RECURRENCE CRITERIA FOR RANDOM WALKS ON COUNTABLE ABELIAN GOUPS

BY

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1. Introduction

Let μ be a random walk or, equivalently, a probability measure on a countable Abelian group G. The random walk μ is said to be recurrent (transient) iff $\sum_{n=1}^{\infty} \mu^n(0) = \infty$ ($< \infty$) where μ^n denotes the n-fold convolution of μ. Probabilistically, recurrence means that the walk starting from the origin 0 revisits the origin infinitely often with probability one while transience means that the origin is visited at most finitely often with probability one. It is desirable to obtain criteria which enable us to decide whether a given walk μ is recurrent or transient. The following criterion is proved in [5].

THEOREM 1.1. *Let G be a countable Abelian group endowed with the discrete topology. Let μ be a random walk on G, $\hat{\mu}(\gamma)$ the Fourier transform of μ defined on the compact character group Γ. μ is recurrent iff*

$$\int \operatorname{Re}\,[1/(1 - \hat{\mu}(\gamma))]\, dP(\gamma) = \infty \tag{1.1}$$

where P is the normalized Haar measure on Γ.

The recurrence criterion (1.1) is thus stated in terms of the transform $\hat{\mu}$. It is natural to ask whether one can obtain criteria in terms of μ itself, i.e. can (1.1) be reinterpreted as a condition on μ. This seems a rather difficult problem and has thus far been done only in certain isolated cases. For instance, if G is the d-dimensional lattice Z^d, then recurrence criteria can be obtained in terms of the first and second moments of the walk [4, P. 83]. More recently, Darling and Erdös have obtained such criteria in case G is the direct sum $Z_2 \oplus Z_2 \oplus \cdots$. The elements of G are the infinite sequences $g = (\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n, \cdot)$ where each $\varepsilon_n = \varepsilon_n(g) \in Z_2$ (the additive group of integers mod 2), only a finite number of ε_n's being distinct from 0. Let g_n be the element for which $\varepsilon_n(g_n) = 1$, $\varepsilon_j(g_n) = 0$ $(j \neq n)$, $1 \leq n < \infty$. We then have the following recurrence criterion [2].

THEOREM 1.2. *Let $\mu(g_n) = p_n > 0$ $(1 \leq n < \infty)$ where $\sum_{n=1}^{\infty} p_n = 1$. Assume, without loss of generality, that $\{p_n\} \varepsilon \downarrow$ and define $f_n = \sum_{j=n}^{\infty} p_j$. μ is recurrent iff $\sum_{n=1}^{\infty} 1/2^n f_n = \infty$.*

In this paper, we obtain recurrence criteria in terms of μ for two classes of countable Abelian groups: (i) the groups $G = Z_{m_1} \oplus Z_{m_2} \oplus \cdots \oplus$

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$Z_{m_n} \oplus \cdots$ and (ii) the infinite subgroups of the group of rationals mod one. In (i), $\{m_n\}$ is any infinite sequence of integers ≥ 2 and Z_{m_n} denotes the additive group of integers mod m_n. The elements of G are the infinite sequences

$$g = (\varepsilon_1(g), \cdots, \varepsilon_n(g), \cdots) \quad \text{where } \varepsilon_n(g) \in Z_{m_n}, \qquad 1 \leq n < \infty,$$

only a finite number of the ε_n's being distinct from 0.

The walks we consider on $Z_{m_1} \oplus Z_{m_2} \oplus \cdots$ are distinguished by the property that their individual steps fall in distinct summands of the group. The measure μ we consider will be specified by the pair of sequences

$$\{p_n, n = 1, 2, \cdots\}, \quad \{\alpha_{nj}, n = 1, 2, \cdots, j = 1, 2, \cdots, m_n - 1\},$$

giving respectively the probability that an individual step lies in the nth component and that the step has the value jg_n (g_n a specified generator of the nth summand) given that it lies in the nth summand. The results we obtain differ in the two cases $\{m_n\}$ bounded, and $\{m_n\}$ unbounded. In either case we will state our recurrence criteria in terms of the sequences

$$f_n = \textstyle\sum_{n=1}^{\infty} p_j, \quad M_n = \prod_{j=1}^{n-1} m_j, \quad 2 \leq n < \infty, \quad M_1 = 1.$$

It is assumed, without loss of generality, that (i) $\{p_n\} \in \downarrow$ and (ii) the walk μ is aperiodic.

In the case $\{m_n\}$ bounded we obtain a criterion (subject to a mild restriction) which is essentially necessary and sufficient for recurrence. Specifically, we show:

THEOREM 2.1. *Let $\{m_n\}$ be bounded; then $\sum_{n=1}^{\infty} 1/M_n f_n = \infty \Rightarrow \mu$ is recurrent.*

The necessity of this recurrence criterion is restricted by a condition (condition (A), of Section 2) requiring that not too much mass is concentrated on proper subgroups of the summands, a strong form of aperiodicity. We then have:

THEOREM 2.2. *Let $\{m_n\}$ be bounded and let μ satisfy condition* (A). *Then μ is recurrent $\Rightarrow \sum_{n=1}^{\infty} 1/M_n f_n = \infty$.*

We show that the conclusion of Theorem 2.2 may be untrue when condition (A) fails to hold.

When $\{m_n\}$ is unbounded our criteria are less complete. The analogue of Theorem 2.1 is shown to hold only with the additional hypothesis that μ is symmetric in the sense that $\alpha_{nj} = \alpha_{n,m_n-j}$ for $1 \leq n < \infty$, $1 \leq j \leq m_n - 1$. Necessary conditions for recurrence are obtained under a variety of additional conditions, but these do not generally coincide with our sufficient condition. An example shows that, in fact, the conclusion of Theorem 2.2 fails in the case $\{m_n\}$ unbounded, even when condition (A) is satisfied.

The methods developed in Sections 2 and 3 to treat class (i) carry over with minor modification to treat the class (ii), which we discuss in Section 4.

A noteworthy feature of the argument is our treatment of $\hat{\mu}(\gamma)$ as a random variable. This is possible in view of the fact that the groups we consider are discrete. The duals of discrete groups are, of course, compact, their Haar measures may be normalized, and functions on the duals are indeed random variables. The power of this approach is that it allows us to employ probabilistic tools to study the convergence and divergence of the integrals in (1.1).

2. Random walks on $Z_{m_1} \oplus Z_{m_2} \oplus \cdots \oplus Z_{m_n} \oplus \cdots$, $\{m_n\}$ bounded

In this section we consider random walks on the group $G = Z_{m_1} \oplus Z_{m_2} \oplus \cdots \oplus Z_{m_n} \oplus \cdots$ where $\{m_n\}$ is assumed to be a bounded sequence. We let g_n be that element of G for which $\varepsilon_j(g_n) = \delta_{jn}$, $1 \leq j < \infty$. The elements jg_n, $0 \leq j \leq m_n - 1$, form an additive cyclic subgroup of G of order m_n which we again designate as Z_{m_n}. We consider random walks μ with the property $\mu(Z_{m_n}) = p_n > 0$, $1 \leq n < \infty$, where $\sum_{n=1}^{\infty} p_n = 1$; i.e. the support of $\mu \subseteq \bigcup_{n=1}^{\infty} Z_{m_n}$. We assume, without loss of generality, that (i) $\{p_n\} \in \downarrow$ and (ii) the walk μ is aperiodic, i.e. the support of μ generates G. (That (i) entails no loss of generality follows from the observation that it may be achieved by a relabeling of indices; as for (ii) we may consider the walk as a walk on the group generated by the support of μ.) Let

$$\alpha_{nj} = \mu(jg_n)/p_n \quad (1 \leq n < \infty, 0 \leq j \leq m_n - 1).$$

In the usual graphic terminology we may describe our walk as one whose steps lie in distinct components of G. A step falls in the nth component subgroup with probability p_n, and given that the step lies in the nth subgroup, it has the value jg_n with probability α_{nj}. We wish to find recurrence criteria in terms of $\{p_n\}$ and $\{\alpha_{nj}\}$. Note that $\sum_{j=0}^{m_n-1} \alpha_{nj} = 1$, $1 \leq n < \infty$. It will be convenient later on to assume that $\alpha_{n0} = 0$, $1 \leq n < \infty$. This can always be done in view of the rather obvious statement whose proof we omit.

THEOREM 2.0. *Let μ be a random walk on the countable group G, $\mu(0) < 1$. Define the random walk ν as the conditional probability $\nu(A) = \mu(A \mid G - \{0\})$ for all subsets A of G. μ is recurrent iff ν is recurrent.*

Now $\mu(0) > 0$ is equivalent to $\alpha_{n0} \neq 0$ for some n. In this case we replace μ by ν where $\nu(0) = 0$ and the new α_{n0}'s are equal to 0.

