VECTOR VALUED KOTHE FUNCTION SPACES III

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This paper is a continuation of [8] and [9].

1. Compactness

We begin with a general situation. The results obtained will yield information about compact sets in v.f.s.'s.

Let X and a family $\{Y_\alpha : \alpha \in A\}$ be Hausdorff topological spaces and let $\sigma_\alpha : X \to Y_\alpha$ be continuous maps. We suppose that $\sigma_\alpha(x_1) = \sigma_\alpha(x_2)$ for each α implies $x_1 = x_2$. Define $Y = \prod_\alpha Y_\alpha$ and $\sigma : X \to Y$ by $\sigma(x) = \{\sigma_\alpha(x)\}$. Then σ is continuous and one to one. A set $S \subseteq X$ is said to be *projectively compact* if $\sigma_\alpha(S)$ is compact for each α. A sequence $(x_n) \subseteq X$ is *projectively convergent* if $\sigma_\alpha(x_n)$ is convergent for each α. Other terms are defined similarly.

The proofs of the following propositions present no difficulties; the proof of Proposition 1.1 uses Tychonoff's theorem and that of Proposition 1.3(2) uses the finite intersection property characterization of compactness (see [6]).

PROPOSITION 1.1 *A set $S \subseteq X$ is compact if and only if*

(1) *S is projectively compact, and*

(2) *every projectively convergent net in S is convergent to a point in S.*

PROPOSITION 1.2. *Suppose A is countable. Then a set $S \subseteq X$ is sequentially compact (respectively relatively sequentially compact) if and only if*

(1) *S is projectively sequentially compact (respectively relatively sequentially compact), and*

(2) *every projectively convergent sequence in S is convergent to a point in S (respectively convergent).*

PROPOSITION 1.3. *Suppose A is countable. Then:*

(1) *If in Y_α the compact (respectively relatively compact) sets are sequentially compact (respectively relatively sequentially compact), then the same is true in X.*

(2) *If in Y_α the sequentially compact sets are compact, then the same is true in X.*

(3) *If in Y_α the countably compact sets are compact, then the same is true in X.*

(4) *If in Y_α the countably compact (respectively relatively countably compact) sets are sequentially compact (respectively relatively sequentially compact), then the same is true in X.*

We now make some applications of the above three propositions.

THEOREM 1.4. *If E is metrizable and $S(E)$ is a v.f.s. with a topology finer than the weak topology induced from $\Omega(E)$, then the compact, sequentially compact,*

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and countably compact sets in $S(E)$ are the same. Furthermore, a set $A \subseteq S(E)$ is compact iff it is weakly compact in $\Omega(E)$ and every sequence in A which converges weakly in $\Omega(E)$ converges in $S(E)$.

Proof. This is a simple application of Propositions 1.1–1.3 to the single continuous injection map $i : S(E) \to \Omega(E)$, using the fact that in the weak topology of a Fréchet space the compact, countably compact, and sequentially compact sets are the same [7, p. 318]. ▌

We omit the similar proof of the following result.

THEOREM 1.5. *If E is metrizable and $S(E)$ is a v.f.s. with a topology finer than that induced from $\Omega(E)$, then a set $A \subseteq S(E)$ is compact iff it is compact in $\Omega(E)$ and every sequence in A which converges in $\Omega(E)$ also converges in $S(E)$.*

Let $S(E)$ be a v.f.s. such that for every compact set $K \subseteq Z$, the map $f \to \int_K f\, d\pi$ of $S(E)$ into $\hat{E}$ is continuous. (Here $\hat{E}$ does not have to have its original topology.) If Z is second countable, so that a countable number of integrals suffice to determine f [8, Corollary 6.3] we could apply Propositions 1.1–1.3 to obtain information about the compact sets in $S(E)$. We omit the details.

In order to apply Theorems 1.4 and 1.5, it is necessary to identify the compact and weakly compact sets in $\Omega(E)$. We do this, for special cases, in the results to follow.

