CANONICAL GENERATORS OF FUCHSIAN GROUPS

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Introduction

Let Γ be a Fuchsian group. A hyperbolic convex fundamental domain of Γ is a fundamental domain of Γ which is also convex in the hyperbolic sense. It is clear that the boundary in the upper half plane H^+ of such a domain consists of hyperbolic lines or line segments, which are called sides. If $A \in \Gamma$ is elliptic or parabolic and generates the stabilizer of its fixed point, then there exists a hyperbolic (H) convex fundamental domain of Γ such that A conjugates a pair of sides. If $A \in \Gamma$ is any transformation which conjugates a pair of sides on some H-convex fundamental domain of Γ, then A will be called a *canonical generator* of Γ. This paper is devoted to the rudimentary study of the canonical generators of a Fuchsian group.

We first find necessary and sufficient conditions for a hyperbolic transformation $A \in \Gamma$ to be a canonical generator. Then for a class of the two generator genus zero groups we exhibit all the canonical generators. We also characterize in terms of traces the free hyperbolic transformations of Γ. In particular we show that they act very much like the parabolic and elliptic elements of a Fuchsian group.

The techniques used in this paper are readily seen to be a generalization of Ford's isometric circles and the Poincare method of generating fundamental domains. However, we note that Theorem 1 is in fact a generalization of Ford's theorem for constructing a fundamental domain for which a given parabolic element of Γ conjugates a pair of sides.

In comparing the various ways of constructing fundamental domains for Γ, we recall that in Ford's method it is essential that ∞ be an ordinary point or the fixed point of some parabolic transformation. This hypothesis on ∞ guarantees the radii of the isometric circles do not approach ∞, which implies the Ford domain is non-void. The following example shows this hypothesis to be necessary even if ∞ is the fixed point of a canonical generator. Let

$$\Gamma = \{\begin{pmatrix} 4 & -5 \\ 1 & -1 \end{pmatrix} = A, \quad \begin{pmatrix} 4 & 0 \\ 0 & 1/4 \end{pmatrix} = B\}.$$

Then Γ is the discrete free product of the cyclic groups $\{A\}$ and $\{B\}$. Moreover, A is a canonical generator. These facts follow from Theorem 1 of [5]. However, $B^n A$ for $n \geq 0$, n an integer, is of the form

$$\begin{pmatrix} * & * \\ (1/4)^n & -(1/4)^n \end{pmatrix}.$$

Thus the isometric circle of $B^{-n}A$ is $(\frac{1}{4})^n \mid z - 1 \mid = 1$ or $\mid z - 1 \mid = 4^n$. We also note that Ford assumed that ∞ was not the fixed point of an elliptic

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transformation. This assumption allowed him to conclude that every transformation has an isometric circle. However, this assumption is somewhat extraneous. For the domain which is exterior to all isometric circles and within a sector determined by the elliptic transformation which generates the stabilizer of ∞ is a fundamental domain.

Our method disposes of the assumption that ∞ be an ordinary point. In fact we will almost always conjugate so that ∞ is the fixed point of a hyperbolic transformation. Moreover, we will know one of the generators which conjugates a pair of sides on the fundamental domain that we construct, whereas in both Ford's method and the Poincare method no information is given about the transformations which conjugate the sides. Incidently, we note the restriction that the center of the Poincare domain not be a fixed point of an elliptic transformation is similarly somewhat extraneous.

DEFINITIONS. Let (z_1, z_2, z_3, z_4) denote the cross ratio which associates z_2, z_3, z_4 with 0, 1, ∞, respectively. For $A \in LF(2, \mathbf{R})$, the group of linear fractional transformations with real entries and determinant 1, let x_A, y_A denote the fixed points of A. For A hyperbolic and $B \in LF(2, \mathbf{R})$ set

$$K(A, B) = 1/1 - (Bx_A, x_A, By_A, y_A) \tag{1}$$

if $\{x_A, y_A\} \cap \{x_B, y_B\} = \emptyset$ and $K(A, B) = 1$ otherwise. We observe that if

$$A = \begin{pmatrix} \rho & 0 \\ 0 & \rho^{-1} \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

then $K = ad$. Also, this definition of K is different from the one in [5], which defined K to be the cross ratio.

