MODULAR QUOTIENT GROUPS

BY
Morris Newman

Introduction

Let

$$\Gamma = \Gamma_t = SL(t, Z)$$

be the t-dimensional modular group,

$$\Gamma(n) = \{A \in \Gamma : A \equiv I \bmod n\},$$

the principal congruence subgroup of Γ of level n, so that $\Gamma(n)$ consists of all elements of Γ congruent elementwise to the identity element modulo n, and

$$G(n, m) = \Gamma(n)/\Gamma(m).$$

Here n, m are arbitrary positive integers such that n divides m. The question which motivated this paper was to determine $G(n, m)'$, the commutator subgroup of $G(n, m)$, and hence to determine the number of 1-dimensional representations of $G(n, m)$. It turns out that for $t > 2$ a complete answer to this question can be given using a result of J. L. Mennicke proved in [4]. This in turn brings out some interesting new relationships involving the principal congruence groups $\Gamma(n)$, and implies a number of other results, such as a necessary and sufficient condition for the solvability of the quotient group $G(n, m)$.

The case $t = 2$ requires a special discussion, and is the motivation for examining the normal subgroups of Γ containing a principal congruence group $\Gamma(n)$. This question had already been studied and answered completely in [3], [5], and [6], with a more natural (but also more restrictive) definition of principal congruence group. In order to obtain similar results in the present situation, limitations must be imposed on m, n, and t.

We list for convenience some important properties of the groups $\Gamma(n)$, $G(n, m)$. These may be found for example in [8] or [9].

Let $(m, n) = \delta$, $[m, n] = \Delta$, so that δ is the greatest common divisor of m and n and Δ the least common multiple of m and n. Then

(1) $$\Gamma(m)\Gamma(n) = \Gamma(\delta),$$

(2) $$\Gamma(m) \cap \Gamma(n) = \Gamma(\Delta),$$

(3) $$G(\delta, m) \cong G(n, \Delta).$$

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A restatement of (3) is

(4) $$G(d, da) \cong G(db, dab)$$

for all positive integers a, b, d such that $(a, b) = 1$.

(5) $$G(\delta, \Delta) \cong G(\delta, m) \times G(\delta, n),$$

where $\times$ stands for direct product.

A restatement of (5) is

(6) $$G(d, dab) \cong G(d, da) \times G(d, db)$$

for all positive integers a, b, d such that $(a, b) = 1$.

Let $m = \prod p^{b_p}$ be the canonical decomposition of m into prime powers. For each prime p dividing m let p^{a_p} be the highest power of p dividing n (so that $a_p \geq 0$, $b_p > 0$). Then

(7) $$G(n, mn) \cong \times G(p^{a_p}, p^{a_p+b_p}),$$

where $\times$ denotes direct product, and is extended over all primes p dividing m.

$G(n, mn)$ is abelian if and only if m divides n. If every prime dividing m also divides n, then the order of $G(n, mn)$ is m^{t^2-1}. If p is a prime dividing n, $G(n, pn)$ is abelian of type $(p, p, \cdots, p)$ and order p^{t^2-1}, and may be thought of as the multiplicative group of matrices

$$I + nE, \ E \text{ modulo } p, \ \mathrm{tr}\ (E) \equiv 0 \bmod p.$$

This group is isomorphic to the additive group of matrices E, E modulo p, tr $(E) \equiv 0 \bmod p$.

As usual, E_{ij} denotes the matrix with 1 in position (i, j) and 0 elsewhere.