Our basic tool in establishing recurrence criteria is Theorem (1.1). As G is discrete, its character group Γ is compact. The normalized Haar measure P is a probability measure on the probability space Γ and $\hat{\mu}(\gamma)$ is a random variable defined on Γ.

Specifically, if $\gamma \in \Gamma$, then by definition

$$\hat{\mu}(\gamma) = \sum_{g \in G} \mu(g)\gamma(g) = \sum_{n=1}^{\infty} p_n \sum_{j=1}^{m_n-1} \alpha_{nj}\, \gamma(jg_n).$$

We introduce the random variables $U_n(\gamma) = \gamma(g_n)$. Then

$$U_n^j(\gamma) = \gamma^j(g_n) = \gamma(jg_n)$$

so that

$$\hat{\mu}(\gamma) = \sum_{n=1}^{\infty} p_n \sum_{j=1}^{m_n-1} \alpha_{nj}\, U_n^j(\gamma).$$

We make the following observations concerning the random variables $\{U_n\}$.

(1) For any positive integer n let $l_1, \cdots, l_n$ denote arbitrary integers satisfying $0 \le l_j \le m_j - 1$, $1 \le j \le n$. Let

$$S_{l_1,\ldots,l_n} = \{\gamma \mid U_1(\gamma) = e^{2\pi i l_1/m_1}, \cdots, U_n(\gamma) = e^{2\pi i l_n/m_n}\}.$$

Then

$$P(S_{l_1,\ldots,l_n}) = 1/m_1 \cdots m_n .$$

To see this we give a concrete description of the character group Γ. For any character γ, we have $[\gamma(g_n)]^{m_n} = \gamma(m_n g_n) = \gamma(0) = 1$, $1 \le n < \infty$. Hence

$$\gamma(g_n) = e^{2\pi i l_n/m_n}, \quad 1 \le n < \infty,$$

where $\{l_n\}$ is a sequence of integers satisfying $0 \le l_n \le m_n - 1$, $1 \le n < \infty$. Conversely, it is readily checked that for any such sequence $\{l_n\}$, the sequence $\gamma(g_n) = e^{2\pi i l_n/m_n}$, $1 \le n < \infty$, has a unique extension as a character $\gamma(g)$ on G. Hence Γ may be identified as the set of sequences

$$(\gamma(g_1), \cdots, \gamma(g_n), \cdots), \quad \gamma(g_n) = e^{2\pi i l_n/m_n} \quad (1 \le n < \infty),$$

where $\{l_n\}$ is any sequence of integers satisfying $0 \le l_n \le m_n - 1$, $1 \le n < \infty$. It follows in particular that each $S_{l_1,\ldots,l_n}$ is non-empty. It is readily checked that

$$S_{l_1,\ldots,l_n} = \gamma S_{0\ldots0} \quad \text{for any } \gamma \in S_{l_1,\ldots,l_n} .$$

Since P is translation invariant we obtain $P(S_{l_1,\ldots,l_n}) = P(S_{0\ldots0})$. Hence the $m_1 \cdots m_n$ numbers $P(S_{l_1,\ldots,l_n})$, $0 \le l_i \le m_i - 1$, $1 \le i \le n$, all equal $1/m_1 \cdots m_n$ as their sum equals 1.

(2) $\{U_n\}$ is a sequence of independent random variables.

This follows from (1) as

$$P\{U_1 = e^{2\pi i l_1/m_1}, \cdots, U_n = e^{2\pi i l_n/m_n}\} = 1/m_1 \cdots m_n$$

while

$$P\{U_n = e^{2\pi i l_n/m_n}\} = \sum_{l_1=1}^{m_1} \cdots \sum_{l_{n-1}=1}^{m_n-1} P\{U_1 = e^{2\pi i l_1/m_1}, \cdots, U_n = e^{2\pi i l_n/m_n}\} = m_1 \cdots m_{n-1}/m_1 \cdots m_n = 1/m_n ,$$

$$1 \le n < \infty.$$

Hence

$$P\{U_1 = e^{2\pi i l_1/m_1}, \cdots, U_n = e^{2\pi i l_n/m_n}\} = P(U_1 = e^{2\pi i l_1/m_1}) \cdots P(U_n = e^{2\pi i l_n/m_n}).$$

(3) $$E(U_n^j) = \int_\Gamma U_n^j(\gamma)\, dP(\gamma) = 1, \quad j = 0$$
$$= 0, \quad 0 < j \le m_n - 1.$$

For

$$E(U_n^j) = \sum_{l=0}^{m_n-1} \int U_n^j(\gamma)\, dP(\gamma) = (1/m_n)\sum_{l=0}^{m_n-1} e^{2\pi i j l/m_n} = 1, \quad j = 0$$
$$= 0, \quad 0 < j \le m_n - 1.$$

It will be useful to write

$$V_n(\gamma) = 1 - \sum_{j=0}^{m_n-1} \alpha_{nj}\, U_n^j(\gamma).$$

We have $1 - \hat{\mu}(\gamma) = \sum_{n=1}^{\infty} p_n V_n(\gamma)$ and in view of (1), (2), (3) we have

(1′) $|V_n - 1| \le 1$.
(2′) $\{V_n\}$ is a sequence of independent random variables.
(3′) $E(V_n) = 1$.

Furthermore, we have

(4′) $V_n = 0 \Leftrightarrow U_n = 1$.

(4′) is proven as follows. Let I_n denote the set of indices j, $1 \le j \le m_n - 1$, for which $\alpha_{nj} > 0$. Since $V_n = 1 - \sum_{j \in S_n} \alpha_{nj}\, U_n^j$ and $|U_n^j| = 1$,

$$V_n = 0 \Leftrightarrow U_n^j = 1 \quad \text{for } j \in I_n .$$

Since (support of μ) $\cap\, Z_{m_n}$ generates Z_{m_n}, the latter condition is equivalent to $U_n = 1$. Thus $V_n \to 0 \Leftrightarrow U_n = 1$.

Since $\hat{\mu}(\gamma) = \sum_{n=1}^{\infty} p_n(1 - V_n(\gamma))$, we conclude from (1′) that $|\hat{\mu}(\gamma)| \le 1$ for $\gamma \in \Gamma$. Furthermore $\hat{\mu}(\gamma) = 1 \Leftrightarrow V_n(\gamma) = 0,\ 1 \le n < \infty$. Using (4′),

$$V_n(\gamma) = 0\ (1 \le n < \infty) \quad \Leftrightarrow \quad \gamma(g_n) = 1\ (1 \le n < \infty) \quad \Leftrightarrow \quad \gamma = e$$

where e is the identity of Γ. The function $w = 1(1 - z)$ maps $|z| \le 1$ onto $\operatorname{Re} w \ge \frac{1}{2}$ with $w(1) = \infty$. Hence

$$\tfrac{1}{2} \le \operatorname{Re}\,[1/(1 - \hat{\mu}(\gamma))] < \infty, \quad \gamma \ne e.$$

We now obtain a sufficient condition for recurrence. We let

$$f_n = \sum_{j=n}^{\infty} p_j, \quad M_n = \prod_{j=1}^{n-1} m_j\ (2 \le n < \infty), \quad M_1 = 1.$$

Theorem 2.1. *Let $\{m_n\}$ be bounded. Then $\sum_{n=1}^{\infty} 1/M_n f_n = \infty \Rightarrow \mu$ is recurrent.*

Proof. Let

$$E_n = \{\gamma \mid V_1(\gamma) = \cdots = V_{n-1}(\gamma) = 0, \quad V_n(\gamma) = 1\}.$$

From our above remarks, $\Gamma - \{e\} = \bigcup_{n=1}^{\infty} E_n$ so that $\sum_{n=1}^{\infty} P(E_n) = 1$ and

$$\int_\Gamma \operatorname{Re}\,[1/(1 - \hat{\mu}(\gamma))]\, dP(\gamma) = \sum_{n=1}^{\infty} \int_{E_n} \operatorname{Re}\,[1/(1 - \hat{\mu}(\gamma))]\, dP(\gamma).$$

For any complex number z for which $\operatorname{Re} z > 0$, we choose $\arg z$ to be that value of the argument for which $|\arg z| < \pi/2$. We have

$$\operatorname{Re}\left[\frac{1}{1 - \hat{\mu}(\gamma)}\right] = \frac{\cos \arg\,[1 - \hat{\mu}(\gamma)]}{|1 - \hat{\mu}(\gamma)|}.$$