$$C_B = \{z \mid \arg Bz = \arg z\} \cap H^+, \tag{2}$$

$$U_B = \{z \mid d(\mathrm{Im}, z) = d(\mathrm{Im}, B(z)\}, \tag{3}$$

$$R_B = \{z \mid \arg Bz = \arg(-z^*)\} \cap H^+, \tag{4}$$

where d is the Poincare metric, z^* is the conjugate of z, and Im is $\{iy \mid y > 0\}$.

We shall use the usual notation S^0 to mean the interior of S and H-line for a semicircle in H^+ which is perpendicular to the real axis. Also a *free hyperbolic transformation*, A, of Γ is one which generates the stabilizer of its two fixed points $\{x_A, y_A\}$, and one of the intervals determined by x_A and y_A in $R \cup \{\infty\}$ is in the set of proper discontinuity. We note here that in the case Γ is finitely presented, Maclachlan [4] has shown this definition to be equivalent to A being a boundary element as defined in [2].

Γ is a Fuchsian group of signature rank 2 if Γ is a triangle group, or a free product of two cyclic subgroups of finite or infinite order, or

$$\Gamma = \{A, B, C \mid C^n = ABA^{-1}B^{-1}C = 1\}.$$

We remark that the rank of a Fuchsian group cannot be determined in the natural way from the presentation in [3]. Thus we need the above definition (see [6]).

If Γ is finitely presented, then a set of generators will be called a *standard generating set* if they yield the standard presentation [3] and come from a fundamental polygon of Γ, i.e., a fundamental domain whose boundary in H^+ consists of a finite number of hyperbolic lines or line segments.

The main theorem

THEOREM 1. *Let Γ be a Fuchsian group. Let A be a hyperbolic transformation of Γ. Then A is canonical generator of Γ if and only if for every $V \in \Gamma$, $V \neq A^p$, p an integer, we have $K(A, V) \notin (0, 1]$. Moreover A is a free hyperbolic transformation of Γ if and only if $K(A, V) \leq 0$ for every $V \in \Gamma$, $V \neq A^p$, p an integer.*

Before proving this theorem we make some observations. The condition $K(A, V) \leq 0$ for every $V \in \Gamma$, $V \notin A^p$ is equivalent to the following: if we pick A, V from the corresponding matrix group in $SL(2, \mathbf{R})$ with the condition that the traces, $\operatorname{tr}(A)$, $\operatorname{tr}(V)$, be non-negative, then for some integer n, $\operatorname{tr}(A^n V) \leq 0$. We easily see that this property is possessed by each elliptic and parabolic transformation of Γ. Moreover, it will be apparent that the following proof can be with minor modifications applied to an elliptic or parabolic transformation.

Lemmas

LEMMA 1. *Let*

$$A = \begin{pmatrix} \rho & 0 \\ 0 & \rho^{-1} \end{pmatrix}, \qquad \rho > 0.$$

(i) *If $V \in LF(2, \mathbf{R})$ satisfies $K(A, V) > 1$, then $C_V = \emptyset$.*
(ii) *If $K(A, V) < 0$, then $R_V = \emptyset$.*

Proof. (i) One easily sees that $C_V = \{z \mid V(z) = \alpha^2 z \text{ for some } \alpha > 0\}$. Setting

$$A_\alpha = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}$$

we get $z \in C_V$ is equivalent to z is the fixed point of A_α^{-1} for some $\alpha > 0$. Since $K(A, V) = K(A_\alpha, V) > 1$ we see that $A_\alpha^{-1} V$ is never an elliptic or the identity. transformation. Thus $C_V = \emptyset$.

(ii) Suppose $K(A, V) < 0$. If $z \in R_V$ then $V(z) = -\alpha^2 z^*$ for some $\alpha > 0$. We may assume

$$V = \begin{pmatrix} a & ad-1 \\ 1 & d \end{pmatrix},$$

where $a > 0$, $d < 0$. Thus $\operatorname{Im}(az) = -\operatorname{Im}(\alpha^2 z^* d)$. Hence $R_V = \emptyset$.