Preliminary matter

LEMMA 1. *Let H be a normal subgroup of the group G. Then*

$$(G/H)' = G'H/H.$$

Proof. The result is an immediate consequence of the fact that if xH, yH are arbitrary elements of G/H, then the commutator of xH, yH is just

$$[xH, yH] = (xH)(yH)(xH)^{-1}(yH)^{-1} = xyx^{-1}y^{-1}H = [x, y]H.$$

LEMMA 2 (Mennicke [4]). *Suppose that $t > 2$, and let i, j be any distinct pair of integers such that $1 \leq i, j \leq t$. Let $\Delta(I + nE_{ij})$ stand for the normal closure of $I + nE_{ij}$ in Γ. Then*

$$\Delta(I + nE_{ij}) = \Gamma(n).$$

LEMMA 3. *Suppose that $t > 2$. Then $I + n^2E_{12}$ belongs to $\Gamma(n)'$, the commutator subgroup of $\Gamma(n)$.*

Proof. The lemma follows from the identity

$$[I + nE_{13}, I + nE_{32}] = (I + nE_{13})(I + nE_{32})(I - nE_{13})(I - nE_{32})$$
$$= I + n^2E_{12},$$

as may be seen from the multiplication law $E_{ij}E_{kl} = \delta_{jk}E_{il}$.

LEMMA 4. $\Gamma(n)' \subset \Gamma(n^2)$.

Proof. Let A, B be any elements of $\Gamma(n)$. Then

$$A \equiv I \bmod n, \qquad B \equiv I \bmod n,$$

so that

$$(A - I)(B - I) \equiv 0 \bmod n^2, \qquad AB \equiv A + B - I \bmod n^2.$$

Similarly,

$$BA \equiv B + A - I \bmod n^2,$$

so that

$$AB \equiv BA \bmod n^2, \qquad [A, B] = ABA^{-1}B^{-1} \equiv I \bmod n^2$$

From this the lemma follows at once.

The results for $t > 2$

Our first result is

THEOREM 1. *Suppose that $t > 2$, and that n is any positive integer. Then $\Gamma(n)' = \Gamma(n^2)$.*

Proof. Because of Lemma 4, we need only show that $\Gamma(n^2) \subset \Gamma(n)'$. By Lemma 2, $\Gamma(n^2) = \Delta(I + n^2E_{12})$. By Lemma 3, $I + n^2E_{12}$ belongs to $\Gamma(n)'$. Hence $\Delta(I + n^2E_{12}) \subset \Gamma(n)'$ and the result follows.

Theorem 1 is certainly false for $t = 2$. For then, if $n > 2$, $\Gamma(n)$ is a free group of finite rank >2, and so $\Gamma(n)'$ is a free group of countably infinite rank, and so is not even of finite index in Γ.

The next theorem is the principal result of this section.

THEOREM 2. *Suppose that $t > 2$, and that m, n are arbitrary positive integers. Put $\delta = (m, n)$. Then*

$$G(n, mn)' = G(n\delta, mn). \tag{8}$$

Proof. By Lemma 1, we have that

$$G(n, mn)' = (\Gamma(n)/\Gamma(mn))' = \Gamma(n)'\Gamma(mn)/\Gamma(mn).$$

By Theorem 1 and (1) we have

$$\Gamma(n)'\Gamma(mn) = \Gamma(n^2)\Gamma(mn) = \Gamma((n^2, mn)) = \Gamma(n\delta).$$

This completes the proof.

As a corollary, we obtain

COROLLARY 1. *The number of 1-dimensional representations of $G(n, mn)$ is just δ^{t^2-1}, where $t > 2$, $\delta = (m, n)$.*

Proof. The number of 1-dimensional representations of $G(n, mn)$ is the order of $G(n, mn)/G(n, mn)'$, and we have

$$G(n, mn)/G(n, mn)' = G(n, mn)/G(n\delta, mn) \cong G(n, n\delta).$$

Since $\delta \mid n$, the order of $G(n, n\delta)$ is just δ^{t^2-1}, which is the desired result.

