VECTOR VALUED KÖTHE FUNCTION SPACES II

BY

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This paper is a continuation of [9].

1. Dual pairs of v.f.s.'s

We say that $(S(E), T(F))$ is a *dual pair of v.f.s.'s* if $S(E)$ and $T(F)$ are v.f.s.'s and if for every $f \in S(E)$ and $g \in T(F)$, $\int \langle f, g \rangle \, d\pi$ exists. The integral $\int \langle f, g \rangle \, d\pi$ will always be understood to be the bilinear form connecting the two spaces. Since $T(F) \supseteq \Gamma(F)$, [9, Proposition 6.2] shows that $T(F)$ separates points of $S(E)$. Since $S(E) \supseteq \Gamma(E)$ it is easy to show that $S(E)$ separates points of $T(F)$.

If $S(E)$ is a v.f.s., the *Köthe dual* of $S(E)$, denoted $S(E)^*$, is the set of all $g \in \bar{\Omega}(E')$ such that $\int \langle f, g \rangle \, d\pi$ exists for all $f \in S(E)$. Since clearly $S(E)^* \supseteq \Gamma(E')$, $S(E)^*$ is a v.f.s. If $T(F)$ is a v.f.s., $T(F)^*$ is defined similarly and is a v.f.s.

There is a lack of symmetry in the concept of a dual pair of v.f.s.'s since $\Omega(E)$ and $\bar{\Omega}(F)$ are defined differently. There is an important case where we do have symmetry.

PROPOSITION 1.1. *Let E be a reflexive separable Banach space and give E' the norm topology. Then $\Omega(E') = \bar{\Omega}(E')$.*

Proof. Comparing the definitions of $\Omega(E')$ and $\bar{\Omega}(E')$, this follows from five facts: (1) If p is the norm on E, then p^0 is the norm on E'; (2) if E is separable, so is E' [8, p. 259]; (3) for a separable Banach space scalar measurability is the same as measurability [1, p. 181]; (4) considering E' as a Banach space, $\Omega(E') = \Omega_0(E')$ [9]; (5) if two measurable functions are scalarly a.e. equal, they are a.e. equal [6, p. 21]. ▌

Remark. Suppose E is a reflexive Banach space. Then Proposition 1.1 is true in the following sense. Suppose g is a member of a class of functions in $\bar{\Omega}(E')$. Then by [2, p. 95, Ex. 25] there is a $g' \equiv g$ such that g' is a member of a class of functions in $\bar{\Omega}(E')$. It is easy to show that the map $g \to g'$ induces a well defined, one to one map of $\bar{\Omega}(E')$ onto $\bar{\Omega}(E')$, giving the result. Using this fact we may drop the hypothesis that E be separable in Theorem 2.7.

I would like to thank the referee for pointing out the following remark.

Remark. If E is a Banach space with a separable dual then Proposition 1.1 is still true. For the unit ball in E, being $\sigma(E'', E')$ dense in the $\sigma(E'', E')$ compact, metrizable, and hence separable unit ball of E'', contains a countable

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set dense in the unit ball of E''. Thus by [1, p. 181] any $g \in \bar{\Omega}(E')$ is norm measurable. By [9, Proposition 3.1] if $g_1 \equiv g_2$ in $\Omega(E')$ then $g_1 = g_2$ a.e. The proof is now similar to the one given in the proposition.

We omit the easy proof of the following proposition.

PROPOSITION 1.2. *Let $(S(E), T(F))$ be a dual pair of v.f.s.'s. Let $f \in S(E)$ and $B \subseteq T(F)$ be solid. Then*

$$\operatorname{Sup}\left\{\left|\int \langle f, g\rangle \, d\pi\right| : g \in B\right\} = \operatorname{Sup}\left\{\int |\langle f, g\rangle| \, d\pi : g \in B\right\}.$$

It follows that if in the above proposition $S(E)$ is solid, then the polar of B is solid.

