \leq C/(\log(n+2))^{1+\varepsilon} \quad (n \geq 0). \tag{3.3}$$

We have, for $h \geq 1$ integer,

$$I = \int_0^1 |\textstyle\sum_{k=1}^{n} e^{2\pi i h\lambda_k x}|^2 \, d\mu(x) = \sum_{p,q=1}^{n} \hat{\mu}(h\lambda_p - h\lambda_q).$$

The number of solutions of $|h\lambda_p - h\lambda_q| \in [j, j+1)$ $(j = 0, 1, 2 \cdots)$ is uniformly bounded by C, for a fixed q. Combining this fact with (3.3), clearly

$$I \leq n \textstyle\sum_{j=0}^{n-1} C/(\log(j+2))^{1+\varepsilon} \leq Cn^2/(\log(n+2))^{1+\varepsilon}.$$

The argument is now exactly like that in [15], [1]. Let

$$f_n(x) = (1/n) \textstyle\sum_{k=1}^{n} e^{2\pi i h\lambda_k x}, \qquad n_k = [e^{k^b}] \quad \text{where} \quad b = 2/(2+\varepsilon).$$

Then

$$\sum_{k=1}^{\infty} \int_0^1 |f_{n_k}(x)|^2 \, d\mu(x) < \infty,$$

so $\sum_{k=1}^{\infty} |f_{n_k}(x)|^2 < \infty$ a.e $(d\mu)$, so $f_{n_k}(x) \to 0$ a.e $(d\mu)$, and since $n_{k+1}/n_k \to 1$, one deduces easily $f_n(x) \to 0$ a.e $(d\mu)$. Since this is true for $h \geq 1$, Weyl's criterion gives $\mu(E(\Lambda)) = 0$ which is absurd, and so (3.1) must be true. The proof of (3.2) is similar, but we use $\lambda_p - \lambda_q \geq c_1^{p-q}$ $(p > q)$. This completes the proof of Theorem 3.1.

If we consider the smaller set $F(\Lambda)$, we find that it is a U^*-set (this is a much stronger assertion than that $F(\Lambda)$ is locally null). Actually we need a rather different hypothesis on Λ, one that is not comparable with (1.1) in general but is much weaker if G has simple structure (see Theorem 3.3).

THEOREM 3.2. *Let G be a locally compact abelian group and $\Lambda = (\lambda_n)_{n=1}^{\infty}$ a sequence in $\hat{G}$. Suppose the relation $m\lambda_n \in K$ has only finitely many solutions (m, n), $m \geq 1$, $n \geq 1$, for each compact set K in $\hat{G}$. Then $F(\Lambda)$ is a U^*-set.*

Theorem 3.3 is a consequence of the following lemmas. Lemma 3.2 is similar to a result of Rajchman [12]. Lemma 3.1 is well known when $G = T$ [16].

LEMMA 3.1. (i) *If E supports no measure $\mu \geq 0$ of mass 1 in $M_0(G)$, E is a U^*-set.*

(ii) *If $E_n (n \geq 1)$ are U^*-sets, so is $\bigcup_{n=1}^{\infty} E_n$.*

LEMMA 3.2. *Let A be an arc of T and let*

$$E(A, \Lambda) = \{x \in G : (x, \lambda_n) \notin A \quad (n = 1, 2 \cdots)\}.$$

Under the hypothesis on Λ of Theorem 3.2, $E(A, \Lambda)$ supports no measure $\mu \geq 0$ in $M_0(G)$.

Proof of Lemma 3.1. In this proof X_B is the indicator function of the set B. We assert that

(3.4) if $\mu \in M_0(G)$ and $h \in L^1(d \mid \mu \mid)$, $dv = hd\mu$, then $v \in M_0(G)$.

To see this, note that $M_0(G)$ is a closed set in $M(G)$. Next, if $t(x)$ is a trigonometric polynomial on G, and $\mu \in M_0(G)$, $dv = t\, d\mu$, clearly $v \in M_0(G)$. Finally if $h \in L^1(d \mid \mu \mid)$ there is such a t with

$$\| t - h \|_{L^1(d|\mu|)} < \varepsilon$$

[11, §31.4]). (3.4) follows at once from these three statements.