PROPOSITION 1.6. *Let Z be a locally compact Abelian group with Haar measure π. Suppose E is a Fréchet space. Then a set $C \subseteq \Omega(E)$ is relatively compact if and only if*

(1) *for every compact set $K \subseteq Z$ and $a \in L^\infty$, the set*

$$\left\{\int_K af\, d\pi : f \in C\right\}$$

is relatively compact in E, and

(2) *given a compact set $K \subseteq Z$, a $p \in P$, and an $\varepsilon > 0$, there is a symmetric neighborhood W of 0 (in Z) such that if $z_0 \in W$ and $f \in C$, then*

$$\int_K p(f(z) - f(z - z_0))\, d\pi < \varepsilon.$$

Proof. Since E is a Fréchet space, all elements of $\Omega(E)$ are functions from Z into E [8, Section 3].

($\Rightarrow$) (1) follows from [8, Theorem 6.1(1)]. We prove (2) in stages.

(a) Suppose $C = \{f\}$, a single function and

$$f = \sum_{j=1}^{n} c(R_j)x_j \in \Gamma(E).$$

Using [4, p. 269], choose a symmetric compact neighborhood W_j such that for $z_0 \in W_j$,

$$\int_K | c(R_j)(z) - c(R_j)(z - z_0) |\, d\pi \leq \varepsilon(np(x_j))^{-1}.$$

Then for $z_0 \in W_1 \cap W_2 \cap \cdots \cap W_n$,

$$\int_K p(f(z) - f(z - z_0))\, d\pi$$

$$\leq \sum_{j=1}^{n} p(x_j) \int_K | c(R_j)(z) - c(R_j)(z - z_0) |\, d\pi \leq \varepsilon.$$

(b) Now suppose $C = \{f\}$ where f is any function in $\Omega(E)$. Let W_1 be any symmetric compact neighborhood of 0. Then $K + W_1$, the image of $K \times W_1$ under the map $(z, w) \to z + w$, is compact. Since $\Gamma(E)$ is dense in $\Omega(E)$ (it separates points of $\Omega(E)' = \Phi(E')$) there is an $f' \in \Gamma(E)$ such that

$$\int_{K+W_1} p(f - f')d\pi \leq \varepsilon.$$

By (a), we can find a symmetric compact neighborhood W_2 of 0 such that if $z_0 \in W_2$, then

$$\int_K p(f'(z) - f'(z - z_0))d\pi \leq \varepsilon.$$

Then for $z_0 \in W_1 \cap W_2$,

$$\int_K p(f(z) - f(z - z_0)) \leq \int_K p(f(z) - f'(z)) + \int_K p(f'(z) - f'(z - z_0))\, d\pi$$

$$+ \int_K p(f'(z - z_0) - f(z - z_0))\, d\pi$$

$$\leq 2\varepsilon + \int_{K+W_1} p(f'(z) - f(z))\, d\pi$$

$$\leq 3\varepsilon.$$

(c) Suppose $C = \{f_i\}$, a finite set. The existence of a W in this case follows easily from (b).

(d) Finally, suppose C is relatively compact. Then C is precompact. Let W_1 be any symmetric compact neighborhood of 0. Since $K + W_1$ is compact, the set

$$V = \{f \in \Omega(E) : \int_{K+W_1} p(f)d\pi \leq \varepsilon\}$$

is a neighborhood in $\Omega(E)$. Let $\{f_j\}$ be a finite set in $\Omega(E)$ such that $C \subseteq \bigcup(f_j + V)$. Let W_2 be the neighborhood for $\{f_j\}$ guaranteed by (c). Let $f \in C$ be arbitrary and choose j' so that $f - f_{j'} \in V$. Then for $z_0 \in W_1 \cap W_2$,

$$\int_K p(f(z) - f(z - z_0))d\pi \leq \int_K p(f(z) - f_{j'}(z))d\pi$$

$$+ \int_K p(f_{j'}(z) - f_{j'}(z - z_0))d\pi$$

$$+ \int_K p(f_{j'}(z - z_0) - f(z - z_0))d\pi$$

$$\leq 3\varepsilon.$$

($\Leftarrow$) Let K, p, and $\varepsilon > 0$ be given. We shall show that C can be covered by a finite number of translates of the set

$$\{f \in \Omega(E) : \int_K p \circ f \, d\pi \leq 2\varepsilon\}.$$

Thus C will be precompact and so relatively compact. Pick a symmetric compact neighborhood W by (2). Let $r(z)$ be a continuous non-negative, real-valued function such that Supp $r \subseteq W$ and $\int r \, d\pi = 1$. Set $M = \operatorname{Sup} r(z)$. For $f \in C$ set