LEMMA 1.3. *Let $(S(E), T(F))$ be a dual pair of v.f.s.'s. Then if $T(F)$ is solid, the solid hull of a point $g \in T(F)$ is weakly compact and if $S(E)$ is solid, the solid hull of a point in $S(E)$ is weakly compact.*

Proof. If $T(F)$ is solid the map $V: L^\infty \to T(F)$ given by $Vb = bg$ has an adjoint $V^*: S(E) \to L^1$ given by $V^*f = \langle f, g\rangle$. Thus V is weakly continuous and so the solid hull of g, which is the image by V of the weakly compact unit ball of L^∞, is weakly compact. The proof of the other part is similar. ∎

If $(S(E), T(F))$ is a dual pair of v.f.s.'s and $T(F)$ is solid, the *normal topology* on $S(E)$ is the topology of uniform convergence on the solid hulls of points in $T(F)$. By the above lemma and the Mackey-Arens theorem, the dual of $S(E)$ under the normal topology is $T(F)$.

PROPOSITION 1.4. *Let $(S(E), T(F))$ be a dual pair of solid v.f.s.'s. Then the solid hull of a weakly bounded set in $S(E)$ or $T(F)$ is again weakly bounded*

Proof. By the comment above, the bounded sets in the weak and normal topologies on $S(E)$ are the same. By the comment following Proposition 1.2, the normal topology has a base of solid sets. Thus the solid hull of a set bounded in the normal topology is bounded. The proof of the other part is similar. ∎

COROLLARY 1.5. *Let $(S(E), T(F))$ be a pair of solid v.f.s.'s. Then the solid hull of a strongly bounded set in $S(E)$ or $T(F)$ is again strongly bounded.*

Proof. By the proposition, the topology $\beta(S(E), T(F))$ has a base of solid sets from which the first part follows. The proof of the second part is similar. ∎

In several theorems that follow, the statement "let $S(E)$ be a v.f.s. with a topology finer than that induced from $\Omega(E)$" is part of the hypotheses. The following proposition gives a sufficient condition that this hypothesis be satisfied.

PROPOSITION 1.6. *For a compact set K and $p \in P$, set*

$$Q(K, p) = \{g \in \Gamma(F) : p^0 \circ g \leq c(K)\}.$$

Let $(S(E), T(F))$ be a dual pair of v.f.s.'s and let $S(E)$ be given a polar topology stronger than the topology of uniform convergence on the sets $Q(K, p)$. Then this topology is stronger than that induced from $\Omega(E)$.

Proof. In the proof of [9 Proposition 6.2] it was shown that

$$Q(K, p)^0 = \left\{f \in S(E) : \int_K p \circ f \, d\pi \leq 1\right\}$$

from which the result follows. ∎

2. Completeness

The following proposition, due to Garling [5, p. 998] is exactly what we need to establish the completeness of many v.f.s.'s.

PROPOSITION 2.1. *Let G be a complete Hausdorff topological vector space with topology defined by a set of seminorms P. Let Q be a set of lower semicontinuous extended valued seminorms on G. Set*

$$H = \{g \in G : q(g) < \infty \text{ for all } q \in Q\}.$$

Then H is complete under the topology given by $P \cup Q$.

THEOREM 2.2. *Let $(S(E), T(F))$ be a dual pair of solid v.f.s.'s with $S(E) = T(F)^*$. Let $S(E)$ have the topology of uniform convergence on a set $\mathcal{B}$ of solid sets in $T(F)$ whose union is $T(F)$. Suppose the topology on $S(E)$ is finer than that induced from $\Omega(E)$. Then $S(E)$ is complete.*