Now suppose E is not a U^*-set and $\mu \in M_0(G)$ is concentrated on E. Then for a suitable closed $F \subset E$, $X_F\, d \mid \mu \mid$ is a positive measure, not zero, supported on E, of form $hd\mu$, $h \in L^1(d \mid \mu \mid)$. This proves (i). To obtain (ii) suppose $E = \bigcup_{n=1}^{\infty} E_n$ is not a U^*-set and let $\mu \in M_0(G)$ be concentrated on E. Then $\mid \mu \mid (E_n) > 0$ some n and $dv = X_{E_n}\, d \mid \mu \mid$ is again of form $dv = hd\mu$, so E_n is not a U^*-set. This proves (ii).

Proof of Lemma 3.2. Suppose if possible $d\mu \geq 0$, nonzero in $M_0(G)$, concentrated on $E(A, \Lambda)$. Then $\hat{\mu}(0) > 0$. Let

$$f(e^{2\pi ix}) = \sum_{n=-\infty}^{\infty} c_n e^{2\pi inx}, \qquad B = \sum_n \mid c_n \mid < \infty, \qquad c_0 > 0,$$

be a 'triangular' function vanishing outside the arc A. We have

$$\sum_{n=-\infty}^{\infty} c_n (x, n\lambda_k) = 0 \quad (k \geq 1)$$

for x in $E(A, \Lambda)$ and integrating termwise w.r.t. $d\mu$,

$$\sum_{n=-\infty}^{\infty} c_n\, \hat{\mu}(n\lambda_k) = 0 \quad (k \geq 1).$$

Thus if K is a compact set in G such that $\mid \hat{\mu}(\gamma) \mid < \varepsilon$ for $\gamma \notin K$,

$$0 = \sum_{n\lambda_k \in K} c_n\, \hat{\mu}(n\lambda_k) + \sum_{n\lambda_k \notin K} c_n\, \hat{\mu}(n\lambda_k).$$

The first sum on the right is $c_0\, \hat{\mu}(0)$ if k is large, by hypothesis on Λ, and the second is at most $B\varepsilon$ in modulus. If ε is small enough, we have a contradiction, and this completes the proof of Lemma 3.2.

We now obtain Theorem 3.3. from Lemmas 3.1 and 3.2 on observing that $F(\Lambda) = \bigcup_{A \subset T} E(A; \Lambda)$, and we can take the union over arcs with rational endpoints.

The hypothesis on Λ in Theorem 3.3 may not be fulfilled even when $\hat{G}$ is discrete and torsion free and Λ is a sequence of distinct points; take for example $\hat{G} = R_d$, the reals with the discrete topology, and $\Lambda = (1/n)_{n=1}^{\infty}$. Then $n\lambda_n = 1$ has infinitely many solutions. On the other hand, we have the following result.

THEOREM 3.3. *Let G be a locally compact abelian group. Suppose G has an open subgroup (topologically isomorphic with) $(T^{\mathbf{b}}/A) \times R^a$, where $\mathbf{b}$ is any cardinal and A is a closed subgroup of $T^{\mathbf{b}}$. If $\lambda_n \to \infty$ in $\hat{G}$, $F(\Lambda)$ is a U^*-set.*

Stronger results are known when $G = T$ [12], [13].

Proof. We have only to show that if K is compact in $\hat{G}$, $m\lambda_n \in K$ has only finitely many solutions (m, n) $(m \geq 1, n \geq 1)$. We can identify restriction of characters to $H = (T^b/A) \times R^a$ with a homomorphism $Q : \hat{G} \to \hat{H}$ which is continuous, open and has compact kernel Γ [14, §2.1]. We show that $mQ(\lambda_n) \in Q(K)$ has only finitely many solutions. Now $Q(\lambda_n) \to \infty$ in $\hat{H}$. For let U be a relatively compact neighbourhood of 0 in $\hat{G}$. Then $Q(U)$ is a relatively compact neighbourhood of 0 in $\hat{H}$. If p is given and q is sufficiently large, $\lambda_q \notin U + \Gamma + \lambda_p$; so $Q(\lambda_q) \notin Q(U) + Q(\lambda_p)$. It follows easily that $Q(\lambda_n) \to \infty$.