Now

$$1 - \hat{\mu}(\gamma) = \sum_{n=1}^{\infty} \sum_{j=1}^{m_n - 1} p_n\, \alpha_{nj}[1 - U_n^j(\gamma)].$$

Suppose that $1 - U_n^j(\gamma) \neq 0$. Then $U_n^j(\gamma) = e^{2\pi i r/m_n}$ for some integer r, $1 \le r \le m_n - 1$. Using the identity $\arg\,(1 - e^{i\theta}) = (\theta - \pi)/2$, $0 < \theta < \pi$ we conclude that

$$|\arg\,[1 - U_n^j(\gamma)]| \le \pi/2 - \pi/m_n.$$

Hence the numbers $p_n\, \alpha_{nj}[1 - U_n^j(\gamma)]$ lie in the angular sector

$$|\theta| \le \pi/2 - \pi/m$$

where $m = \max_{1 \le n < \infty} m_n$. It follows that for $\gamma \neq e$,

$$|\arg\,(1 - \hat{\mu}(\gamma))| \le \pi/2 - \pi/m \quad \text{and} \quad \cos \arg\,[1 - \hat{\mu}(\gamma)] \ge \sin \pi/m.$$

Furthermore for $\gamma \in E_n$, $1 - \hat{\mu}(\gamma) = \sum_{j=n}^{\infty} p_j\, V_j(\gamma)$ so that $|1 - \hat{\mu}(\gamma)| \le 2 \sum_{j=n}^{\infty} p_j = 2f_n$. Hence for $\gamma \in E_n$,

$$\operatorname{Re}\left[\frac{1}{1 - \hat{\mu}(\gamma)}\right] \ge \frac{1}{2}\left(\sin \frac{\pi}{m}\right)\frac{1}{f_n}$$

so that

$$\int_\Gamma \operatorname{Re}\left[\frac{1}{1 - \hat{\mu}(\gamma)}\right] dP(\gamma) \ge \frac{1}{2}\left(\sin \frac{\pi}{m}\right) \sum_{n=1}^{\infty} \frac{1}{f_n} P(E_n)$$

Property (4′) implies

$$E_n = \{\gamma \mid U_1(\gamma) = \cdots = U_{n-1}(\gamma) = 1, \quad U_n(\gamma) \neq 1\}$$

and property (2′) that

$$P(E_n) = (1/M_n)(1 - 1/m_n) \ge 1/2M_n.$$

Hence

$$\int_{\Gamma} [1/(1 - \hat{\mu}(\gamma))]\, dP(\gamma) \geq \frac{1}{4} (\sin \pi/m) \sum_{n=1}^{\infty} 1/M_n f_n ,$$

from which the theorem follows.

We now proceed to show that $\sum_{n=1}^{\infty} 1/M_n f_n = \infty$ is necessary for recurrence. To prove this we impose the following condition on the walk.

Condition (A). There exists c, $0 < c < 1$, such that $\sum_{j \notin H} \alpha_{nj} \leq c$ where $1 \leq n < \infty$ and H is any proper subgroup of $Z_{m_n} = \{0, 1, \cdots, m_n - 1\}$.

Roughly speaking, condition A demands that μ is not too concentrated on proper subgroups of the Z_{m_n}'s and may be viewed as a strong form of aperiodicity. Indeed (A) implies that μ is aperiodic. It is easily seen that it is equivalent to aperiodicity iff a finite number of the m_n's are composite. We use (A) to establish the following estimate needed later on.

$$(2.1) \qquad X_n(\gamma) = \text{Re}\,(V_n(\gamma)) \geq \alpha/m_n^2, \quad \gamma \in E_n \text{ where } \alpha = 8(1 - c)$$

(2.1) is proven as follows.

$$X_n = \sum_{j=1}^{m_n-1} \alpha_{nj}[1 - \text{Re}\,(U_n^j)]$$

so that on E_n, X_n assumes the values

$$\sum_{j=1}^{m_n-1} \alpha_{nj}[1 - \cos 2\pi jl/m_n], \quad 1 \leq l \leq m_n - 1.$$

Let $H_l = \{j \mid jl \not\equiv 0 \pmod{m_n}\}$, $1 \leq l \leq m_n - 1$. H_l is a proper subgroup of Z_{m_n}. We have

$$1 - \cos 2\pi jl/m_n = 0 \quad \text{for } j \in H_l$$

while

$$1 - \cos 2\pi jl/m_n \geq 1 - \cos 2\pi/m_n \quad \text{for } j \notin H_l .$$

Using the estimate $1 - \cos \theta \geq 2\theta^2/\pi^2$, $0 \leq \theta \leq \pi$, we have

$$1 - \cos 2\pi jl/m_n \geq 8/m_n^2.$$

Thus for given l,

$$\sum_{j=1}^{m_n-1} \alpha_{nj}[1 - \cos 2\pi jl/m_n] \geq (8/m_n^2) \sum_{j \notin H_l} \alpha_{nj}$$

and we conclude from (A) that $X_n(\gamma) \geq \alpha/m_n^2$ for $\gamma \in E_n$.

We obtain the following result.

THEOREM 2.2. *Let $\{m_n\}$ be bounded and let μ satisfy condition* (A). *Then μ is recurrent* $\Rightarrow \sum_{n=1}^{\infty} 1/M_n f_n = \infty$.

To prove the above theorem we establish two lemmas, the first one being of some independent interest.

LEMMA 2.1. *Let $\{X_n\}$ be a sequence of independent random variables with $E(X_n) = 0$ and $|X_n| \leq 1$, $1 \leq n < \infty$. Let $S_n = \sum_{i=1}^{n} X_i$, $1 \leq n < \infty$.*

Then for each $\varepsilon > 0$, $\sum_{j=1}^{\infty} P[|S_n| \geq \varepsilon n \text{ for some } n \geq j] \leq C_\varepsilon$ *where* C_ε *is a positive constant independent of* $\{X_n\}$.

Proof. We observe that for any choice of $a_1, \cdots, a_n \geq 0$ we have

$$|E(X_1^{a_1} \cdots X_n^{a_n})| \leq 1.$$

Since $E(X_1^{a_1} \cdots X_n^{a_n}) = E(X_1^{a_1}) \cdots E(X_n^{a_n})$, we also have

$$E(X_1^{a_1} \cdots X_n^{a_n}) = 0$$

whenever some $a_i = 1$. Hence

$$E(S_n^6) = \sum_{1 \leq i \leq n} E(X_i^6) + 15 \sum_{1 \leq i, j \leq n} \mathrm{w}(X_i^4 X_j^2) + 20 \sum_{1 \leq i < j \leq n} E(X_i^3 X_j^3) + 90 \sum_{1 \leq i < j < k \leq n} E(X_i^2 X_j^2 X_k^2)$$

$$\leq n + 15\, C(n, 2) + 20\, C(n, 2) + 90\, C(n, 3)$$

$$\leq 50\, n^3.$$

Using Chebycheff's inequality, we obtain

$$P(|S_n| \geq \varepsilon_n) \leq E(S_n^6)/\varepsilon^6 n^6 \leq (50/\varepsilon^6)(1/n^3)$$

so that

$$P[|S_n| \geq \varepsilon_n \quad \text{for some} \quad n \geq j] \leq \sum_{n=j}^{\infty} P[|S_n| \geq \varepsilon_n]$$

$$\leq (50/\varepsilon^6) \sum_{n=j}^{\infty} 1/n^3 \leq 100/\varepsilon^6 j^2.$$

Hence

$$\sum_{j=1}^{\infty} P[|S_n| \geq \varepsilon_n \text{ for some } n \geq j] \leq (100/\varepsilon^6) \sum_{j=1}^{\infty} 1/j^2 = 50\pi^2/3\varepsilon^6.$$

For any sequence of positive constants $\{c_n\}$ we introduce the sets

$$A_{nr} = \{\gamma \mid f_n > r c_n \operatorname{Re}(1 - \hat{\mu}(\gamma))\}, \quad 1 \leq n, r < \infty.$$

LEMMA 2.2. *Suppose that* $\sum_{n=1}^{\infty} P(A_{nr} \mid E_n) < C$ *where* C *is a positive constant independent of* n. *Then* $\sum_{n=1}^{\infty} c_n/M_n f_n < \infty \Rightarrow \mu$ *is transient.*

Proof. Since $\operatorname{Re}[1/(1 - \hat{\mu})] \leq 1/\operatorname{Re}[1 - \hat{\mu}]$, it suffices to show that

$$\sum_{n=1}^{\infty} c_n/M_n f_n < \infty \quad \Rightarrow \quad E(1/\operatorname{Re}[1 - \hat{\mu}]) < \infty$$

($E(f)$ stands as an abbreviation for $\int_\Gamma f(\gamma)\, dP(\gamma)$). We use the elementary inequality

$$E(X) \leq \sum_{r=1}^{\infty} P(X > r) + 1,$$

valid for any non-negative random variable X. Let $\Psi = \operatorname{Re}[1 - \hat{\mu}]$. We have

$$E(\Psi^{-1} \mid E_n) = \frac{c_n}{f_n} E\left(\frac{f_n}{c_n \Psi} \,\middle|\, E_n\right) \leq \frac{c_n}{f_n}\left[\sum_{r=1}^{\infty} P(A_{nr} \mid E_n) + 1\right] \leq (C + 1)\frac{c_n}{f_n}.$$

Hence

$$E(\Psi^{-1}) = \sum_{n=1}^{\infty} E(\Psi^{-1} \mid E_n)P(E_n) \leq (C+1)\sum_{n=1}^{\infty} c_n/M_n f_n ,$$

proving the lemma.