$$f^*(z) = \int f(z - w) r(w) d\pi(w).$$

(This is sort of a convolution.) Then $f^*(z) \in E$. Also, for z_0 fixed and $p_0 \in P$,

$$\begin{aligned} p_0(f^*(z_0) - f^*(z_0 + z)) &= p_0\left[\int (f(z_0 - w) - f(z_0 + z - w)) r(w) d\pi(w)\right] \\ &\leq \int p_0[(f(z_0 - w) - f(z_0 + z - w)) r(w)] d\pi(w) \\ &\leq M \int_W p_0(f(z_0 - w) - f(z_0 + z - w)) d\pi(w) \\ &= M \int_{W + z_0} p_0(f(-w) - f(z - w)) d\pi(w) \\ &= M \int_{-W - z_0} p_0(f(w) - f(w - z)) d\pi(w) \end{aligned}$$

which by (2) can be made arbitrarily small, uniformly for $f \in C$, for z in a sufficiently small neighborhood of 0. Thus f^* is continuous and in fact the set $C^* = \{f^* : f \in C\}$ is equicontinuous.

If we let $\mathcal{C}(E)$ be the set of all continuous functions from Z into E equipped with the topology of uniform convergence on compact sets, we have shown that $C^* \subseteq \mathcal{C}(E)$ and is equicontinuous. For z_0 fixed,

$$\begin{aligned} f^*(z_0) &= \int f(z_0 - w) r(w) d\pi = \int_W f(z_0 - w) r(w) d\pi \\ &= \int_{z_0 + W} f(-w) r(w - z_0) d\pi = \int_{-z_0 - W} f(w) r(z_0 - w) d\pi. \end{aligned}$$

Thus by (1), $\{f^*(z_0) : f^* \in C^*\}$ is relatively compact in E. By Ascoli's Theorem [6, p. 233], C^* is relatively compact in $\mathcal{C}(E)$ and so is relatively compact in the weaker topology of $\Omega(E)$. Thus C^* can be covered by a finite number of translates of

$$(*) \qquad \{f \in \Omega(E) : \int_K p \circ f \, d\pi \leq \varepsilon\}$$

But

$$\begin{aligned}\int_K p(f - f^*)d\pi &= \int_K p(f(z) - \left[\int_Z f(z-w)r(w)d\pi(w)\right] d\pi(w)d\pi(z) \\ &= \int_K p\left(\int_Z [f(z)r(w) - f(z-w)r(w)]d\pi(w)\right) d\pi(z) \\ &\leq \int_K \int_Z r(w)p(f(z) - f(z-w))d\pi(w)d\pi(z) \\ &= \int_Z r(w) \int_K p(f(z) - f(z-w))d\pi(z)d\pi(w) \\ &= \int_W r(w) \int_K p(f(z) - f(z-w))d\pi(z)d\pi(w) \\ &\leq \varepsilon\end{aligned}$$

by the choice of W and r. The last inequality together with $(*)$ show that C can be covered as claimed. (A proof of the p-measurability of $f(z-w)$ which allows the use of Fubini's Theorem above is similar to the proof of the analogous result for real valued functions found in [3, p. 634].) ▌

Remark. The only reason for restricting E to be a Fréchet space in the proposition is that expressions such as $f(z - z_0)$ and $f(z - w)$ used in the proof are then defined since the elements of $\Omega(E)$ are functions. If the definition of these expressions is extended to all of $\Omega(E)$ and certain relationships between these extensions are shown (c.f. [8, Proposition 5.1]), then the proposition can be proved for a general E.

PROPOSITION 1.7. *If E is a separable reflexive Banach space then the following statements about a set $C \subseteq \Omega(E)$ are equivalent*:

(1) *C is weakly relatively compact.*

(2) *For every $g \in \bar{\Phi}(E')$, the set $\langle C, g\rangle$ is weakly relatively compact in Ω.*

(3) *For every $g \in \bar{\Phi}(E')$, compact set K, and $\varepsilon > 0$, $\langle C, g\rangle$ is bounded in Ω and there is a $\delta > 0$ such that if $R \subseteq K$ is measurable and $\pi(R) < \delta$, and $f \in C$, then $\int_R |\langle f, g\rangle| \, d\pi < \varepsilon$.*

Proof. (2) $\Leftrightarrow$ (3) is just the characterization of the weakly relatively compact sets in Ω given in [2, p. 98].