Now $\hat{H}$ can be identified with $B \times R^a$ where B is a subgroup of the (discrete) weak direct product $Z^{\mathbf{b}}$ [14, §2.2]. Thus $Q(K) \subset F \times [-C, C]^a$ for some $C > 0$ where F is finite in B. It remains to show that if $Q(\lambda_n) \to \infty$ in $B \times R^a$,

$$mQ(\lambda_n) \in F \times [-C, C]^a \tag{3.5}$$

has only finitely many solutions (m, n) $(m \geq 1, n \geq 1)$. In fact, if (m_j, n_j) are distinct solutions $(j \geq 1)$ it is clear that n_j is bounded, so m_j is unbounded. But this obviously implies $Q(\lambda_{n_j}) = (0, Z_{n_j})$ for large j, where $Z_{n_j} \to \infty$ in R^a. This contradicts (3.5), and so there are only finitely many such (m, n) and the proof of Theorem 3.3 is complete.

4. Some properties of the groups $\Omega_{\mathbf{a}}$

The following description of the groups $\Omega_{\mathbf{a}}$ is amplified in [11, §§10 and 25]. Let $\mathbf{a}$ be any fixed sequence of integers $(a_n)_{n=-\infty}^{\infty}$ where $a_n > 1$, all n. Let $\Omega_{\mathbf{a}}$ be the set of all $x = (x_n)$ in the Cartesian product

$$\textstyle\prod_{n \in Z} \{0, 1, \cdots a_n - 1\}$$

such that only finitely many x_n $(n < 0)$ are nonzero, and with the following addition: if $x, y \in \Omega_{\mathbf{a}}$, $x + y$ is $z = (z_n)$ where

$$x_{m_0} \neq 0, \quad x_n = 0 \quad \text{for} \quad n < m_0, \qquad y_{n_0} \neq 0, \quad y_n = 0 \quad \text{for} \quad n < n_0,$$

$$p_0 = \min(m_0, n_0)$$

$$z_n = 0 \quad \text{for} \quad n < p_0;$$

$$x_{p_0} + y_{p_0} = t_{p_0} a_{p_0} + z_{p_0}, \quad z_{p_0} \in \{0, 1 \cdots, a_{p_0} - 1\}, \quad t_{p_0} = 0 \quad \text{or} \quad 1;$$

if $z_{p_0}, \cdots z_k$ and $t_{p_0}, \cdots t_k$ are defined $(k > p_0)$,

$$x_{k+1} + y_{k+1} + t_k = t_{k+1}\, a_{k+1} + z_{k+1},$$

$$z_{k+1} \in \{0, 1, \cdots a_{k+1} - 1\}, \qquad t_{k+1} \text{ integer.}$$

Let Λ_k be the set of all x_m in $\Omega_{\mathbf{a}}$ such that $x_n = 0$ for $n < k$ $(k \in Z)$. We write $\Delta_{\mathbf{a}}$ for Λ_0. With the topology whose subbasis is

$$\{x + \Lambda_k : x \in \Omega_{\mathbf{a}},\, k \in Z\},$$

$\Omega_{\mathbf{a}}$ becomes a locally compact abelian, 0-dimensional, σ-compact, metrizable group in which the Λ_k $(k \in Z)$ are compact open subgroups forming a neighbourhood base of 0.

Let $\mathbf{a}^*$ be the sequence: $a_n^* = a_{-n}$ for $n \in Z$. The dual group of $\Omega_{\mathbf{a}}$ can be identified with $\Omega_{\mathbf{a}^*}$ under the following map: $y \leftrightarrow \chi_y$; $y \in \Omega_{\mathbf{a}^*}$, $\chi_y \in \hat{\Omega}_{\mathbf{a}}$,

$$\chi_y(x) = \exp\{2\pi i \textstyle\sum_{n=m}^{-k-1} x_n \sum_{s=n}^{-k-1} y_{-s}/a_n\, a_{n+1} \cdots a_s\}$$

$(x_j = 0$ for $j < m$; $y_l = 0$ for $l \leq k)$.