LEMMA 2. *Let $V \in LF(2, \mathbf{R})$. Then $U_V = R_V \cup C_V$.*

Proof. This lemma follows from the observation that the distance from a point to a line is measured along the perpendicular to the line through the given point.

LEMMA 3. *Let*

$$A = \begin{pmatrix} \rho & 0 \\ 0 & \rho^{-1} \end{pmatrix}, \qquad \rho > 0$$

and $V \in LF(2, \mathbf{R})$ *satisfy* $K(A, V) < 0$. *Then* $C_V \cap \mathrm{Im} = \emptyset$.

Proof. Now $z \in C_V$ if and only if z is the fixed point of $A_\alpha^{-1} V$ for some $\alpha > 0$.

We conjugate A, V so that

$$A \to A_1 = \begin{pmatrix} a & c \\ c & a \end{pmatrix} \quad \text{and} \quad V \to V_1 = \begin{pmatrix} a_1 & b_1 \\ c_1 & a_1 \end{pmatrix},$$

where a_1, $a \geq 0$ and $c_1 > | b_1 |$. Now Im goes onto $\{ | z | = 1\} \cap H^+$. An easy calculation shows that 1 is outside the isometric circle $| c_1 z - a_1 | = 1$ is equivalent to $K(A, V) < 0$. To complete the proof we note that A_α goes onto a matrix of the form

$$A_{1\alpha} = \begin{pmatrix} \dfrac{\alpha + \alpha^{-1}}{2} & \gamma \\ \gamma & \dfrac{\alpha + \alpha^{-1}}{2} \end{pmatrix}.$$

Hence $A_{1\alpha}^{-1} V_1$ is elliptic is equivalent to the isometric circles of $A_{1\alpha}$ and V_1 having non-void intersection and the fixed point of $A_{1\alpha}^{-1} V_1$ in H^+ is the point of intersection. Thus C_V after a conjugation is the isometric circle $| c_1 z - a_1 | = 1$.

LEMMA 4. *Let*

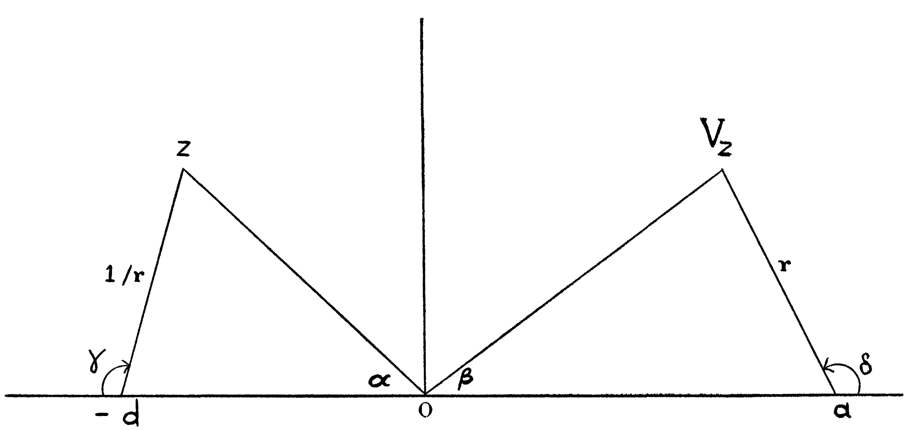
$$A = \begin{pmatrix} \rho & 0 \\ 0 & \rho^{-1} \end{pmatrix}, \qquad \rho > 0$$

and $V \in LF(2, \mathbf{R})$. *If* $K(A, V) > 1$. *Then* $R_V \cap \mathrm{Im} = \emptyset$.

Proof. We may assume

$$V = \begin{pmatrix} a & b \\ 1 & d \end{pmatrix} \quad \text{with} \quad ad > 1,$$

i.e., the fixed points of V and A separate each other. It will suffice to show that $R_V = \{z \,\|\, z + d \,| = \sqrt{(d/a)}\}$ (see [5]). Now consider the following diagram.