Another noteworthy corollary of Theorem 2 is the following:

COROLLARY 2. *If $t > 2$ and $(m, n) = 1$ then $G(n, mn)$ is a perfect group and so not solvable.*

It is clear that Theorem 2 provides an effective means of determining precisely when $G(n, m)$ is solvable. It is also clear from (7) that it is only necessary to consider the case $n = p^a$, $m = p^b$, p prime. In this connection we prove

LEMMA 5. *Suppose that $t > 2$. Let p be a prime, $a \geq 0$, $b > 0$. Then $G(p^a, p^{a+b})$ is solvable if and only if $a \neq 0$.*

Proof. Suppose first that $a = 0$. Then $G(p^a, p^{a+b}) = G(1, p^b)$ and so is not solvable by Corollary 2. Now suppose that $a > 0$. If $b \leq a$, then $G(p^a, p^{a+b})$ is abelian and hence certainly solvable. Suppose that $b > a$. Then a unique positive integer n exists such that

$$2^n a < a + b \leq 2^{n+1} a.$$

A simple calculation now shows that

$$G(p^a, p^{a+b})^{(k)} = G(p^{2^k a}, p^{a+b}), \quad 1 \leq k \leq n.$$

But now $G(p^{2^n a}, p^{a+b})$ is abelian, since

$$G(p^{2^n a}, p^{a+b}) = G(p^{2^n a}, p^{2^n a} p^{b-(2^n-1)a}),$$

and

$$b - (2^n - 1)a \leq 2^n a.$$

Hence $G(p^a, p^{a+b})^{(n+1)}$ is trivial and the result follows. Lemma 5, together with (7), implies the following result:

THEOREM 3. *Suppose that $t > 2$. Then the group $G(n, mn)$ is solvable if and only if each prime dividing m also divides n.*

A comment of some interest implied by the proof of Lemma 5 is that if $G(n, mn)$ is solvable then the length of its derived series is at most $O(\log m)$.

Another corollary, previously proved in [1] by another method, is the following:

COROLLARY 3. *Suppose that $t > 2$, $1 \leq a \leq b - 1$. Then no two of the $b - 1$ groups $G(p^a, p^{a+b})$ are isomorphic, although they are all of the same order $p^{b(t^2-1)}$.*

Proof. By Corollary 1, the number of 1-dimensional representations of

$G(p^a, p^{a+b})$ is $p^{a(t^2-1)}$, since $(p^a, p^b) = p^a$. Since these numbers are all different for $1 \leq a \leq b - 1$, no two of the groups can be isomorphic. This concludes the proof.

Some inclusion theorems

We now go on to some inclusion theorems for the groups $\Gamma(n)$ (and all dimensions t) which are of interest in themselves and which will be used to prove results analogous to the preceding ones for $t = 2$. We must consider the structure of $G(n, np)$ more closely, where p is a prime dividing n.

Let G be the additive abelian group of all $t \times t$ matrices E over $GF(p)$ with $\operatorname{tr}(E) = 0$. Then G is of type $(p, p, \cdots, p)$ and order p^{t^2-1}, and the generators of G may be taken as

$$(9) \qquad \begin{aligned} V_{ij} &= E_{ij}, & & i \neq j, \\ &= E_{ii} - E_{i+1,i+1}, \quad 1 \leq i \leq t-1, & & i = j. \end{aligned}$$

G may also be described as the additive abelian group generated by the normal closure in $SL(t, GF(p))$ of the matrix

$$V_{12} = E_{12}.$$

Thus a subgroup H of G which contains V_{12}, and for which

$$UHU^{-1} \subset H \quad \text{for all} \quad U \in SL(t, GF(p)),$$

must be all of G.

If $p \mid n$ then $G(n, np)$ is isomorphic to G and the generators of $G(n, np)$ may be taken modulo np as $I + nV_{ij}$, where the V_{ij} are given by (9).

Let Δ be a normal subgroup of Γ such that

$$\Gamma(n) \supset \Delta \supset \Gamma(np),$$

and assume that $\Delta \neq \Gamma(np)$. If we can show that Δ contains

$$I + nV_{12} = I + nE_{12},$$

it will follow that Δ must be $\Gamma(n)$, by the preceding remarks. For this to occur some restriction on p is necessary, and what we will prove is the following:

THEOREM 4. *Let p be an odd prime such that $(p, t) = 1$ and $p \mid n$. Then if Δ is a normal subgroup of Γ such that $\Gamma(n) \supset \Delta \supset \Gamma(np)$, Δ must be $\Gamma(n)$ or $\Gamma(np)$.*

Proof. Assume that $\Delta \neq \Gamma(np)$. Then Δ must contain an element $I + nE$ such that $E \not\equiv 0 \bmod p$.