(1) $\Rightarrow$ (2). The map $T : \Omega(E) \to \Omega$ given by $Tf = \langle f, g\rangle$ has an adjoint $T^* : \Phi \to \bar{\Phi}(E')$ given by $T^*b = bg$ and so is weakly continuous. The result follows.

(3) $\Rightarrow$ (1) is more difficult. Since $\langle C, g\rangle$ is bounded for every $g \in \bar{\Phi}(E')$ we have that $\int \langle C, g\rangle d\pi$ is bounded for every $g \in \bar{\Phi}(E')$ and so C is bounded in $\Omega(E)$. Let G be the strong dual of $\bar{\Phi}(E')$ and let $\bar{C}$ be the closure of C in G under the weak topology induced from $\bar{\Phi}(E')$. Now $\bar{C}$ is compact in this weak topology since it is contained in the bipolar of C which is the polar of a $\beta(\bar{\Phi}(E'), \Omega(E))$ neighborhood. Let $\varphi \in \bar{C}$. We now apply [8, Theorem

7.1] (with E and E' switched; see [9, Proposition 1.1]) to show that $\varphi \in \Omega(E)$, thus completing the proof. If π does not have compact support then by [9, Proposition 2.5], $\bar{\Phi}(E')$ is, under the strong topology, the strict inductive limit of spaces $\bar{\Phi}_{K_n}(E')$. Thus for any compact set K, the set

$$D = \{g \in \bar{\Phi}(E') : \|g\| \leq c(K)\},$$

which is contained in and bounded in some $\bar{\Phi}_{K_n}(E')$ is strongly bounded in $\bar{\Phi}(E')$ [10, p. 129]. If π has compact support, [9, Proposition 2.5] again shows that D is bounded. Thus ϕ, which is strongly continuous, is bounded on D. This gives condition (2) of [8, Theorem 7.1]. For condition (1) let $(f_\alpha) \subseteq C$ be a net such that $f_\alpha \to \phi$ weakly. Fix $g \in \bar{\Phi}(E')$ with Supp $g \subseteq K$, a compact set. Suppose $R_j \uparrow R$ and set $S_j = R - R_j$. Then

$$\phi(g|_R) - \phi(g|_{R_j}) = \phi(g|_{S_j}) = \phi(g|_{K\cap S_j}) = \lim_\alpha \int_{K\cap S_j} \langle f_\alpha, g\rangle d\pi \to 0$$

as $j \to \infty$ by (3). ▌

2. The spaces $\Lambda(E)$ and $\Sigma^0(E')$

If Λ is a solid scalar v.f.s. (e.g. $\Lambda = L^p$) we set

$$\Lambda(E) = \{f \in \Omega(E) : p \circ f \in \Lambda \text{ for all } p \in P\}$$

and

$$\Lambda^0(E') = \{g \in \bar{\Omega}(E') : g = bg_0 \text{ with } b \in \Lambda \text{ and } p^0(g_0(z)) \leq 1 \text{ a.e. for some } p \in P\}$$

Since $p^0 \circ g$ is not necessarily well defined for $g \in \bar{\Omega}(E')$ [8, example following Theorem 3.2], the definition of $\Lambda^0(E')$ is not complete. We shall make the agreement, here and in similar cases later, that the representation for g need only hold for one function in the class. If E is separable, then $p^0 \circ g$ is well defined [8, Theorem 3.1] and so in this case we have

$$\Lambda^0(E') = \{g \in \bar{\Omega}(E') : p^0 \circ g \in \Lambda \text{ for some } p \in P\}.$$

Using the remarks following [9, Proposition 1.1] it is easy to show that if E is a Banach space and E' is separable, then $\Lambda^0(E') = \Lambda(E')$ and if E is a reflexive Banach space then $\Lambda^0(E')$ can be identified with $\Lambda(E')$.