Proof. By Proposition 1.2, a set of seminorms defining the topology on $S(E)$ is given by $\text{Sup}_{b \in B} \int |\langle f, g\rangle| \, d\pi$ with $B \in \mathcal{B}$. This defines a set of (possibly extended valued) seminorms on $\Omega(E)$. We claim that these are lower semicontinuous. Since the supremum of a family of lower semicontinuous functions is lower semicontinuous, it is sufficient to show that $f \to \int |\langle f, g\rangle| \, d\pi$ is lower semicontinuous. We know Z can be expressed as a countable union of compact sets so by the monotone convergence theorem it is sufficient to show that

$$f \to \int_K |\langle f, g\rangle| \, d\pi$$

is lower semicontinuous for each K. Similarly, if we set $g|_K = bg_0$ where $b \in \Omega$ and $p^0 \circ g_0 \leq 1$. then it is sufficient to show that $f \to \int_{K'} |\langle f, g\rangle| \, d\pi$ is lower semicontinuous for any K' such that $b|_{K'}$ is bounded. But this is clear and the claim is established. Since $S(E) = T(F)^*$ and since the sets in $\mathcal{B}$ have union $T(F)$ we have

$$S(E) = \left\{f \in \Omega(E) : \text{Sup}_{g \in B} \int |\langle f, g\rangle| \, d\pi < \infty \quad \text{for all} \quad B \in \mathcal{B}\right\}.$$

Thus Proposition 2.1 applies and $S(E)$ is complete. ∎

COROLLARY 2.3. *Let $(S(E), T(F))$ be a dual pair of solid v.f.s.'s with $S(E) = T(F)^*$. Then $S(E)$ is complete under the strong topology.*

Proof. By Proposition 1.4, the strong topology is the topology of uniform convergence on all *solid* weakly bounded sets in $T(F)$. By Proposition 1.6, the topology on $S(E)$ is finer than the topology induced from $\Omega(E)$ since the sets $Q(K, p)$ of Proposition 1.6 are easily seen to be weakly bounded. Thus the theorem applies and $S(E)$ is complete. ∎

PROPOSITION 2.4. *Let $(S(E), T(E'))$ be a dual pair of solid v.f.s.'s with $S(E) = T(E')^*$ and $T(E') \supseteq \bar{\Phi}(E')$. Then $S(E)$ is complete under the Mackey topology.*

Proof. Give $S(E)$ the topology ξ of uniform convergence on the set $\mathcal{B}$ of solid hulls of points in $T(E')$ and the absolutely convex hulls of the solid hulls, $Q'(K, p)$, of the sets $Q(K, p)$ of Proposition 1.6. The sets in $\mathcal{B}$ are solid and absolutely convex. By Proposition 1.6 the topology ξ is stronger than the topology induced from $\Omega(E)$. Thus Theorem 2.2 applies and $S(E)$ is complete under ξ. By Lemma 1.3 the solid hulls of points in $T(E')$ are weakly compact. The set $Q'(K, p)$ is weakly relatively compact in $\bar{\Phi}(E')$, being contained in the polar of a neighborhood in $\Omega\ (E)$. Thus $Q'(K, p)$ is also weakly relatively compact in $T(E')$. By the Mackey-Arens theorem, ξ is coarser than the Mackey topology and so $S(E)$ is also complete under the Mackey topology. ∎

PROPOSITION 2.5. *Let E be a separable Banach space.*

(1) *If Z is compact, then $\Omega(E)$ is a Banach space and the norm topology on $\bar{\Phi}(E')$, the dual of $\Omega(E)$, is $\||\| g \||\|_\infty$.*

(2) *If π does not have compact support, there is a sequence $K_1 \subseteq K_2 \cdots$ of compact sets such that $\bar{\Phi}(E')$ is, under the strong topology, the strict inductive limit of the Banach spaces $\bar{\Phi}_{K_n}(E')$.*

Proof. (1) $\Omega(E)$ is a Banach space with norm $\int \| f \|\, d\pi$. For any $g \in \bar{\Phi}(E')$ the function g is $\sigma(E', E)$ measurable and the function $\| g \|$ is measurable [9, Proposition 3.1]. Thus by [9, Lemma 5.2], for any $a \in L^1$,

$$\int |a| \| g \|\, d\pi = \operatorname{Sup}\left\{ \left| \int \langle f, g\rangle\, d\pi \right| : f \in \Omega(E),\ \| f \| \leq |a| \right\}.$$