Proof of Theorem 2.2. We must show that for bounded $\{m_n\}$,

$$\sum_{n=1}^{\infty} m_n^2/M_n f_n < \infty \Rightarrow \mu$$

is transient, provided (A) is fulfilled. This establishes Theorem 2.2, as

$$\sum_{n=1}^{\infty} 1/M_n f_n < \infty \quad \Leftrightarrow \quad \sum_{n=1}^{\infty} m_n^2/M_n f_n < \infty \text{ for bounded } \{m_n\}.$$

In view of Lemma 2.2, it suffices to demonstrate the existence of a constant C such that $\sum_{r=1}^{\infty} P(A_{nr} \mid E_n) < C$, $1 < n < \infty$, for the choice $c_n = m_n^2/\alpha$. On E_n we have $\Psi = \sum_{j=n}^{\infty} p_j X_j$. Let $S_{nj} = \sum_{i=1}^{j} X_{n+i}$, $1 \leq n, j < \infty$. Using summation by parts we obtain

$$\Psi(\gamma) = p_n X_n(\gamma) + \sum_{j=1}^{\infty} S_{nj}(\gamma)[p_{n+j} - p_{n+j+1}], \quad \gamma \in \Gamma.$$

It follows from (2.1) that

$$rc_n \Psi(\gamma) \geq rp_n + rc_n \sum_{j=1}^{\infty} S_{nj}(\gamma)[p_{n+j} - p_{n+j+1}], \quad \gamma \in E_n . \tag{2.2}$$

Using summation by parts again, $f_n = p_n + \sum_{j=1}^{\infty} j[p_{n+j} - p_{n+j+1}]$ from which we conclude

$$f_n \leq rp_n + \sum_{j=r}^{\infty} [p_{n+j} - p_{n+j+1}] \tag{2.3}$$

It follows from (2.2) and (2.3) that for $\gamma \in E_n$,

$$f_n > rc_n \Psi(\gamma) \quad \Rightarrow \quad rc_n S_{nj}(\gamma) < j \text{ for some } j \geq r.$$

Since $c_n = m_n^2/8(1-c) \geq \frac{1}{2}$, $1 \leq n < \infty$, we conclude that for $r \geq 3$,

$$(A_{nr} \cap E_n) \subseteq (B_{nr} \cap E_n)$$

where

$$B_{nr} = \{\gamma \mid S_{nj}(\gamma) < (2/3)j \text{ for some } j \geq r\}.$$

Let $X_n' = X_n - 1$, $1 \leq n < \infty$, $S_{nj}' = \sum_{i=1}^{j} X_{n+i}'$. We have

$$B_{nr} \subseteq B_{nr}' \quad \text{where} \quad B_{nr}' = \{\gamma \mid \mid S_{nj}'(\gamma) \mid > j/3 \quad \text{for some} \quad j \geq r\}.$$

The sequence $\{X_n\}$ clearly has the properties (1′)–(4′) satisfied by $\{V_n\}$. It follows that for given $n \geq 1$, the random variables $\{X_{n+1}', \cdots, X_{n+j}'\}$ defined on E_n are independent and $E(X_{n+j}' \mid E_n) = E(X_{n+j} - 1) = 0$, $\mid X_{n+j}' \mid \leq 1$. We may therefore apply Lemma 2.1 to conclude that

$$\sum_{r=1}^{\infty} P(B_{nr}' \mid E_n) \leq C_{1/3}, \qquad 1 \leq n < \infty.$$

Hence

$$\sum_{r=1}^{\infty} P(A_{nr} \mid E_n) \leq 2 + C_{1/3}, \quad 1 \leq n < \infty$$

and we may choose $C = 2 + C_{1/3}$.

The question arises as to whether Theorem 2.2 remains true in case μ fails to satisfy condition (A). The answer is no. Let $\{m_n\}$ be any bounded sequence of integers ≥ 2 containing an infinite number of composite integers. We produce an aperiodic recurrent walk μ on $Z_{m_1} \oplus Z_{m_2} \oplus \cdots \oplus Z_{m_n} \oplus \cdots$ for which $\sum_{n=1}^{\infty} 1/M_n f_n < \infty$. (The walk μ will of course violate condition (A).)

We first require the following:

LEMMA 2.3. *Let* $0 < a_n \leq b_n$ $(1 \leq n < \infty)$ *and* $b_n/a_n \uparrow \infty$. *Let* $\sum_{n=1}^{\infty} a_n$, $\sum_{n=1}^{\infty} b_n < \infty$. *Then there exists* $\{x_n\}$ *such that* x_n *is a strictly increasing sequence* $\uparrow \infty$, $\sum_{n=1}^{\infty} a_n x_n < \infty$, $\sum_{n=1}^{\infty} b_n x_n = \infty$.

Proof. Let $A_n = \sum_{j=n}^{\infty} a_j$, $B_n = \sum_{j=n}^{\infty} b_j$. Then $\lim_{n\to\infty} B_n/A_n = \infty$. Let $u_n = x_n - x_{n-1}$, $1 \leq n < \infty$, where x_0 is defined to be 0. Using summation by parts we obtain

$$\sum_{j=1}^{n} a_j x_j = \sum_{j=1}^{n} (A_j - A_{n+1})u_j, \quad \sum_{j=1}^{n} b_j x_j = \sum_{j=1}^{n} (B_j - B_{n+1})u_j.$$

Since $\lim_{n\to\infty} B_n/A_n = \infty$ we can certainly find a sequence $\{v_n\}$ such that

$$v_n > 0, \quad \sum_{n=1}^{\infty} v_n < \infty, \quad \sum_{n=1}^{\infty} (B_n/A_n)v_n = \infty.$$

Let $u_n = v_n/A_n$. Thus $u_n > 0$, $\sum_{n=1}^{\infty} A_n u_n < \infty$ $\sum_{n=1}^{\infty} B_n u_n = \infty$. Since $B_n u_n = O(u_n)$ we must also have $\sum_{n=1}^{\infty} u_n = \infty$. It follows that $x_n = \sum_{j=1}^{n} u_j$ is a strictly increasing positive sequence $\uparrow \infty$. Since

$$\sum_{j=1}^{n} a_j x_j \leq \sum_{j=1}^{\infty} A_j u_j,$$

we conclude $\sum_{n=1}^{\infty} a_n x_n < \infty$. For any positive integer n,

$$\lim_{n\to\infty} \sum_{j=1}^{n} (B_j - B_{n+1})u_j = \sum_{j=1}^{n} B_j u_j.$$

Hence $\liminf \sum_{j=1}^{n} b_j x_j \geq \sum_{j=1}^{n} B_j u_j$. Since n is arbitrary we conclude $\sum_{n=1}^{\infty} b_n x_n = \infty$.