Besides the L^p spaces, examples of scalar v.f.s.'s include the Orlicz spaces [12] and general Banach function spaces [13]. Spaces of the form $\Lambda(E)$ have been studied by Gregory [5] when Z is the set of natural numbers and π is the counting measure. Các [1] has studied the spaces $\Lambda(E)$ when E is a Banach space.

If (Λ, Σ) is a dual pair of solid v.f.s.'s and $f \in \Lambda(E)$ and $g \in \Sigma^0(E')$, set $g = bg_0$ where $b \in \Sigma$ and $p^0(g_0(z)) \leq 1$. Then

$$\left| \int \langle f, g\rangle \, d\pi \right| \leq \int p \circ f \, | b | \, d\pi < \infty.$$

Thus $(\Lambda(E), \Sigma^0(E'))$ is a dual pair of v.f.s.'s. We shall find that the dual pair $(\Lambda(E), \Sigma^0(E'))$ inherits many of the properties of the dual pair (Λ, Σ), especially when E is normed.

If Λ has a solid topology, we topologize $\Lambda(E)$ with the set of seminorms $\{q(p \circ f)\}$ where $p \in P$ and q is the gauge of a solid absolutely convex neighborhood in Λ. It is easy to show that the seminorms $q(p \circ f)$ *are* seminorms and generate a solid topology on $\Lambda(E)$.

PROPOPOSITION 2.1. (1) $\Lambda^*(E) = \Lambda^0(E')^*$.
(2) *If E is a separable normed space,* $\Lambda(E)^* = \Lambda^{*0}(E')$.

Proof. (1) If $f \in \Omega(E)$, $p \in P$, and $b \in \Lambda$, we have by [8, Lemma 5.3],

$$(*)\quad \int p \circ f \mid b \mid d\pi = \text{Sup}\left\{\int \mid \langle f, g\rangle \mid d\pi : g \in \Lambda^0(E') \text{ with } g = bg_0 \text{ where } p^0 \circ g_0 \leq 1\right\}$$

and both sides of the equality are finite if every entry in the supremum is finite. Thus

$$\begin{aligned} f \in \Lambda^0(E')^* &\Leftrightarrow \int |\langle f, g\rangle| d\pi < \infty \text{ for every } g \in \Lambda^0(E')^* \\ &\Leftrightarrow \int p \circ f\, |b| d\pi < \infty \text{ for every } p \in P \text{ and } b \in \Lambda \\ &\Leftrightarrow f \in \Lambda^*(E). \end{aligned}$$

(2) This is proved as is (1) except that [8, Lemma 5.2] is used. ∎

Remark. Equality (2) is not true for general Λ and E. For let $\Lambda = \Phi$. Then $\Phi(E)^* = \bar{\Omega}(E')$. But in general $\Omega^0(E') \neq \bar{\Omega}(E')$ unless E is normed or Z is compact. See also Proposition 2.5.

PROPOSITION 2.2. *Let (Λ, Σ) be a dual pair of solid v.f.s.'s and let $B \subseteq \Sigma$ be a solid set whose polar has gauge q. Then for any $f \in \Lambda(E)$,*

$$q(p \circ f) = \text{Sup}\left\{\left|\int \langle f, g\rangle d\pi\right| : g \in \Sigma^0(E') \text{ where } g = bg_0 \text{ with } b \in B \text{ and } p^0(g_0(z)) \leq 1\right\}.$$

This implies that if a solid topology on Λ is a polar topology induced from Σ, then the topology on $\Lambda(E)$ is a polar topology induced from $\Sigma^0(E')$.

Proof. The result is obtained by taking a supremum on both sides of the equality $(*)$ in the proof above, as b runs through all elements of B.

The identification of $L^p(E)'$ $(1 \leq p < \infty)$ when E is a separable Banach space is well known [4]. The case of a general E has been studied in [11].