Suppose first that E is diagonal modulo p. Then E cannot be scalar modulo p. For if $E \equiv aI \bmod p$, then $\operatorname{tr}(E) \equiv ta \bmod p$. But $\operatorname{tr}(E) \equiv 0 \bmod p$ and $(t, p) = 1$. Hence $a \equiv 0 \bmod p$, which implies that $E \equiv 0 \bmod p$, a contradiction. If follows that the diagonal entries of E contain at least two elements which are distinct modulo p, and which may be taken as the (1, 1) and (2, 2) elements, after a suitable conjugacy by generalized permutation matrices of $SL(t, GF(p))$ has been performed. Thus we have

$$E = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \dotplus D,$$

where $a \not\equiv b \bmod p$.

Now

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a & b-a \\ 0 & b \end{bmatrix}.$$

Put

$$U = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \dotplus I_{t-2} = I + E_{12}.$$

Then

$$UEU^{-1} - E = (b-a)E_{12},$$

and since $b - a \not\equiv 0 \bmod p$, it follows that Δ must contain $I + nE_{12}$. In this case then we may conclude that Δ is all of $\Gamma(n)$.

Next assume that E is not diagonal modulo p. Then after a suitable conjugacy by generalized permutation matrices of $SL(t, GF(p))$ has been performed, we may assume that the (2, 1) element of E is $\not\equiv 0 \bmod p$. Write

$$E = \begin{bmatrix} A & B \\ C & D \end{bmatrix},$$

where A is 2×2. Put $U = -I_2 \dotplus I_{t-2}$. Then

$$E_1 = UEU^{-1} = \begin{bmatrix} A & -B \\ -C & D \end{bmatrix}, \qquad E + E_1 = 2(A \dotplus D).$$

Since we are assuming that p is odd, we can conclude that Δ must contain $I + n(A \dotplus D)$. Write

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

where $c \not\equiv 0 \bmod p$. Then for any x,

$$A_x = \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -x \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a + xc & b - xa + xd - x^2c \\ c & d - xc \end{bmatrix},$$

$$A_x - A = x\begin{bmatrix} c & -a + d - xc \\ 0 & -c \end{bmatrix}.$$

If we assume that $(x, p) = 1$, it follows that Δ contains

$$I + n\left(\begin{bmatrix} c & -a + d - xc \\ 0 & -c \end{bmatrix} \dotplus 0\right).$$

Choosing $x = 1, 2$ (as we may since p is odd) and subtracting, we find that Δ contains $I + ncE_{12}$; and since $(c, p) = 1$, Δ must also contain $I + nE_{12}$.

Thus in this case also we can conclude that Δ must be all of $\Gamma(n)$. This concludes the proof.

We next prove

THEOREM 5. *Let p be an odd prime such that $(p, t) = 1$ and $p \mid n$. Then if Δ is a normal subgroup of Γ such that*

$$\Gamma(n) \supset \Delta \supset \Gamma(np^2), \tag{10}$$

Δ must be $\Gamma(n)$, $\Gamma(np)$, $\Gamma(np^2)$.

Proof. Intersecting and producting by $\Gamma(np)$ in (10) and using (1) and (2) we find that

$$\Gamma(n) \supset \Delta\Gamma(np) \supset \Gamma(np), \qquad \Gamma(np) \supset \Delta \cap \Gamma(np) \supset \Gamma(np^2).$$

Theorem 4 now implies that

$$\Delta\Gamma(np) = \Gamma(n), \Gamma(np), \qquad \Delta \cap \Gamma(np) = \Gamma(np), \Gamma(np^2).$$