We now construct a recurrent random walk μ for which $\sum_{n=1}^{\infty} 1/M_n f_n < \infty$. We specify the p_n's and α_{nj}'s which define μ. For each composite m_n choose d_n to be a proper divisor of m_n, $1 < d_n < m_n$. For m_n prime let $d_n = m_n$. Let $D_n = d_1 \cdots d_{n-1}$, $2 \leq n < \infty$, $D_1 = 1$. Thus $2^n \leq D_n \geq M_n$ and $\lim_{n\to\infty} M_n/D_n = \infty$, as there is an infinite number of composite m_n. It follows from Lemma 2.3 that there exists $\{x_n\}$, where $\{x_n\}$ is positive and strictly increasing to ∞, such that

$$\sum_{n=1}^{\infty} x_n/M_n < \infty \quad \text{and} \quad \sum_{n=1}^{\infty} x_n/D_n = \infty.$$

We may assume that $x_1 = 1$. Let $f_n = 1/x_n$, $p_n = f_n - f_{n+1}$, $1 \leq n < \infty$. We then have

$$p_n > 0, \quad \sum_{n=1}^{\infty} p_n = 1, \quad \sum_{n=1}^{\infty} 1/M_n f_n < \infty, \quad \sum_{n=1}^{\infty} 1/D_n f_n = \infty.$$

We now define the α_{nj}'s. Let $\{\varepsilon_n\}$ denote a decreasing sequence of numbers in (0, 1) to be specified later. For m_n prime choose $\{\alpha_{nj}\}$ as any sequence satisfying

$$\alpha_{nj} = \alpha_{n\, m_n-j} > 0, \quad 1 \leq j \leq m_n - 1, \quad \sum_{j=1}^{m_n-1} \alpha_{nj} = 1.$$

For m_n composite choose $\{\alpha_{nj}\}$ as any sequence satisfying

$$\alpha_{nj} = \alpha_{n\, m_n-j} > 0, \quad 1 \leq j \leq m_n - 1,$$

and

$$\alpha_{nj} = (1 - \epsilon_n)/(d_n - 1) \quad \text{for } j = ke_n\,, \quad 1 \leq k \leq m_n - 1,$$

where $e_n = m_n/d_n$. μ is aperiodic as $\alpha_{nj} > 0$ for $1 \leq n < \infty$, $1 \leq j \leq m_n - 1$. Using the requirement $\alpha_{nj} = \alpha_{n\, m_n-j}$, we obtain

$$\bar{V}_n = \sum_{j=1}^{m_n-1} \alpha_{nj}(1 - \bar{U}_n^j) = \sum_{j=1}^{m_n-1} \alpha_{nj}(1 - U_n^{m_n-j})$$
$$= \sum_{j=1}^{m_n-1} \alpha_{n\, m_n-j}(1 - U_n^j) = V_n$$

so that V_n is real. Thus $1 - \hat{\mu}$ is real and $\int_\Gamma \operatorname{Re}\,[1/(1 - \hat{\mu})]\, dP$. We choose the ε_n's so that the latter integral diverges, thus obtaining a recurrent walk μ for which $\sum 1/M_n f_n < \infty$.

Let $V_n = V_n' + V_n''$ where

$$V_n' = \sum_{e_n \mid j} \alpha_{nj}(1 - U_n^j), \qquad V_n'' = \sum_{e_n \nmid j} \alpha_{nj}(1 - U_n^j),$$
$$1 \leq j \leq m_n - 1.$$

(For m_n prime, $e_n = 1$ divides all j. In this case V_n'' is defined to be 0.) We have

$$|\, V_n'' \,| \leq 2\sum_{e_n \nmid j} \alpha_{nj} < 2\varepsilon_n\,.$$

U_n assumes the values $2\pi i l/m_n$, $0 \leq l \leq m_n - 1$. If $d_n \mid l$, then $V_n' = 0$, i.e. $V_n' = 0$ whenever $U_n \in Z_{e_n}$, the symbol Z_{e_n} denoting the e_n e_nth roots of 1. For m_n composite, let

$$F_{nk} = E_n \cap \{\gamma \mid U_n(\gamma) \in Z_{e_n}, \cdots, U_{n+k-1}(\gamma) \in Z_{e_{n+k-1}}, U_{n+k}(\gamma) \notin Z_{e_{n+k}}\},$$
$$1 \leq n, \quad k < \infty.$$

Then

$$P(F_{nk}) = \frac{1}{m_1 \cdots m_{n-1}} \frac{e_n - 1}{m_n} \frac{1}{d_{n+1} \cdots d_{n+k-1}} \left(1 - \frac{1}{d_{n+k}}\right)$$
$$= \left(1 - \frac{1}{e_n}\right)\left(1 - \frac{1}{d_{n+k}}\right) \frac{1}{e_1 \cdots e_{n-1}} \frac{1}{D_{n+k}}.$$

$e_n \geq 2$ as m_n is composite and $d_{n+k} \geq 2$ by definition. Hence

$$P(F_{nk}) \geq \frac{1}{4} \frac{1}{e_1 \cdots e_{n-1}} \frac{1}{D_{n+k}}, \quad 1 \leq n < \infty,$$

whenever m_n is composite.

On F_{nk} we have $V_j = 0$, $1 \leq j < n$, $|\, V_j \,| < 2\varepsilon_n$, $n \leq j < n + k$. Thus on F_{nk},

$$1 - \hat{\mu} = \sum_{j=n}^{\infty} p_j V_j \leq 2\varepsilon_n \sum_{j=n}^{n+k-1} p_j + 2\sum_{j=n+k}^{\infty} p_j \leq 2(\varepsilon_n f_n + f_{n+k}).$$

We conclude that

$$\int_{E_n} \frac{1}{1-\hat{\mu}}\,dP \geq \sum_{k=1}^{\infty} \int_{F_{nk}} \frac{1}{1-\hat{\mu}}\,dP$$
$$\geq \frac{1}{8}\,\frac{1}{e_1 \cdots e_{n-1}} \sum_{k=1}^{\infty} \frac{1}{\varepsilon_n f_n + f_{n+k}}\,\frac{1}{D_{n+k}}.$$

Let

$$g_n(x) = \sum_{k=1}^{\infty} \frac{1}{f_n\, x + f_{n+k}}\,\frac{1}{D_{n+k}}.$$

$g_n(x)$ is well defined for $x > 0$, as

$$g_n(x) \leq (1/f_n\, x) \textstyle\sum_{n=1}^{\infty} 1/D_n < \infty.$$

Since $\sum_{n=1}^{\infty} 1/f_n\, D_n = \infty$, $\lim_{x\to 0+} g_n(x) = \infty$. We may therefore choose $\{\varepsilon_n\}$ as any decreasing sequence for which $g_n(\varepsilon_n) > 1$, $1 \leq n < \infty$, and obtain

$$\int_{E_n} 1/(1-\hat{\mu})\,dP \geq 1$$

whenever m_n is composite. As there are an infinite number of composite m_n, we obtain

$$\int_{\Gamma} 1/(1-\hat{\mu})\,dP = \infty$$

so that the random walk μ is recurrent

We summarize the above discussion:

THEOREM 2.3. *For $\{m_n\}$ bounded, recurrence of μ is equivalent to*

$$\textstyle\sum_{n=1}^{\infty} 1/M_n f_n = \infty$$

iff only a finite number of the m_n's are composite.

3. Random walks on $Z_{m_1} \oplus Z_{m_2} \oplus \cdots \oplus Z_{m_n} \oplus \cdots$, $\{m_n\}$ unbounded

We now obtain various necessary and sufficient conditions for the recurrence of μ in case $\{m_n\}$ is unbounded.

THEOREM 3.1. *Let μ be symmetric, i.e. $\alpha_{nj} = \alpha_{n m_n - j}$ for $1 \leq n < \infty$, $1 \leq j \leq m_n - 1$. Then $\sum_{n=1}^{\infty} 1/M_n f_n = \infty \Rightarrow \mu$ is recurrent.*

Proof. $\hat{\mu}$ is real valued as μ is symmetric. Hence

$$\int_{\Gamma} \mathrm{Re}\left[\frac{1}{1-\hat{\mu}}\right] dP = \int_{\Gamma} \frac{1}{1-\hat{\mu}}\,dP = \sum_{n=1}^{\infty} \int_{E_n} \frac{1}{1-\hat{\mu}}\,dP.$$

On E_n, $1 - \hat{\mu} = \sum_{j=n}^{\infty} p_j V_j$ so that $0 \leq 1 - \hat{\mu} \leq 2f_n$.
Hence

$$\int_{E_n} \frac{1}{1-\hat{\mu}}\, dP \geq \frac{1}{2f_n} P(E_n) \geq \frac{1}{4M_n f_n}.$$

It follows that $\sum_{n=1}^{\infty} 1/M_n f_n = \infty \Rightarrow \mu$ is recurrent.

Remarks. For $\{m_n\}$ bounded, the above result holds also when μ is asymmetric (Theorem 2.1). We don't know whether this is the case for $\{m_n\}$ unbounded.

THEOREM 3.2. (i) *Let μ satisfy condition* (A). *Then μ is recurrent* $\Rightarrow \sum_{n=1}^{\infty} m_n^2/M_n f_n = \infty$.

(ii) *Let $\alpha_{nj} = 0$ for $1 \leq n < \infty, j \neq 1, m_n - 1$. Then μ is recurrent* $\Rightarrow \sum_{n=1}^{\infty} m_n/M_n f_n = \infty$.