Taking the supremum over a in the unit ball of L^1 we have

$$\||\| g \||\|_\infty = \operatorname{Sup}\left\{ \left| \int \langle f, g\rangle\, d\pi \right| : f \in \Omega(E),\ \int \| f \|\, d\pi \leq 1 \right\}.$$

(2) Let $K_1 \subseteq K_2 \subseteq \cdots$ be a sequence of compact sets such that

$$K_n \subseteq K_{n+1}^0 \quad \text{and} \quad \bigcup_{n=1}^\infty K_n = Z.$$

We may assume that $\pi(K_{n+1} - K_n) > 0$ and so $\bar{\Phi}_{K_n}(E')$ is a proper subspace of $\bar{\Phi}_{K_{n+1}}(E')$. Now $\bar{\Phi}(E') = \bigcup_{n=1}^{\infty} \bar{\Phi}_{K_n}(E')$ and so the inductive limit topology on $\bar{\Phi}(E')$ by the spaces $\bar{\Phi}_{K_n}(E')$ exists and is a strict inductive limit; it remains to show that this topology is the strong topology. Set $R_n = K_n - K_{n-1}$. Let $\Omega_n(E)$ (respectively $\bar{\Phi}_n(E')$) be the set of all restrictions of functions in $\Omega(E)$ (respectively $\bar{\Phi}(E')$) to R_n and give $\Omega_n(E)$ norm $\int_{R_n} \| f \| \, d\pi$. Since a function $f : Z \to E$ is measurable iff $f|_{R_n}$ is measurable for all n [1, p. 175], we see that $\Omega(E) = \prod_{n=1}^{\infty} \Omega_n(E)$ algebraically and topologically. Now by arguments similar to those of [9, Theorem 4.1] and (1) of the present theorem (or by [2, p. 47]), $\Omega_n(E)' = \bar{\Phi}_n(E')$ and the norm on $\bar{\Phi}_n(E')$ is $\||\| g \||\|_\infty$. Thus $\Omega(E)' = \bar{\Phi}(E')$ is the direct sum of the $\bar{\Phi}_n(E')$ [11, p. 93] and the direct sum topology on $\Phi(E')$ is the strong topology [11, p. 100]. Thus we must show that the direct sum topology on $\bar{\Phi}(E')$ coincides with the strict inductive limit topology. Let (a_n) be an arbitrary decreasing sequence of positive numbers. Set

$$N_1(a_n) = \bigcup_{n=1}^{\infty} \{g \in \bar{\Phi}(E') : \operatorname{Supp} g \subseteq K_n ,\ \||\| g \||\|_\infty \le a_n\}$$

and

$$N_2(a_n) = \bigcup_{n=1}^{\infty} \{g \in \bar{\Phi}(E') : g|_{R_n} \equiv g,\ \||\| g \||\|_\infty \le a_n\}.$$

Let $\langle N_1\rangle(a_n)$ and $\langle N_2\rangle(a_n)$ be the absolutely convex envelopes of $N_1(a_n)$ and $N_2(a_n)$. Then $\langle N_1\rangle(a_n)$ and $\langle N_2\rangle(a_n)$ are arbitrary elements of a base of neighborhoods for the strict inductive limit and direct sum topologies on $\bar{\Phi}(E')$ respectively. Clearly

$$\langle N_2\rangle(a_n) \subseteq \langle N_1\rangle(a_n).$$

Now we claim that $N_1(n^{-1}a_n) \subseteq \langle N_2\rangle(a_n)$, completing the proof. Let $g \in N_1(n^{-1}a_n)$. Then for some m, $\operatorname{Supp} g \subseteq K_m$ and $\||\| g \||\|_\infty \le m^{-1}a_m$. Then

$$g = \sum_{j=1}^{m} m^{-1}(mg \mid_{R_j}).$$

But

$$\||\| mg|_{R_j} \||\|_\infty \le m(m^{-1}a_m) \le a_j$$

since (a_n) is decreasing and so $g \in \langle N_2\rangle(a_n)$. ▮

LEMMA 2.6. *If E is a separable reflexive Banach space, $\Omega(E)$ is weakly sequentially complete.*