The dual group of $\Delta_{\mathbf{a}}$ can be identified with

$$Z(\mathbf{a}^\infty) = \{t = \exp\,(2\pi i l/a_0 \cdots a_r),\ l \text{ integer}\},$$

a discrete subgroup of T, under the mapping $t \leftrightarrow \chi_t$,

$$\chi_t(x) = t^{x_0 + a_0 x_1 + \cdots + (a_0 a_1 \cdots a_{r-1}) x_r}.$$

The main theorem of this section is

THEOREM 4.1. *Let* $\mathbf{a}$ *be any sequence as above. Then*
(i) $\Omega_{\mathbf{a}}$ *is a Weyl group,*
(ii) $\Omega_{\mathbf{a}}$ *is almost torsion free.*

Before giving this proof we state a very simple proposition. Recall that $F_k(G)$ is the set of x in G for which $kx = x + \cdots + x$ (k summands) is 0.

PROPOSITION 4.1. *Suppose* $F_m(G)$ *has at most* s *elements and* $F_n(G)$ *at most* t *elements. Then* $F_{mn}(G)$ *has at most* st *elements.*

Proof. Let $x \in F_{mn}(G)$, then $nx \in F_m(G)$. If $nx = y$ for a fixed y, clearly x can take at most t values in G. Since y can take at most s values in $F_m(G)$, x is one of at most st elements of G.

Proof of Theorem 4.1. To prove (i) it is enough to show that $\Omega_{\mathbf{a}}$ has a compact open subgroup H with almost torsion free dual. Take $H = \Delta_{\mathbf{a}}$, then $\hat{H}$ is algebraically a subgroup of T, which is almost torsion free. This proves (i).

To prove (ii) it is enough, by Proposition 4.1, to show that $F_p(\Omega_{\mathbf{a}})$ is finite for each prime p. By Theorem 4 of [3] and the fact that $\Omega_{\mathbf{a}^*}$ is a Weyl group, the groups $F_p(\Omega_{\mathbf{a}})$ are compact $(p = 2, 3, 5 \cdots)$. Thus it is enough to prove each $F_p(\Omega_a)$ discrete. We divide the prime numbers into two classes:

(a) for arbitrarily large positive values of n, a_n is divisible by p;
(b) for sufficiently large n, say $n \geq n_0$, p does not divide a_n.

In case (a) we shall show

$$F_p(\Omega_a) = \{0\}. \tag{4.1}$$

Suppose $x \in \Omega_a$, say $x \in \Lambda_n$, and $px = 0$. If

$$x = (\cdots, 0, x_n, x_{n+1}, \cdots, x_k, \cdots)$$

we write

$$x(k) = x_n + a_n x_{n+1} + \cdots + (a_n \cdots a_{k-1})x_k \quad (k > n). \tag{4.2}$$

$x(k)$ is a positive integer, and if $x(k)$ is divisible by $a_n \cdots a_{k-1}$ then

$$x_n = \cdots = x_{k-1} = 0.$$

Thus to prove $x = 0$, we have only to show that for arbitrarily large values of k,

$$a_n \cdots a_{k-1} \mid x(k). \tag{4.3}$$

Now from the definition of addition in Ω_a, $px = 0$ implies that $px(k)$ is a multiple of $a_n \cdots a_k$ for each $k > n$, say

$$px(k) = qa_n \cdots a_k \quad (q \geq 1 \text{ integer})$$

For arbitrarily large values of k we can cancel p from a_k and obtain (4.3). This shows $x = 0$ and (4.1) follows.

In case (b) we shall show that for the n_0 in question,

$$F_p(\Omega_a) \cap \Lambda_{n_0} = \{0\}. \tag{4.4}$$

Since Λ_{n_0} is open, it follows that $F_p(\Omega_a)$ is discrete, once we prove (4.4).

Suppose $x \in F_p(\Omega_a) \cap \Lambda_{n_0}$, $x \neq 0$, and let x_n $(n \geq n_0)$ be the first nonzero coordinate of x. Clearly px_n is a multiple of a_n. But p and a_n are coprime, and $0 \leq x_n \leq a_n - 1$. We conclude that $x_n = 0$. This is absurd, (4.4) is proved and the proof of Theorem 4.1 is complete.