Proof. In proving Theorem 2.2 we have actually shown that

$$\sum_{n=1}^{\infty} m_n^2/M_n f_n < \infty \Rightarrow \mu$$

is transient, which is part (i) of the above theorem. Part (ii) is proven in a similar fashion. Let

$$E_{nl} = \{V_1 = 0, \cdots, V_{n-1} = 0, \quad V_n = 1 - \cos 2\pi l/m_n\},$$
$$1 \leq n < \infty, \quad 1 \leq l \leq m_n - 1.$$

Since $\alpha_{nj} = 0$ for $j \neq 1, m_n - 1$, $X_n(\gamma)$ assumes the values $1 - \cos 2\pi l/m_n$, $0 \leq l \leq m_n - 1$. Since $1 - \cos\theta \geq 2\theta^2/\pi^2$ for $0 \leq \theta \leq \pi$, and $\cos\theta = \cos(2\pi - \theta)$ we have

$$1 - \cos\theta \geq 8 \operatorname{Min}[(\theta/2\pi)^2, \quad (1 - \theta/2\pi)^2], \quad 0 \leq \theta \leq 2\pi.$$

Hence

$$X_n(\gamma) \geq 8 \operatorname{Min}[(l/m_n)^2, \quad (1 - l/m_n)^2], \quad \gamma \in E_{nl}. \tag{3.1}$$

Let $c_{nl} = (1/8) \operatorname{Max}[(m_n/l)^2, (m_n/(m_n - l))^2]$. It follows from (3.1) that

$$rc_{nl}\Psi_n(\gamma) \geq rp_n + rc_{nl}\sum_{j=1}^{\infty} S_{nj}(\gamma)[p_{n+j} - p_{n+j+1}], \quad \gamma \in E_{nl}. \tag{3.2}$$

Let $A_{nlr} = \{\gamma \mid f_n > rc_{nl}\Psi\}$, $1 \leq n, r < \infty$, $1 \leq l \leq m_n - 1$. Arguing as in the proof of Theorem 2.2, we conclude from (3.2) and (2.3) that

$$\sum_{r=1}^{\infty} P(A_{nlr} \mid E_{nl}) \leq C$$

where C is a positive constant independent of n and l. Hence

$$\begin{aligned} E(\Psi^{-1} \mid E_{nl}) &\leq \frac{c_{nl}}{f_n} E\left(\frac{f_n}{c_{nl}\Psi} \,\middle|\, E_{nl}\right) \\ &\leq \frac{c_{nl}}{f_n}\left[\sum_{r=1}^{\infty} P(A_{nlr} \mid E_{nl}) + 1\right] \\ &\leq (C+1)\frac{c_{nl}}{f_n}. \end{aligned}$$

We have

$$E(\Psi^{-1}) = \sum_{n=1}^{\infty} \sum_{l=1}^{m_n-1} E(\Psi^{-1} \mid E_{nl}) P(E_{nl})$$
$$\leq (C+1) \sum_{n=1}^{\infty} 1/M_n\, m_n f_n \sum_{l=1}^{m_n-1} c_{nl}\,.$$

Since

$$\sum_{l=1}^{m_n-1} c_{nl} = \frac{1}{8} \sum_{l=1}^{m_n-1} \text{Max}\left[\left(\frac{m_n}{l}\right)^2, \left(\frac{m_n}{m_n - l}\right)^2\right]$$
$$\leq \frac{m_n^2}{8} \sum_{l=1}^{m_n-1} \left[\frac{1}{l^2} + \frac{1}{(m_n - l)^2}\right]$$
$$\leq \frac{m_n^2}{4} \sum_{l=1}^{\infty} \frac{1}{l^2} = \frac{\pi^2 m_n^2}{24},$$

we conclude $E(\Psi^{-1}) \leq (C+1)(\pi^2/24) \sum_{n=1}^{\infty} m_n/M_n f_n$. Hence

$$\sum_{n=1}^{\infty} m_n/M_n f_n < \infty \quad \Rightarrow \quad E(\Psi^{-1}) < \infty$$

so that $\sum_{n=1} m_n/M_n f_n < \infty \Rightarrow \mu$ is recurrent.

We now give an example showing that for $\{m_n\}$ unbounded,

$$\sum_{n=1}^{\infty} 1/M_n f_n = \infty$$

is no longer necessary for recurrence even though condition (A) prevails. Indeed we give an example of *a* recurrent walks satisfying the hypothesis of Theorem 3.2 (ii) (in which case (A) is satisfied) for which $\sum_{n=1}^{\infty} 1/M_n f_n < \infty$ so that $\sum_{n=1}^{\infty} 1/M_n f_n = \infty$ is not necessary for recurrence.

Let $p_n = C(s) n^s/n!$ where s is to be specified and $C(s)$ so chosen that $\sum_{n=1}^{\infty} p_n = 1$. Let $m_n = n$ and $\alpha_n = \alpha_{n\, n-1} = \frac{1}{2}$, $1 \leq n < \infty$. Thus $M_n = (n-1)!$. It is easily verified that $f_n \sim p_n$ (i.e. $\lim_{n\to\infty} f_n/p_n = 1$) so that $1/M_n f_n \sim 1/C(s) n^{s-1}$. Hence $\sum_{n=1}^{\infty} 1/M_n f_n < \infty$ for $s > 2$. Using $1 - \cos\theta \leq \theta^2/2$ we have

$$X_n(\gamma) = 1 - \cos 2\pi l/n \leq 2\pi^2 l^2/n^2, \quad \gamma \in E_{nl}\,.$$

Thus $\Psi(\gamma) \leq 2\pi^2 C(s) l^2 n^{s-2}/n! + 2 f_{n+1}$ on E_{nl}. Since $f_{n+1} \sim C(s) n^{s-1}/n!$, we conclude that there exists a positive constant $C_1(s)$ independent of n, l for which

$$\Psi(\gamma) \leq C_1(s)(n^{s-2}/n!)(l^2 + n), \quad \gamma \in E_{nl}\,. \tag{3.3}$$

Hence

$$\int_{E_{nl}} \frac{1}{\Psi}\, dP \geq \frac{1}{C_1(s)}\, n! \quad \frac{n^{2-s}}{l^2 + n} P(E_{nl}) = \frac{1}{C_1(s)} \frac{n^{2-s}}{l^2 + n}$$

and

$$\int_{E_n} \frac{1}{\Psi}\, dP \geq \frac{1}{C_1(s)} n^{2-s} \sum_{l=1}^{n-1} \frac{1}{l^2 + n}\,.$$

Since

$$\sum_{l=1}^{n-1} \frac{1}{l^2 + n} \sim \int_0^n \frac{dx}{x^2 + n} = \frac{1}{\sqrt{n}} \quad \arctan \quad \sqrt{n} \sim \frac{\pi}{2\sqrt{n}},$$

we conclude that there exists a positive constant $C_2(s)$ independent of n for which

$$(3.4) \qquad \int_{E_n} (1/\Psi)\, dP \geq C_2(s)/n^{s-3/2}, \quad 1 \leq n < \infty,$$

so that $\int_\Gamma (1/\Psi)\, dP = \infty$ for $s \leq 5/2$. Hence for $2 < s \leq 5/2$, μ is recurrent while $\sum_{n=1}^{\infty} 1/M_n f_n < \infty$.

Remark. It is also possible to construct transient random walks satisfying the hypotheses of Theorem 3.2 (ii) for which $\sum_{n=1}^{\infty} m_n/M_n f_n = \infty$. Thus $\sum_{n=1}^{\infty} m_n/M_n f_n = \infty$ is not sufficient for recurrence of these random walks.