Proof. Let (f_n) be a weakly Cauchy sequence. Then for $g \in \bar{\Phi}(E')$ and every $b \in L^\infty$ we have $bg \in \bar{\Phi}(E')$ and so $\int \langle f_n , g\rangle b \, d\pi$ is a Cauchy sequence of scalars. Thus $\langle f_n , g\rangle$ is weakly Cauchy in L^1 and so has a weak limit [4, *p.* 92]. Define $\Psi_n : \bar{\Phi}(E') \to L^1$ by $\Psi_n(g) = \langle f_n , g\rangle$ and $\Psi : \bar{\Phi}(E') \to L^1$ by $\Psi(g) = = \lim \langle f_n , g\rangle$. We claim that the Ψ_n are continuous when $\bar{\Phi}(E')$ has the strong dual topology and L^1 has the weak topology. Assume that π does not have compact support; if it has compact support the proof is simpler. Let (K_j) be a sequence of compact sets such that by Proposition 2.5, $\bar{\Phi}(E')$ is the strict inductive limit of the $\bar{\Phi}_{K_j}(E')$. Since each $\bar{\Phi}_{K_j}(E')$ is a Banach space, $\bar{\Phi}(E')$ is bornological and so sequential continuity of Ψ_n will ensure continuity. Let $g_k \to 0$ in $\bar{\Phi}(E')$. Then $(g_k) \subseteq \bar{\Phi}_{K_r}(E')$ for some r [11, p. 129] and $g_k \to 0$ in $\bar{\Phi}_{K_r}(E')$. Thus for any $b \in L^\infty$,

$$\begin{aligned}\lim_{k\to\infty}\left|\int \Psi_n(g_k)b\ d\pi\right| &= \lim_{k\to\infty}\left|\int \langle f_n, g_k\rangle b\ d\pi\right| \\ &\leq \|b\|_\infty \int_{K_r} \|f_n\|\ d\pi \lim_{k\to\infty} |||g_k|||_\infty \\ &= 0\end{aligned}$$

establishing the continuity of Ψ_n. Now $\bar{\Phi}(E')$, being an inductive limit of barrelled spaces, is barrelled. Thus by the Banach-Steinhaus Theorem, Ψ is continuous.

We now apply [9, Theorem 7.1] (with E and E' switched—see Proposition 1.1) to $\phi(g) = \int \Psi(g)\, d\pi$. For condition (1) of that theorem let $R_j \uparrow R$. Then

$$\begin{aligned}\phi(g|_{R_j}) &= \int \Psi(g|_{R_j})\ d\pi \\ &= \int \lim_{n\to\infty} \langle f_n, g|_{R_j}\rangle\ d\pi \\ &= \lim_{n\to\infty} \int \langle f_n, g|_{R_j}\rangle\ d\pi \\ &= \lim_{n\to\infty} \int_{R_j} \langle f_n, g\rangle\ d\pi \\ &= \int_{R_j} \lim_{n\to\infty} \langle f_n, g\rangle\ d\pi \\ &= \int_{R_j} \Psi(g)\ d\pi.\end{aligned}$$

Similarly, $\phi(g|_R) = \int_R \Psi(g)\, d\pi$. By the dominated convergence theorem, $\phi(g|_{R_j}) \to \phi(g|_R)$.