If $\Delta\Gamma(np) = \Gamma(np)$ then $\Gamma(np) \supset \Delta \supset \Gamma(np^2)$, which implies that $\Delta = \Gamma(np)$ or $\Gamma(np^2)$. If $\Delta \cap \Gamma(np) = \Gamma(np)$, then $\Gamma(n) \supset \Delta \supset \Gamma(np)$, which implies that $\Delta = \Gamma(n)$ or $\Gamma(np)$. Assume then that

$$\Delta\Gamma(np) = \Gamma(n), \qquad \Delta \cap \Gamma(np) = \Gamma(np^2).$$

Then

$$\Gamma(n)/\Gamma(np) \cong \Delta/\Gamma(np^2).$$

Since $\Gamma(n)/\Gamma(np)$ is abelian of type $(p, p, \cdots, p)$, the same must also be true of $\Delta/\Gamma(np^2)$. In particular, the pth power of any element of Δ must belong to $\Gamma(np^2)$.

Let $A = I + nE$ be any element of Δ. Since $p \mid n$, we have

$$A^p = (I + nE)^p \equiv I + npE \bmod np^2.$$

But this implies that $E \equiv 0 \bmod p$, which in turn implies that $A \in \Gamma(np)$. Thus $\Delta \subset \Gamma(np)$, and the proof in this case is completed precisely as before. This concludes the proof.

We now use these results to prove

THEOREM 6. *Let m, n be positive integers such that $(m, 2t) = 1$, and each prime dividing m also divides n. Let Δ be a normal subgroup of Γ such that*

$$\Gamma(n) \supset \Delta \supset \Gamma(nm). \tag{11}$$

Then $\Delta = \Gamma(nd)$, for some divisor d of m.

Proof. The proof will be by induction on n and on $\sigma_0(m)$, the number of divisors of m. We note that if m and n satisfy the hypotheses of the theorem, then so do m_1 and n_1, where m_1 is any divisor of m and n_1 any multiple of n.

If $\sigma_0(m) \leq 3$ then $m = 1$, p, or p^2 for some prime p, and the theorem is true in these cases by Theorems 4 and 5.

Now assume the theorem proved for all m and n satisfying the hypotheses of the theorem such that $\sigma_0(m) < k$, where $k \geq 4$. Let m and n satisfy the hypotheses of the theorem and suppose that $\sigma_0(m) = k$. Producting in (11) with $\Gamma(nd)$, where d is any proper divisor of m, we obtain

$$\Gamma(n) \supset \Delta\Gamma(nd) \supset \Gamma(nd).$$

Since d is a proper divisor of m the induction hypothesis implies that

$$\Delta\Gamma(nd) = \Gamma(n\delta), \quad \delta \mid d.$$

Then

$$\Gamma(n\delta) \supset \Delta \supset \Gamma(nd).$$

If $\delta > 1$ we get our conclusion from the induction hypothesis, with n replaced by $n\delta$ and d replaced by d/δ. We may assume therefore that

$$\Delta\Gamma(nd) = \Gamma(n), \quad d \mid m, 1 < d < m. \tag{12}$$

Similarly, intersecting with $\Gamma(nd)$ in (11), we obtain

$$\Gamma(nd) \supset \Delta \cap \Gamma(nd) \supset \Gamma(nm).$$

The induction hypothesis implies (with n replaced by nd and m by m/d) that

$$\Delta \cap \Gamma(nd) = \Gamma(nd\delta), \quad \delta \mid (m/d).$$

Thus

$$\Gamma(n) \supset \Delta \supset \Gamma(nd\delta).$$

If $\delta < m/d$, so that $d\delta < m$, we again get our conclusion from the induction hypothesis, with m replaced by $d\delta$. We may assume therefore that

$$\Delta \cap \Gamma(nd) = \Gamma(nm), \quad d \mid m, 1 < d < m. \tag{13}$$

But now (12) and (13) imply that $\Gamma(n)/\Gamma(nd) \cong \Delta/\Gamma(nm)$, so that $(\Gamma(n):\Gamma(nd))$ is independent of d. But $(\Gamma(n):\Gamma(nd)) = d^{t^2-1}$, and d assumes at least 2 different values, since d may be any proper divisor of m and $\sigma_0(m) \geq 4$. Hence (12) and (13) cannot both hold, and the result is true for all m and n satisfying the hypotheses of the theorem such that $\sigma_0(m) = k$. This concludes the proof.