The problem of obtaining conditions both necessary and sufficient for recurrence seems to be rather difficult for unbounded $\{m_n\}$ and is not resolved here. Nevertheless, such conditions are easily obtained for the uniform walk. μ is said to a uniform random walk provided $\alpha_{nj} = 1/(m_n - 1)$, $1 \leq n < \infty$, $1 \leq j \leq m_n - 1$. In this case we have:

THEOREM 3.3. *Let μ be the uniform walk on $Z_{m_1} \oplus Z_{m_2} \oplus \cdots \oplus Z_{m_n} \oplus \cdots$. Then μ is recurrent iff $\sum_{n=1}^{\infty} 1/M_n f_n = \infty$.*

Proof. Since μ is symmetric, we conclude from Theorem 3.1 that

$$\sum_{n=1}^{\infty} 1/M_n f_n = \infty \Rightarrow \text{recurrence}.$$

Conversely we show that $\sum_{n=1}^{\infty} 1/M_n f_n < \infty \Rightarrow$ transience. We have

$$X_n = (1/(m_n - 1)) \sum_{j=1}^{m_n-1} [1 - U_n^j],$$

and

$$\begin{aligned} (1/(m_n - 1)) \sum_{j=1}^{m_n-1} U_n^j &= 1 && \text{if } U = 1 \\ &= -1/(m_n - 1) && \text{if } U_n \neq 1. \end{aligned}$$

Hence

$$\begin{aligned} X_n &= 0 && \text{if } U = 1 \\ &= m_n/(m_n - 1) && \text{if } U_n \neq 1 \end{aligned}$$

so that on E_n, $X_n = m_n/(m_n - 1)$. It follows that inequality (2.2) holds with the choice $c_n = 1$, $1 \leq n < \infty$, from which we conclude

$$\sum_{n=1}^{\infty} P(A_{nr} \mid E_n) < C, \quad 1 \leq n < \infty,$$

for some positive C independent of n. Lemma 2.2 therefore yields

$$\sum_{n=1}^{\infty} 1/M_n f_n < \infty \Rightarrow \mu$$

is transient.

4. Random walks on subgroups of the rationals mod one

We now obtain recurrence criteria for random walks on the subgroups of infinite order of the rationals mod one. (We denote the latter as Q/Z.) Most of the analysis of Sections 2 and 3 apply here, the essential new feature being that $1 - \hat{\mu}$ is no longer a sum of independent random variables.

We give a brief description of these groups and their duals. The group Q/Z may be identified as the set of complex numbers $e^{2\pi i r}$, r rational. Let G be an infinite subgroup of Q/Z. G has a denumerable set of generators $\{x_n\}$. Let G_n be the subgroup of G generated by $x_1, \cdots, x_n$. It is readily seen that G_n is cyclic. Let $[G_n : G_{n-1}] = m_n$, $1 \leq n < \infty$, where $G_0 = \{1\}$. Thus G_n consists of $m_1 \cdots m_n$ numbers $e^{2\pi i j/m_1 \cdots m_n}$, $0 \leq j \leq m_1 \cdots m_n - 1$, while $G = U_{n=1}^{\infty} G_n$ consists of the numbers $e^{2\pi i j/m_1 \cdots m_n}$, $1 \leq n < \infty$, $0 \leq j \leq m_1 \cdots m_n - 1$. We denote G_n by $Z(m_1, \cdots, m_n)$ and G by $Z(m_1, \cdots, m_n, \cdots)$. Thus the infinite subgroups of the rationals mod one are precisely the $Z(m^{\infty})$ groups discussed in [3, p. 403].

We now describe the dual of $Z(m_1, \cdots, m_n, \cdots)$. Let $g_n = e^{2\pi i/m_1 \cdots m_n}$, $1 \leq n < \infty$. g_n generates $Z(m_1, \cdots, m_n)$ and $\{g_n\}$ is a set of generators for $Z(m_1, \cdots, m_n, \cdots)$. Any character $\gamma(g)$ of $Z(m_1, \cdots, m_n, \cdots)$ is thus determined by the values $\{\gamma(g_n)\}$. Since $g_1^{m_1} = 1$, $g_n^{m_n} = g_{n-1}$, $2 \leq n < \infty$, we have the relations

$$[\gamma(g_1)]^{m_1} = 1, \quad [\gamma(g_n)]^{m_n} = \gamma(g_{n-1}), \quad 2 \leq n < \infty. \tag{4.1}$$

Conversely, it is easily verified that any sequence $\{\gamma(g_n)\}$ satisfying (4.1) can be uniquely extended as a character $\gamma(g)$ on G. Thus the dual of $Z(m_1, \cdots, m_n, \cdots)$ may be identified as the set of sequences $\{\gamma(g_1), \cdots, \gamma(g_n), \cdots\}$ satisfying the relations (4.1).

Let μ be a random walk on G. Without loss of generality, let μ be aperiodic. We may therefore choose a set of generators $\{x_n\}$ of G satisfying $\mu(x_n) > 0$, $1 \leq n < \infty$. Let G_n denote the subgroup of G generated by $x_1, \cdots, x_n$; we may assume that G_n is a proper subgroup of G_{n+1} ($1 \leq n < \infty$). Denote, as above, G_n by $Z(m_1, \cdots, m_n)$ and G by $Z(m_1, \cdots, m_n, \cdots)$. Thus, without loss of generality, we assume that (support of μ) $\cap\, Z(m_1, \cdots, m_n)$ generates $Z(m_1, \cdots, m_n)$, $1 \leq n < \infty$.

Let

$$Z'(m_1, \cdots, m_n) = Z(m_1, \cdots, m_n) - Z(m_1, \cdots, m_{n-1}), \ 2 \leq n < \infty,$$
$$Z'(m_1) = Z(m_1) - \{1\}.$$

Thus $Z'(m_1, \cdots, m_n)$ consists of the numbers $e^{2\pi i j/m_1 \cdots m_n}$, $j \in S_n$, where S_n denotes the set of positive integers which are not multiples of m_n. We define $\{p_n\}$, $1 \leq n < \infty$, and $\{\alpha_{nj}\}$, $1 \leq n < \infty$, $j \in S_n$, as

$$p_n = \mu[Z'(m_1, \cdots, m_n)], \quad \alpha_{nj} = \mu(e^{2\pi i j/m_1 \cdots m_n})/p_n.$$

We may assume $\mu(1) = 0$ (in view of Theorem 2.0). Thus $p_n > 0$, $1 \leq n < \infty$, $\sum_{n=1}^{\infty} p_n = 1$, $\sum_{j \in S_n} \alpha_{nj} = 1$, $1 \leq n < \infty$. Let $\hat{\mu}(\gamma)$ denote the Fourier transform of μ. We then have

$$\hat{\mu}(\gamma) = \sum_{g \in G} \mu(g)\gamma(g) = \sum_{n=1}^{\infty} \sum_{j \in S_n} \alpha_{nj}\, \gamma(g_n^j), \quad \gamma \in \Gamma.$$

We seek recurrence criteria for μ in terms of $\{p_n\}$ and $\{\alpha_{nj}\}$. Let

$$U_n(\gamma) = \gamma(g_n), \quad V_n(\gamma) = \sum_{j \in S_n} \alpha_{nj}[1 - U_n^j(\gamma)], \quad X_n(\gamma) = \mathrm{Re}\,(V_n(\gamma)).$$

We establish:

LEMMA 3.1. *$\{V_n\}$ satisfies the following properties:*
(1) $|V_n - 1| \le 1$.
(2) $E(V_n \mid U_1, \cdots, U_{n-1}) = 1, \quad 1 \le n < \infty$.
(3) $V_1 = \cdots = V_n = 0 \Leftrightarrow U_1 = \cdots = U_n = 1$.
A similar statement applies to the sequence $\{X_n\}$.

Proof. (1) $1 - V_n = \sum_{j \in S_n} \alpha_{nj} U_n^j$ so that

$$|1 - V_n| \le 1, \quad |1 - X_n| \le 1,$$

(2) Let $\zeta_1, \cdots, \zeta_n$ be any set of numbers satisfying $\zeta_1^{m_1} = 1$, $\zeta_k^{m_k} = \zeta_{k-1}\ 2 \le k \le n$. It is easily verified that

$$P\{U_1(\gamma) = \zeta_1, \cdots, U_n(\gamma) = \zeta_n\} = 1/m_1 \cdots m_n .$$

It follows that

$$E(U_n^j \mid U_1 = \zeta_1, \cdots, U_{n-1} = \zeta_{n-1}) = (1/m_1 \cdots m_n) \sum_{\zeta^{m_n} = \zeta_{n-1}} \zeta^j.$$

The sum

$$\sum_{\zeta^{m_n} = \zeta_{n-1}} \zeta^j$$

extends over the m_n m_nth roots of 1 and equals 0 for $m_n \nmid j$. We conclude that

$$E(U_n^j \mid U_1, \cdots, U_{n-1}) = 0, \quad j \in S_n .$$

Hence

$$\begin{aligned} E(V_n \mid U_1, \cdots, U_{n-1}) &= E(1 - \textstyle\sum_{j \in S_n} \alpha_{nj} U_n^j \mid U_1, \cdots, U_{n-1}) \\ &= 1 - \textstyle\sum_{j \in S_n} \alpha_{nj} E(U_n^j \mid U_1, \cdots, U_{n-1}) = 0. \end{aligned}$$

A similar proof works for the X_n's.