THEOREM 2.3 *Let (Λ, Σ) be a dual pair of solid scalar v.f.s.'s with $\Lambda^* = \Sigma$. Let Λ be given a solid polar topology from Σ. Then*

$$\Lambda' = \Sigma \quad \Leftrightarrow \quad \Lambda(E)' = \Sigma^0(E').$$

Proof. ($\Rightarrow$) By [9, Thorem 3.5], if $R_i \uparrow R$ and $a \in \Lambda$ then $a|_{R_i} \to a|_R$. An easy calculation shows that if $f \in \Lambda(E)$ then $f|_{R_i} \to f|_R$. Now let $\phi \in \Lambda(E)'$. Then there is a $p \in P$ and a continuous seminorm q on Λ which is the gauge of a solid set such that

$$(*) \qquad q(p \circ f) \leq 1 \quad \Rightarrow \quad |\phi(f)| \leq 1.$$

If $K \subseteq Z$ is compact, the set $\{a \in \Lambda : |a| \leq c(K)\}$ is $\sigma(\Lambda, \Sigma)$ bounded and so bounded in Λ. Thus there is an M such that for any $f \in \Lambda(E)$ with $p \circ f \leq c(K)$ we have $q(p \circ f) \leq M$. By $(*)$, $|\phi(f)| \leq M$ for any such f. By [8, Theorem 7.1], there is a $g \in \bar{\Omega}(E')$ such that $\phi(f) = \int \langle f, g \rangle d\pi$. Furthermore, an inspection of the proof of that theorem shows that $g = bg_0$ where $b \in \Omega$, $b \geq 0$, and $p^0 \circ g_0 \leq 1$ and that for any relatively compact measurable set R,

$$\int_R b d\pi = \operatorname{Sup} \{\phi(\textstyle\sum_i c(R_i)x_i)\}$$

where the supremum is taken over all countable partitions $\{R_i\}$ of R and x_i satisfies $p(x_i) \leq 1$ and $\phi(c(R_n)x_i) \geq 0$. Now let $a \in \Lambda$ with $a \geq 0$ be fixed. Let $a' = \sum a_i c(R_i)$ be a simple function satisfying $0 \leq a' \leq a$. Then

$$\int a'b \, d\pi = \textstyle\sum_i a_i \int_{R_i} b \, d\pi = \sum_i a_i \operatorname{Sup} \{\phi \, (\sum_j c(R_{ij})x_{ij})\}$$
$$= \operatorname{Sup} \{\phi \, (\textstyle\sum_{ij} a_i c(R_{ij})x_{ij})\} \leq q(p \circ f)$$

by $(*)$, using the fact that q is the gauge of a solid set. By [8, Lemma 5.2 (1)], $\int abd\pi < \infty$ and so $b \in \Sigma = \Lambda^*$ and thus $g \in \Sigma^0(E')$.

We have shown that $\Lambda(E)' \subseteq \Sigma^0(E')$. The reverse inclusion is easy to show.

($\Leftarrow$) If $\Lambda' \neq \Sigma$ then by [9, Theorem 3.5] there is an $a \in \Lambda$ and a sequence $R_i \uparrow R$ such that $a|_{R_i} \to a|_R$ is false. Then if $x \neq 0$ in E, $a(z)x|_{R_i} \to a(z)x|_R$ is false in $\Lambda(E)$. Since by Proposition 2.2, the topology on $\Lambda(E)$ is a polar topology induced from $\Sigma^0(E')$, [9, Proposition 3.3] implies that $\Lambda(E)' \neq \Sigma^0(E')$. ∎

Remark. Using the techniques of [1, Proposition 10] we may prove that $\Lambda(E)' = \Sigma^0(E')$ when Λ has the normal topology even if $\Lambda^* \neq \Sigma$.

THEOREM 2.4. *Let (Λ, Σ) be a dual pair of solid v.f.s.'s with $\Lambda = \Sigma^*$. Let Λ be given the topology of uniform convergence on a set of solid sets of Σ whose union is Σ. Then $\Lambda(E)$ is complete.*

Proof. By Proposition 2.2, the topology on $\Lambda(E)$ is a solid polar topology induced from $\Sigma^0(E')$. By Proposition 2.1, $\Lambda(E) = \Sigma^0(E')^*$. Since the topology on Λ is finer than that induced from Ω, the topology on $\Lambda(E)$ is finer than that induced from $\Omega(E)$. Thus [9, Theorem 2.2] $\Lambda(E)$ is complete. ∎

We are now able to extend the equality of Proposition 2.1 (2) to a larger class of spaces.