Now let a compact set K be given. Then for some m, $K \subseteq K_m$. Since Ψ is continuous, it is bounded on the unit ball of $\bar{\Phi}_{K_m}(E')$ from which condition (2) follows. By [9, Theorem 7.1], there is an $f \in \Omega(E)$ such that $\phi(g) = \int \langle f, g\rangle\, d\pi$. Thus

$$\int \langle f, g\rangle\ d\pi = \int \lim_{n\to\infty} \langle f_n, g\rangle\ d\pi = \lim_{n\to\infty} \int \langle f_n, g\rangle\ d\pi.$$

Thus $f_n \to f$ weakly in $\Omega(E)$ and $\Omega(E)$ is weakly sequentially complete. ∎

THEOREM 2.7. *Let E be a reflexive separable Banach space. Let*

$(S(E),\ T(E'))$ be a dual pair of solid v.f.s.'s with $S(E) = T(E')^*$ and $T(E') \supseteq \bar{\Phi}(E')$. Then $S(E)$ is weakly sequentially complete.

Proof. Let (f_n) be a weakly Cauchy sequence. Then (f_n) is weakly Cauchy in $\Omega(E)$ since $T(E') \supseteq \bar{\Phi}(E') = \Omega(E)'$. Thus by Lemma 2.6, $f_n \to f$ weakly in $\Omega(E)$. Now let $g \in T(E')$ be arbitrary. We shall show that

$$\int \langle f_n, g\rangle \, d\pi \to \int \langle f, g\rangle \, d\pi$$

thus completing the proof since it will then follow that $f \in S(E) = T(E')^*$. As in the proof of Lemma 2.6, $\langle f_n, g\rangle \to G$, say, weakly in L^1. Let $K_1 \subseteq K_2 \subseteq \cdots$ be a sequence of compact sets such that $\bigcup K_j = Z$ and set

$$R_j = \{z : \| g(z) \| \leq j\} \cap K_j .$$

Then $\bigcup R_j = Z$. Now $c(R_j)g \in \bar{\Phi}(E')$ and so we know from above that

$$\langle f_n, c(R_j)g\rangle \to \langle f, c(R_j)g\rangle$$

weakly in L^1. Thus $\langle f, g\rangle |_{R_j} = G|_{R_j}$. Therefore $\langle f, g\rangle = G$ and so

$$\int \langle f_n, g\rangle \, d\pi \to \int G \, d\pi = \int \langle f, g\rangle \, d\pi. \qquad \blacksquare$$

3. Duals of v.f.s.'s

In this section we study the relationship between the topological and Köthe duals of v.f.s.'s.

PROPOSITION 3.1. *Suppose $S(E)$ is a solid locally convex v.f.s. which is barrelled. Suppose $S(E)' \supseteq \bar{\Phi}(E')$. Then $S(E)' \supseteq S(E)^*$.*

Proof. Let $g \in S(E)^*$ be given. Given a compact set K, choose a sequence of relatively compact sets $R_1 \subseteq R_2 \subseteq \cdots$ such that $\pi(K - R_j) \leq 1/j$ and $g|_{R_j} \in \bar{\Phi}(E')$. This can be done by the definitions of $\bar{\Omega}(E')$ and $\bar{\Phi}(E')$. By hypothesis, $g|_{R_j} \in S(E)'$. Also

$$\int \langle f, g|_{R_j}\rangle \, d\pi \to \int \langle f, g|_K\rangle \, d\pi$$

by the dominated convergence theorem. Thus by the Banach-Steinhaus theorem, $g|_K \in S(E)'$. Similarly, by expressing Z as a countable union of compact sets we may prove $g \in S(E)'$, thus completing the proof. $\blacksquare$

PROPOSITION 3.2. *Suppose $C \subseteq L^1$ is weakly compact and suppose $R_j \uparrow R$. Then*

$$\int_{R_j} |a| \, d\pi \to \int_R |a| \, d\pi$$

uniformly for $a \in C$.