Results for $t = 2$

From now on we assume that $t = 2$. We remark that Γ and Γ' are no longer equal in this case, but $(\Gamma:\Gamma') = 12$, and $\Gamma' \supset \Gamma(12)$ (see [2] for example).

We first prove

LEMMA 6. *Let m be a positive integer such that $(m, 6) = 1$. Then*

$$G(1, m)' = G(1, m).$$

Proof. By Lemma 1, $G(1, m)' = (\Gamma/\Gamma(m))' = \Gamma'\Gamma(m)/\Gamma(m)$. Now $\Gamma' \supset \Gamma(12)$, and so $\Gamma'\Gamma(m) \supset \Gamma(12)\Gamma(m) = \Gamma((12, m)) = \Gamma$. Hence

$\Gamma'\Gamma(m) = \Gamma$, and the conclusion follows.

We next prove

LEMMA 7. *Let p be a prime >2. Let a, b be integers such that $a > 0, b > 0$. Then*

$$G(p^a, p^{a+b})' = G(p^{a+\min(a,b)}, p^{a+b}). \tag{14}$$

Proof. If $b \leq a$ $G(p^a, p^{a+b})$ is abelian, and so $G(p^a, p^{a+b})'$ is trivial. In this case $a + \min(a, b) = a + b$ and (14) holds.

Now suppose that $b > a$. Then

$$G(p^a, p^{a+b})' = \Gamma(p^a)'\Gamma(p^{a+b})/\Gamma(p^{a+b}).$$

The group $H = \Gamma(p^a)'\Gamma(p^{a+b})$ is a normal subgroup of Γ such that

$$\Gamma(p^{2a}) \supset H \supset \Gamma(p^{a+b}).$$

Furthermore, it is clear that $\Gamma(p^a)'$ is not contained in $\Gamma(p^{2a+c})$ for any positive c (for example, the commutator of

$$\begin{bmatrix} 1 & p^a \\ 0 & 1 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 0 \\ p^a & 1 \end{bmatrix}$$

does not belong to $\Gamma(p^{2a+c})$ for any positive c). But then the same is true for H, and Theorem 6 implies that H must be $\Gamma(p^{2a})$. It follows that

$$G(p^a, p^{a+b})' = G(p^{2a}, p^{a+b}).$$

Since $b > a$, $a + \min(a, b) = 2a$ and so (14) holds in this case as well. This concludes the proof.

Combining these lemmas, we have

THEOREM 7. *Let p be a prime >3. Let a, b be integers such that $a \geq 0$, $b > 0$. Then*

$$G(p^a, p^{a+b})' = G(p^{a+\min(a,b)}, p^{a+b}).$$

Using Theorem 7, formula (7), and elementary properties of direct products, we can show

THEOREM 8. *Suppose that $t = 2$, and that (m, n) are arbitrary positive integers such that $(m, 6) = 1$. Put $\delta = (m,n)$. Then*

$$G(n, mn)' = G(n\delta, mn). \tag{15}$$

(16) *The number of 1-dimensional representations of $G(n, mn)$ is δ^3.*

We omit the proof, which is straightforward.

The classical modular group

Finally, we make one or two comments about the classical modular group $\Gamma = PSL(2, Z)$.

Let Γ^n be the fully invariant subgroup of Γ generated by the n-th powers of the elements of Γ. Then the only normal subgroups of Γ containing Γ' are Γ, Γ^2, Γ^3, Γ' (see [7] for a proof of this statement). Furthermore $(\Gamma:\Gamma^2) = 2$, $(\Gamma:\Gamma^3) = 3$, $(\Gamma:\Gamma') = 6$. On the basis of this information, and following the procedure of Lemma 6, we have

THEOREM 9. *Let* $\Gamma = PSL(2, Z)$, *n a positive integer. Then the number of 1-dimensional representations of* $G(1, n) = \Gamma/\Gamma(n)$ *is just* $(n, 6)$.

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