PROPOSITION 2.5. *Let E be a metrizable space. Let (Λ, Σ) be a dual pair of scalar v.f.s.'s with $\Lambda^* = \Sigma$ and $\Sigma^* = \Lambda$ and suppose that Λ is metrizable under the topology $\tau(\Lambda, \Sigma)$. Then $\Lambda(E)^* = \Sigma^0(E')$.*

Proof. Let Λ be given the topology $\tau(\Lambda, \Sigma)$. This is a solid topology [9, Corollary 3.6] and by Theorem 2.4, $\Lambda(E)$ is complete. Since Λ and E are metrizable, $\Lambda(E)$ is metrizable and so barrelled. By [9, Proposition 3.1], $\Lambda(E)^* \subseteq \Lambda(E)'$. But by Theorem 2.3, $\Lambda(E)' = \Sigma^0(E') \subseteq \Lambda(E)^*$ and so $\Lambda(E)^* = \Sigma^0(E')$. ∎

LEMMA 2.6. *Let (Λ, Σ) be a dual pair of solid scalar v.f.s.'s with $\Lambda^* = \Sigma$. Then, (1) a set $A \subseteq \Lambda(E)$ is*

$$\sigma(\Lambda(E), \Sigma^0(E'))$$

bounded iff for every $p \in P$, $p(A)$ is $\sigma(\Lambda, \Sigma)$ bounded, and (2) if E is a separable normed space, a set $B \subseteq \Sigma^0(E')$ is

$$\beta(\Sigma^0(E'), \Lambda(E))$$

bounded iff $\|B\|$ is $\beta(\Sigma, \Lambda)$ bounded.

Proof. (1) Let Λ be given the normal topology. By [9, Lemma 1.3]. $\Lambda' = \Sigma$ and so by Theorem 2.3, $\Lambda(E)' = \Sigma^0(E')$. Now A is bounded in $\Lambda(E)$ iff $p(A)$ is bounded in Λ for every $p \in P$. But $p(A)$ is bounded iff $p(A)$ is $\sigma(\Lambda, \Sigma)$ bounded and A is bounded iff it is $\sigma(\Lambda(E), \Sigma^0(E'))$ bounded.

(2) Let C be any solid $\sigma(\Lambda, \Sigma)$ bounded set. By [8, Lemma 5.2], for any $a \in \Lambda$ and $g \in \Sigma^0(E')$ we have

$$\int |a| \, \| g \| \, d\pi = \text{Sup } \{ \, | \int \langle f, g \rangle \, d\pi \, | : f \in \Lambda(E) \text{ and } \| f \| \leq a \}.$$

Thus

$$\text{Sup } \{ \int | a| \, \| g \| \, d\pi : a \in C \text{ and } g \in B \}$$

$$= \text{Sup } \{ \, | \int \langle f, g \rangle d\pi \, | : a \in \Lambda(E), \| f \| \in C, \text{and } g \in B \}.$$

Using [9, Proposition 1.4], the left hand side of this equality is finite for every C iff $\|B\|$ is $\beta(\Sigma, \Lambda)$ bounded, while the right side, using part (1), is finite for every C iff B is $\beta(\Sigma^0(E'), \Lambda(E))$ bounded. ∎

COROLLARY 2.7. *Let (Λ, Σ) be a dual pair of solid v.f.s.'s with $\Lambda^* = \Sigma$ and $\Sigma^* = \Lambda$. Let E be a separable normed space. Then (1) a set $B \subseteq \Sigma^0(E')$ is*

$$\sigma(\Sigma^0(E'), \Lambda(E))$$

bounded iff $\|B\|$ is $\sigma(\Sigma, \Lambda)$ bounded, and (2) a set $A \subseteq \Lambda(E)$ is

$$\beta(\Lambda(E), \Sigma^0(E'))$$

bounded iff $\|A\|$ is $\beta(\Lambda, \Sigma)$ bounded.

Proof. (1) Let Λ be given the normal topology. By [9, Lemma 1.3], $\Lambda' = \Sigma$ and so by Theorem 2.3, $\Lambda(E)' = \Sigma^0(E')$. By Theorem 2.4, Λ and $\Lambda(E)$ are complete. Thus [10, p. 72] the strongly and weakly bounded sets in $\Sigma^0(E')$ and Σ are the same and the results follows from Lemma 2.6 (2).