Proof. Let $\varepsilon > 0$ be given. By [4, Théorème 4] choose a compact set K

such that $\int_{Z-K} |a|\, d\pi < \varepsilon/2$ for $a \in C$. By [4, Théorème 4] again, choose j_0 so that if $j \geq j_0$ we have

$$\int_{K\cap R} |a|\, d\pi - \int_{K\cap R_j} |a|\, d\pi < \varepsilon/2$$

for $a \in C$. Thus if $a \in C$ and $j \geq j_0$,

$$\int_{R} |a|\, d\pi - \int_{R_j} |a|\, d\pi < \varepsilon$$

and the result follows.

PROPOSITION 3.3. *Let $(S(E), T(F))$ be a dual pair of solid v.f.s.'s.*

(1) *If ρ is the guage of the polar of a solid weakly bounded set B in $T(F)$ and $R_j \uparrow R$ then $\rho(f|_{R_j}) \to \rho(f|_R)$*

(2) *If $S(E)$ is given the topology of uniform convergence on the solid hulls of the $\sigma(T(F), S(E))$ compact sets and $R_j \uparrow R$ then $f|_{R_j} \to f|_R$.*

Proof. (1) By Proposition 1.2 and the remark following that proposition,

$$\rho(f) = \operatorname{Sup}\left\{ \int \left| \langle f, g\rangle \right| d\pi : g \in B\right\}.$$

Thus by the monotone convergence theorem

$$\rho(f|_{R_j}) = \operatorname{Sup}\left\{ \int_{R_j} |\langle f, g\rangle|\, d\pi\right\} \to \operatorname{Sup}\left\{ \int_{R} |\langle f, g\rangle|\, d\pi = \rho(f|_R).\right.$$

(2) Fix $f \in S(E)$. The map $V : T(F) \to L^1$ given by $Vg = \langle f, g\rangle$ has an adjoint $V^* : L^\infty \to S(E)$ given by $V^*b = bf$. Thus V is weakly continuous. Let C be a $\sigma(T(F), S(E))$ compact set. By the continuity of V, $\langle f, C\rangle$ is a $\sigma(L^1, L^\infty)$ compact set. Thus, if C' is the solid hull of C, then $\langle f, C'\rangle$, which is the solid hull of $\langle f, C\rangle$ is, by [4, Théorème 4], $\sigma(L^1, L^\infty)$ compact. Applying Proposition 3.2 we obtain the result. ▮

Example. A seminorm on $S(E)$ which is not the gauge of a solid set may fail to have the property of Proposition 3.3(1) even though the seminorm is induced from $T(F)$. Let Z be the natural numbers with π the counting measure. Let E and E' be the real field. Let $S(E) = l^\infty$ and $T(E') = l^1$. For every n let $g_n \in l^1$ be defined by

$$\begin{aligned} g_n(i) &= 1 \quad && \text{if } i = n \\ &= -1 && \text{if } i = n+1 \\ &= 0 && \text{otherwise.} \end{aligned}$$

Let $B = \{g_n\}$. Then B is $\sigma(l^1, l^\infty)$ bounded. Let ρ be the gauge of B^0. Let $f \in l^\infty$ be defined by $f(i) \equiv 1$. Then $\rho(f) = 0$ but $\rho(f|_{[1,i]}) = 1$ for any i.

PROPOSITION 3.4. *Let E be a normed space and $S(E)$ a solid locally convex* *v.f.s. For a compact set K define*

$$A(K) = \{f \in S(E) : \|f\| \leq c(K)\}.$$

(1) *Assume $S(E) \supseteq \Phi(E)$ and the topology on $S(E)$ is solid. Let $T(E') \supseteq \bar{\Phi}(E')$ be a v.f.s. Then the topology on $S(E)$ is a polar topology induced from $T(E')$ $\Leftrightarrow$ if ρ is the gauge of a solid neighborhood and $R_j \uparrow R$ then $\rho(f|_{R_j}) \to \rho(f|_R)$, $A(K)$ is bounded for every K, and $S(E)' \supseteq T(E')$.*