(3) We make use of the assumption that (support of μ) $\cap Z(m_1, \cdots, m_n)$ generates $Z(m_1, \cdots, m_n)$, $1 \le n < \infty$ which may be restated as follows. Let H_n denote the positive integers $\le m_1 \cdots m_n$ which are multiples of m_n. Let $I_n \subseteq S_n$ consist of those integers j for which $\alpha_{nj} > 0, j \in S_n$. Then $I_n \cup S_n$ generates the additive group of integers mod $m_1 \cdots m_n$. Since $V_n = \sum_{j \in S_n} \alpha_{nj}[1 - U_n{}^j]$, we have $V_n = 0 \Leftrightarrow U_n{}^j = 1, j \in I_n$. Since I_1 generates the integers mod m_1 and $U_1^{m_1} = 1$, we have $U_1^j = 1\ (j \in I_n) \Leftrightarrow U_1 = 1$. Thus $V_1 = 0 \Leftrightarrow U_1 = 0$. Suppose, by hypothesis for induction, that $V_1 = \cdots = V_{n-1} = 0 \Leftrightarrow U_1 = \cdots = U_{n-1} = 1$. Thus

$$V_1 = \cdots = V_n = 0 \Leftrightarrow U_1 = \cdots = U_{n-1} = 1 \quad \text{and} \quad U_n^j = 1\ (j \in I_n).$$

Now $U_n^{m_n} = U_{n-1} = 1$ so that we also have $U_n^j = 1\ (j \in H_n)$. Since $H_n \cup I_n$ generate the integers mod $m_1 \cdots m_n$, we conclude

$$V_1 = \cdots = V_n = 0 \quad \Leftrightarrow \quad U_1 = \cdots = U_n = 1.$$

As $X_n = 1 \Leftrightarrow V_n = 1$, we obtain a similar result for $\{X_n\}$.

Let $E_n = \{\gamma \mid V_1 = 0, \cdots, V_{n-1} = 0, V_n \neq 0\}, 1 \leq n < \infty$. We conclude from property (3) that

$$P(E_n) =)1/m_1 \cdots m_{n-1})(1 - 1/m_n)$$

exactly as in Section 2.

We now state several recurrence criteria. We again let

$$M_n = m_1 \cdots m_{n-1} 2 \leq n < \infty, \quad M_1 = 1.$$

THEOREM 4.1. *Let* $\sum_{n=1}^{\infty} 1/M_n f_n = \infty$. μ *is recurrent provided* (i) $\{m_n\}$ *is bounded or* (ii) μ *is symmetric, i.e.* $\alpha_{nj} = \alpha_{n, m_1 \dots m_n - j}$, $1 \leq n < \infty$, $1 \leq j \leq m_1 \cdots m_n - 1$.

(i), (ii) are the respective analogs of Theorems 2.1, 3.1. The proofs are omitted as they are identical with the proofs of these former theorems. We just observe that on E_n, $U_n^{m_n} = 1$, as this fact is essential for the proof of part (i).

In order to state our next result, we stipulate the following condition on μ which is just the analog of condition (A) in Section 2.

Condition (A′). There exists c, $0 < c < 1$, such that $\mu(H)/p_n \leq c$ for all proper subgroups H of $Z(m_1, \cdots, m_n)$, $1 \leq n < \infty$.

Reasoning as in Section 2, condition (A') yields the inequality (2.1).

THEOREM 4.2. *Let* $\{p_n\} \in \downarrow$ *and* $\{m_n\}$ *bounded. Let* μ *satisfy condition* (A′). *Then* μ *is recurrent* $\Rightarrow \sum_{n=1}^{\infty} 1/M_n f_n = \infty$.

Proof. We try to mimic the proof of Theorem 2.2. Let $\alpha = 8(1 - c)$ be the constant appearing in inequality (2.1) and let $m = \text{Max}_{1 \leq n < \infty} m_n < \infty$. Define $S_{nj} = \sum_{i=1}^{j} X_{n+i}$ for $1 \leq n, j < \infty$. Arguing exactly as in the proof of Theorem 2.2 it suffices to demonstrate the existence of a constant C such that

$$\sum_{r=1}^{\infty} P(B_{nr} \mid E_n) < C, \quad 1 \leq n < \infty,$$

where $B_{nr} = \{\gamma \mid S_{nj}(\gamma) < j/2m \text{ for some } j \geq r\}$. The latter inequality no longer follows from Lemma 2.1 as the sequences of random variables $\{X_n\}$ fail to be independent. To prove the inequality we compare the X_n's with another sequence of independent random variables $\{Y_n\}$ defined as follows.

Let $\zeta_1, \cdots, \zeta_n$ be any set of numbers satisfying $\zeta_1^{m_1} = 1$, $\zeta_k^{m_k} = \zeta_{k-1}$, $2 \leq k \leq n$, and let

$$S_{\zeta_1, \dots, \zeta_n} = \{\gamma \mid U_1(\gamma) = \zeta_1, \cdots, U_n(\gamma) = \zeta_n\}.$$

For given $\zeta_1, \cdots, \zeta_{n-1}$ we have $E(X_n \mid U_1 = \zeta_1, \cdots, U_{n-1} = \zeta_{n-1}\} = 1$ so that we may choose one value of ζ_n, call it ξ_n, such that $X_n \geq 1$ on $S_{\zeta_1, \dots, \zeta_{n-1}, \xi_n}$. For given $\zeta_1, \cdots, \zeta_{n-1}$ we define

$$\begin{aligned} Y_n(\gamma) &= 1, \quad \gamma \in S_{\zeta_1, \dots, \zeta_{n-1}, \xi_n} \\ &= 0, \quad \gamma \in S_{\zeta_1, \dots, \zeta_n} \quad \text{where} \quad \zeta_n \neq \xi_n. \end{aligned}$$

It is readily checked that $\{Y_n\}$ is a sequence of independent random variables on Γ with $0 \le Y_n \le 1$, $E(Y_n) = 1/m_n$ for $1 \le n < \infty$. Let

$$Z_n = Y_n - 1/m_n, \quad 1 \le n < \infty.$$

The Z_n's form a sequence of independent random variables with $E(Z_n) = 0$, $|Z_n| \le 1$ for $1 \le n < \infty$. Since $Y_n \le X_n$ we conclude that

$$\begin{aligned} B_{nr} &= \{\gamma \mid \textstyle\sum_{i=1}^{j} X_{n+i}(\gamma) < j/2m \quad \text{for some} \quad j \ge r\} \\ &\subseteq \{\gamma \mid \textstyle\sum_{i=1}^{j} Y_{n+i}(\gamma) < j/2m \quad \text{for some} \quad j \ge r\} \\ &\subseteq \{\gamma \mid |\textstyle\sum_{i=1}^{j} Z_{n+i}(\gamma)| > j/2m \quad \text{for some}\ j \ge r\}. \end{aligned}$$

For $n \ge 1$, the sequence $\{Z_{n+1}, \cdots, Z_{n+j}, \cdots\}$ satisfies the hypotheses of Lemma 2.1. Hence

$$\sum_{r=1}^{\infty} P(B_{nr}) \le \sum_{n=1}^{\infty} P\{\, |\textstyle\sum_{i=1}^{j} Z_{n+i}(\gamma)| > j/2m \quad \text{for some} \quad j \ge r\} \le C/2m$$

for $1 \le n < \infty$, thus proving Theorem 4.2.

We remark that Theorem 4.2 permits us to construct a random walk on the additive group of rationals Q which is topologically recurrent under the usual topology but not pointwise recurrent, i.e., for any $\varepsilon > 0$, the walk visits the interval $(-\varepsilon, \varepsilon)$ infinitely often with probability one and yet the walk visits the origin at most finitely often with probability one. Let $\{m_n\}$ be a bounded sequence of integers ≥ 2. Let $\mu(\pm 1/m_1 \cdots m_n) = p_n/2$ where $\{p_n\} \in \downarrow$ and $\sum_{n=1}^{\infty} p_n = 1$. Suppose that $\sum_{n=1}^{\infty} 1/m_1 \cdots m_n f_n < \infty$. Considered as a walk on Q/Z, μ is pointwise transient according to Theorem 4.2. Hence μ is certainly pointwise transient as a walk on Q. Now

$$\int_{-\infty}^{\infty} |x|\, d\mu(x) = \sum_{n=1}^{\infty} p_n/m_1 \cdots m_n \le \sum_{n=1}^{\infty} 1/m_1 \cdots m_n f_n < \infty.$$

Since $\int_{-\infty}^{\infty} x\, d\mu(x) = 0$, we conclude from a result of Chung and Fuchs [1] that μ is topologically recurrent.

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