We note that in general $\bigcup_{n=1}^{\infty} F_n(\Omega_a)$ may be infinite:

PROPOSITION 4.2. *Let a_n $(n \in Z)$ be distinct prime numbers. Then $F_{a_n}(\Omega_a)$ has a_n elements $(n \in Z)$.*

Proof. There are a_n elements of the form $x = (\cdots, 0, x_n, x_{n+1}, \cdots)$ such that $a_n x = 0$. Namely, x_n can take any value $0, 1, \cdots, a_{n-1}$, and once x_n is given the succeeding coordinates are uniquely defined. For the congruence

$$a_n x_{n+k} + z \equiv 0 \pmod{a_{n+k}}$$

has exactly one solution x_{n+k} in $\{0, 1, \cdots a_{n+k} - 1\}$, whatever $k \geq 1$ and $z \geq 0$ are.

On the other hand an element of the form $(\cdots, 0, x_j, x_{j+1}, \cdots)$ $(j < n)$ $(x_j \neq 0)$ cannot have order a_n. This is proved in the same way as (4.4). The proof of Proposition 4.2 is complete.

Here is a simple corollary of Theorem 4.1.

THEOREM 4.2. *Let* $\mathbf{a}$ *be a sequence such that any prime* p*, which divides some* a_m*, divides* a_n *for arbitrarily large positive values of* n*. Then* $\Omega_{\mathbf{a}}$ *is torsion free.*

An example is $a = (\cdots, p, p, p, \cdots)$ where p is an integer ≥ 2. In case p is prime this special case has number theoretic importance, and harmonic analysis has been carried out in important recent work of Y. Meyer and J-P. Schreiber. See Meyer's book *Algebraic numbers and harmonic analysis*, North Holland, 1972.

Proof. Exactly as in case (a) of Theorem 4.1 one shows that $F_p(\Omega_{\mathbf{a}}) = \{0\}$ for every prime p that divides some a_m. If p divides no a_n, one shows as in case (b) of Theorem 4.1. that $F_p(\Omega_{\mathbf{a}}) \cap \Lambda_{n_0} = \{0\}$ for every n_0, which implies $F_p(\Omega_{\mathbf{a}}) = 0$. Theorem 4.2 is proved.

Similarly, under the following condition, $F_{\mathbf{a}} = \bigcup_{n=1}^{\infty} F_n(\Omega_{\mathbf{a}})$ is a discrete subgroup of Ω_a.

(4.5) There is an integer n_0 such that for every $n \in Z$ either $n \mid a_k$ for arbitrarily large positive k, or n is coprime with a_k for $k \geq n_0$.

For integers n of the first type clearly $F_n(\Omega_{\mathbf{a}})$ is $\{0\}$.

For integers n of the second type, our method (b) shows $F_n(\Omega_{\mathbf{a}}) \cap \Lambda_{n_0} = \{0\}$. So $F_{\mathbf{a}} \cap \Lambda_{n_0} = \{0\}$. Since $F_{\mathbf{a}}$ is a group, it must be a discrete subgroup of $\Omega_{\mathbf{a}}$. In case (e.g.) $\mathbf{a} = (\cdots, 11, 7, 5, 3, 2, 2, 2, 2, \cdots)$, $F_{\mathbf{a}}$ is clearly an infinite discrete subgroup.

A question which I cannot answer in general is whether $\Omega_{\mathbf{a}}$ is ever (topologically isomorphic with) $H \times D$ where H is a compact group and D is a discrete group. The following condition on $\mathbf{a}$ ensures that this is not the case.

(4.6) Except for a finite set S of positive integers n, either n divides a_k for arbitrarily large positive k, or n is coprime with every a_k.

For in this case, clearly $F_{\mathbf{a}}$ is finite and we deduce from the following proposition that $\Omega_{\mathbf{a}}$ has no infinite discrete subgroup.

PROPOSITION 4.3. *The elements of a discrete subgroup* D *of any group* $\Omega_{\mathbf{a}}$ *have finite order.*

Proof. Since D is discrete, $D \cap \Lambda_n = \{0\}$ for some n. If x is in $\Omega_{\mathbf{a}}$ and k is a suitable integer, $kx \in \Lambda_n$. In case $x \in D$, this implies $kx = 0$, and the proposition is proved.

We obtain from Theorem 2.2 the following result.