(2) The proof is similar to that of Lemma 2.6(2) and is omitted. ▌

PROPOSITION 2.8. *Let E be a separable normed space. Let (Λ, Σ) be a dual pair of solid scalar v.f.s.'s with $\Lambda^* = \Sigma$ and $\Sigma^* = \Lambda$. Let Λ be given a solid polar topology of the dual pair. Then Λ has the topology $\beta(\Lambda, \Sigma)$ iff $\Lambda(E)$ has the topology $\beta(\Lambda(E), \Sigma^0(E'))$.*

Proof. By [9, Proposition 2.4] the polars of the solid weakly bounded sets in Σ and $\Sigma^0(E')$ form a base for the topologies

$$\beta(\Lambda, \Sigma) \quad \text{and} \quad \beta(\Lambda(E), \Sigma^0(E')).$$

The result follows from Proposition 2.2 and Corollary 2.7(1). ▌

COROLLARY 2.9. *Let E be a separable normed space. Let Λ be a solid scalar v.f.s. with $\Lambda = \Lambda^{**}$ and let Λ be given a solid topology of the dual pair (Λ, Λ^*). Then Λ is barrelled iff $\Lambda(E)$ is barrelled.*

Proof. By Theorem 2.3, $\Lambda(E)$ has a topology of the dual pair

$$(\Lambda(E), \Lambda^{*0}(E')).$$

Since a space is barrelled iff it has the strong topology from its dual, the result follows from the proposition. ▌

PROPOSITION 2.10. *Let E be a reflexive separable Banach space. Let (Λ, Σ) be a dual pair of solid scalar v.f.s.'s with $\Lambda^* = \Sigma$ and $\Sigma^* = \Lambda$. Let Λ be given a topology of the dual pair. Then Λ is semireflexive iff $\Lambda(E)$ is semi-reflexive.*

Proof. By Theorem 2.3, $\Lambda(E)$ has a topology of the dual pair

$$(\Lambda(E), \Sigma^0(E')).$$

Let Σ be given the topology $\beta(\Sigma, \Lambda)$. Then (with E and E' switched—see [9, Proposition 1.1]) Proposition 2.8 shows that $\Sigma^0(E')$ has the topology $\beta(\Sigma^0(E'), \Lambda(E))$ and so by Theorem 2.3, $\Sigma^0(E')' = \Lambda(E)$ iff $\Sigma' = \Lambda$, i.e., $\Lambda(E)$ is semi-reflexive iff Λ is semireflexive. ▌

PROPOSITION 2.11. *Let E be a separable reflexive Banach space. Let Λ be a solid scalar v.f.s. with $\Lambda = \Lambda^{**}$ and let Λ be given a topology of the dual pair (Λ, Λ^*). Then Λ is reflexive iff $\Lambda(E)$ is reflexive.*

Proof. Since a locally convex space is reflexive iff it is barrelled and semi-reflexive [7, p. 302], the proposition follows from Corollary 2.9 and Proposition 2.10. ▌

3. The Spaces $L^p(E)$

We give a list of some properties of the spaces $L^p(E)$. Let q be the conjugate index to p.

(a) $L^p(E)$ is complete (Theorem 2.4).

(b) The topology on $L^p(E)$ is a polar topology induced from $(L^q)^0(E')$ (Proposition 2.2).

(c) $(L^q)^0(E')^* = L^p(E)$ (Proposition 2.1).

(d) If E is metrizable and $1 \leq p < \infty$, $L^p(E)^* = (L^q)^0(E')$ (Proposition 2.5).

(e) If E is separable and normed, $L^\infty(E)^* = (L^1)^0(E')$ (Proposition 2.1).

(f) If $1 \leq p < \infty$, $L^p(E)' = (L^q)^0(E')$ (Theorem 2.3).

(g) If $L^{\infty\prime} \neq L^1$, then $L^\infty(E)' \neq (L^1)^0(E')$ (Theorem 2.3).

(h) If E is a reflexive separable Banach space, $L^p(E)$ is weakly sequentially complete (with respect to the dual pair $(L^p(E), (L^q)^0(E'))$ [9, Theorem 2.7].

(i) If E is normed and $1 \leq p < \infty$, then $(L^q)^0(E')$ is quasicomplete when given the topology of uniform convergence on the compact sets of $L^p(E)$ [9, Proposition 3.7].

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