If $|z + d| = \sqrt{(d/a)} = 1/r$. Then $|V(z) - a| = \sqrt{(a/d)} = r$ (cf. [5]) and $\gamma = \delta$. Hence, by similar triangles $\alpha = \beta$ and $z \in R_V$. If $z \in R_V$, then $\alpha = \beta$ and as always $\gamma = \delta$ (cf. [5]). Therefore by similar triangles $r^2 = a/d$ or $r = \sqrt{(a/d)}$ and $|z + d| = \sqrt{(d/a)}$.

Remarks. If $V \neq A^r$ for any integer r then
(1) C_V, R_V are H-lines,
(2) $C_V \cap \text{Im} \neq \emptyset$ if and only if $1 > K(A, V) > 0$.

Proof of the main theorem

Sufficiency. Let

$$A = \begin{pmatrix} \rho & 0 \\ 0 & \rho^{-1} \end{pmatrix} \in \Gamma$$

satisfy $K(A, V) \notin (0, 1]$, for all $V \in \Gamma$ such that $V \neq A^p$. For each $V \in \Gamma$ such that $K(A, V) \neq 0, 1$, let $D_V = \{z \mid d(\text{Im}, z) < d(\text{Im}, Vz)\}$. We observe D_V is open and its boundary is U_V. Also since $\text{Im} \cap U_V = \emptyset$ we have $\text{Im} \subseteq D_V$. We note here that $V(H^+ \backslash D_V)^\circ = D_{V^{-1}}$ and $V(U_V) = U_{V^{-1}}$.

We now consider the two exceptional cases. If $K(A, V) = 1$ then $V = A^p$ for some integer p. This case will be handled when we choose the circles for A. If $K(A, V) = 0$, then since Γ is Fuchsian, V is an elliptic transformation of order 2 whose fixed points are on the imaginary axis. In this case we note that $U_V = H^+$. We choose D_V to be $\{\text{Re}(z) > 0\} \cap H^+$. To avoid problems later we note that up to conjugation there are at most two elliptic elements V of order 2 such that $K(A, V) = 0$. Moreover if E_1, E_2 are two such elliptic transformations, then $E_1 E_2 = A^p$, some integer p.

Let

$$F = \bigcap_{V \in \Gamma, V \neq A^p} D_V \cap \{z \mid 1 < |z| < \rho^2\}$$

We claim that F is a fundamental domain for Γ and that A conjugates a pair of sides on $\bar{F}$. It is clear that the Γ images of F are all disjoint. To prove $F \neq \emptyset$, F is open, and that A conjugates a pair of sides on $\bar{F}$, it will suffice to show that every Euclidean line segment of the form $L_\theta = \{te^{i\theta} \mid 1 < t < \rho^2\}$, where $0 < \theta < \pi$, intersects only a finite number of D_V. One observes here that if $|\theta - \pi/2| < |\theta' - \pi/2|$, then L_θ intersects fewer D_V than $L_{\theta'}$.

To see that L_θ intersects at most a finite number of D_V, we first observe that if there exists a $z \in H^+$ such that z belongs to infinitely many $D_V \cup U_V$ then Γ is not a Fuchsian group. Now if L_θ intersects infinitely many D_V we see that such a z exists. For the radii of the U_V which intersect L_θ are at least $\sin\theta$ and the centers of the U_V accumulate in $R \cup \{\infty\}$.

To show that any point $z \in H^+$ can be mapped into $\bar{F}$ we use the same technique as Ford. Let $V \in \Gamma$ be such that $d(\text{Im}, Vz) \leq d(\text{Im}, Bz)$ for all $B \in \Gamma$. Then for an appropriate n, $A^n Vz \in \bar{F}$.