THEOREM 4.3. *Let* **a** *be any sequence as above. Let* $y^{(n)}$ $(n \geq 1)$ *be a sequence in* $\Omega_{\mathbf{a}*}$ *satisfying* (1.1). *Then the sequence*

$$\chi_{y^{(n)}}(x) \quad (n \geq 1)$$

has discrepancy $D(n; x)$ *satisfying*

$$D(n; x) = o(n^{-1/3} (\log n)^{\varepsilon+1/2})$$

for every $\varepsilon > 0$, *for almost all* x *w.r.t. Haar measue on* $\Omega_{\mathbf{a}}$.

Haar measure on $\Omega_{\mathbf{a}}$ is described in [11, §15]. For a subset of Λ_n it is simply the product measure $\prod_{j=n}^{\infty} \mu_j$ where μ_j assigns mass $1/a_j$ to each point of $\{0, 1, \cdots, a_j - 1\}$.

Proof of Theorem 4.3. It is enough to deal with discrepancy of sequences

$$\chi_{y^{(n)}}(x - x_0)$$

a.e in the compact open subgroup $\Delta_{\mathbf{a}}$. This point is covered in Theorem 2.3. Since $\hat{\Delta}_{\mathbf{a}}$ is a subgroup of T, $Q(h) \leq h$ inthe notation of Theorem 2.2 and it is easy to complete the proof.

We end this paper with some remarks on earlier papers in the series.

(a) In Theorem 1.1 (and 2.1, 2.3) we cannot delete the word 'local'. In fact, $E(\Lambda)$ can have infinite Haar measure. Let $G = R_d \times R$. It is not hard to show [11, §11.33] that Haar measure $m(A)$ of a set $A \subset R_d \times \{0\}$ is 0 if A is countable, $+\infty$ if A is uncountable.

Let $\lambda_n = (0, n) \in (R_d)^{\wedge} \times R$. Clearly $(\lambda_n)_{n=1}^{\infty}$ satisfies (1.1), but $(x, \lambda_n) = 1$ on $R_d \times \{0\}$ and so $R_d \times \{0\} \subset E(\Lambda)$. This shows $E(\Lambda)$ has infinite Haar measure.

(b) Condition (12) at the end of [3] does not now seem to me to lead to a proof of $\mu(E(\Lambda)) = 0$ without very strong assumptions about the sequence $\gamma_1, \gamma_2, \cdots$.

(c) Corollary 3.2 in [2] can be considerably improved by imitating the method of Theorem 4 of [10], providing the hypothesis on G (which reduces to '$F_2(G)$ has Haar measure 0') is strengthened.

THEOREM 4.4. *Suppose G is a locally compact abelian group having at most countably many elements of finite order. Let K be any compact set in G which supports a continuous measue in $M(G)$. Then there is a sequence Λ in $\hat{G}$ for which K is appropriate.*

The hypothesis on G ensures that if μ is a continuous measure in $M(G)$, there are continuous measures μ_n in $M(G)$ such that $\hat{\mu}_n(\gamma) \equiv \hat{\mu}(n\gamma)$. (See Lemma 2 of [4]: one puts $\mu_n(E) = \mu(\{x : nx \in E\})$.) We leave it to the interested reader to check the details.

(d) It follows from work of Déschamps-Gondim [6] that if Λ is a topological Sidon set in $\hat{G}$, where G is a connected locally compact abelian group, then $E(\Lambda)$ is uncountable and dense in G. For this we only need the result that every compact neighbourhood of 0 is appropriate for Λ; see [10], [2] for the application of this to prove our assertion about $E(\Lambda)$.

(e) In [3] I stated that I could find no *direct* proof that $F_m(\hat{G})$ is compact if G satisfies the hypothesis of (1.1). Here is such a proof.

PROPOSITION 4.4. *Let G be a locally compact abelian group, suppose B is an open subgroup of G and $F_m(\hat{B})$ is finite. Then $F_m(\hat{G})$ is compact* ($m = 1, 2 \cdots$).

Proof. Let A be the annihilator of B in $\hat{G}$; then A is compact since B is open. Now as topological groups, $\hat{B} = \hat{G}/A$. So if $F_m(\hat{B})$ is finite, the relation

$$m(\gamma + A) = A \quad \text{or} \quad m\gamma \in A$$

only has solutions γ from finitely many cosets of A in $\hat{G}$. In particular the equation $m\gamma = 0$ only has a compact set of solutions in $\hat{G}$. This completes the proof.

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