Necessity. Let $V \in \Gamma$ be such that $0 < K(A, V) < 1$. Then $C_V \cap \text{Im} \neq \emptyset$. Hence the origin is inside the H-line C_V. Now let P be a hyperbolic convex fundamental domain such that A conjugates a pair of sides s_1, s_2. We observe that s_1, s_2 must be arcs of two circles which bound discs that contain the origin. Let θ_1, θ_2 be such that if $\theta_1 < \theta < \theta_2$, then the ray $r_\theta = \{te^{i\theta} \mid t > 0\}$ intersects the boundary of P only on s_1, s_2. Now $r_\theta \cap C_V \neq \emptyset$ for all θ. Let $z_\theta \in r_\theta \cap C_V$. Then for some integers m, n, we have $A^n V(z_\theta) \in \bar{P}$. Hence, $A^{n-m}V(z_\theta) = z_\theta$, or $A^{n-m\pm1}V(z_\theta) = z_\theta$. Noting that this can occur at most countably many times we see that P is not a fundamental domain for Γ.

To complete the proof of the theorem we now observe:

(i) If A is a hyperbolic transformation and $V \in LF(2, \mathbf{R})$ satisfies $1 > K(A, V) > 0$, then $K(A, VAV^{-1}) > 1$.

(ii) If A is a hyperbolic transformation and V_1, $V_2 \in LF(2, \mathbf{R})$ satisfy $K(A, V_1) \leq 0$, $K(A, V_2) \leq 0$ but $(V_1 y_A, x_A, V_2 y_A, y_A) < 0$, then $K(A, V_1 V_2) > 0$ and $K(A, V_1^{-1} V_2) > 0$. Now the fact that A is a free hyperbolic transformation if and only if $K(A, V) \leq 0$ for all $V \in \Gamma$, for which $V \neq A^p$ is apparent.

Applications

THEOREM 2. *Let Γ be a Fuchsian group of signature rank 2 of genus 0.*

(1) *If Γ is a triangle group presented by*

$$\Gamma = \{E_1, E_2 \mid E_1^p = E_2^q = (E_1 E_2)^r = 1\}$$

where $1/p + 1/q + 1/r < 1$, then E_1, E_2, and $E_1 E_2$ are the only canonical generators up to conjugacy classes.

(2) *If Γ is the free product of two cyclic groups, $\Gamma = \{A\} * \{B\}$, where* $\operatorname{tr}(A) \geq 0$, $\operatorname{tr}(B) \geq 0$ *and* $\operatorname{tr}(AB) \leq -2$ (*cf.* [5]) *then A, B and AB are the only canonical generators of Γ up to conjugacy classes.*

Proof. (1) Let T be any canonical generator not equal to a conjugate of E_1, E_2 or $E_1 E_2$. Then T is hyperbolic. Let C be the axis of T. Conjugate T in Γ so that C passes through the fundamental domain of Γ determined by E_1, E_2 and $E_1 E_2$. Observe that at least one of x_{E_1}, x_{E_2}, $x_{E_1E_2}$, $x_{E_2E_1}$, but not all are in the half disc determined by C.

Let U, W be two distinct elliptic transformations from a choice of $\{E_1, E_2, E_1 E_2, E_2 E_1\}$ whose fixed point lies on one side of C, and S one of the elliptic generators whose fixed point lies on the other side of C. Now either $US = W$, or $U^{-1}S = W$ (or we may commute U and W). Thus applying the observation at the end of Theorem 1, we get $K(T, S) > 0$. Since T is elliptic $1 > K(T, S)$.

(2) This part is exactly as (1) only an elliptic fixed point may be replaced by either a parabolic fixed point or the two fixed points of a free hyperbolic transformation, Q.E.D.

COROLLARY. *In a genus zero Fuchsian group of signature rank 2 the only standard set of generators are those such that* $\operatorname{tr}(A) \geq 0$, $\operatorname{tr}(B) \geq 0$ *and* $\operatorname{tr}(AB) \leq 0$.

PROPOSITION 1. *Let Γ be a Fuchsian group and $T \in \Gamma$ be of minimal trace. Then T is a canonical generator.*

Proof. Follows from Theorem 1 and [5].

Remark. Fenchel and Nielsen have shown [1] that every finitely generated Fuchsian group has an element of minimal trace.

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