(2) *We have $S(E)' \subseteq S(E)^*$ $\Leftrightarrow$ $R_j \uparrow R$ implies $f|_{R_j} \to f|_R$ and $A(K)$ is bounded for every K.*

Proof. (1) ($\Rightarrow$) We first show that $A(K)$ is bounded in the topology $\beta(S(E), T(E'))$ and thus in any weaker topology. It follows from Proposition 3.1 that $\Omega^* = \Phi$. It follows from [9, Lemma 5.3] that $\Phi(E) = \bar{\Omega}(E')^*$. (We shall prove a more general result in [10].) Give $\Phi(E)$ the Mackey topology $\tau(\Phi(E), \bar{\Omega}(E'))$. By Proposition 2.4, $\Phi(E)$ is complete and so in its dual, $\bar{\Omega}(E')$, the weakly and strongly bounded sets coincide. Now let B be any $\sigma(T(E'), S(E))$ bound set. Then B is $\sigma(\bar{\Omega}(E'), \Phi(E))$ bounded and so by the above $\beta(\bar{\Omega}(E'), \Phi(E))$ bounded. Clearly $A(K) \subseteq \Phi(E)$ and is $\sigma(\Phi(E), \bar{\Omega}(E'))$ bounded. Thus

$$\operatorname{Sup}\left\{\left|\int \langle f, g\rangle \, d\pi\right| : f \in A(K), g \in B\right\} < \infty$$

and so $A(K)$ is $\beta(S(E), T(E'))$ bounded.

The condition on ρ follows from Proposition 3.3(1). The containment $S(E)' \supseteq T(E')$ is clear.

($\Leftarrow$) Let B be a solid closed absolutely convex neighborhood and ρ its gauge. We shall show that B is closed in $\Omega(E)$. It will then follow that B is $\sigma(\Omega(E), \bar{\Phi}(E'))$ and so $\sigma(S(E), T(E'))$ closed. Thus $B^{00} = B$ and the result will follow.

To this end, let $(f_n) \subseteq B$ satisfy $f_n \to f \in \Omega(E)$. Then using [1, p. 131, Théoremè 3, 2°] and a diagonal process there is a subsequence, again denoted (f_n) such that $f_n \to f$ a.e. Let a compact set K and positive integer k be given. Then by Egoroff's theorem [1, p. 175] there is a compact set $K_k \subseteq K$ with $\pi(K - K_k) \leq 1/k$ and $f_n \to f$ uniformly on K_k. We may assume $K_k \subseteq K_{k+1}$. Now

$$|\rho(f_n|_{K_k}) - \rho(f|_{K_k})| \leq \rho((f - f_n)|_{K_k}) \to 0$$

as $n \to \infty$ since $A(K)$ is bounded. Since $\rho(f_n|_{K_k}) \leq 1$ it follows that $\rho(f|_{K_k}) \leq 1$. But by hypothesis $\rho(f|_{K_k}) \to \rho(f|_K)$ and so $\rho(f|_K) \leq 1$. By expressing $Z = \bigcup K_i$ where the K_i are compact and $K_i \uparrow Z$ we find that $\rho(f) \leq 1$, i.e., $f \in B$. Thus B is closed in $\Omega(E)$.

(2) ($\Rightarrow$) Proposition 3.3(2) and the Mackey-Arens theorem show that $R_j \uparrow R$ implies $f|_{R_j} \to f|_R$. The set $A(K)$ is clearly weakly bounded and so bounded.

($\Leftarrow$) This follows immediately from [9, Theorem 7.1]. ∎

By combining the hypotheses of Propositions 3.1 and 3.4(2) one can obtain conditions under which $S(E)' = S(E)^*$. We turn however to the nicer situation in which the topology on $S(E)$ is a polar topology induced from $S(E)^*$.

THEOREM 3.5. *Let E be a Banach space. Let $(S(E), T(E'))$ be a dual pair of solid v.f.s.'s with $S(E)^* = T(E')$ and $S(E) \supseteq \Phi(E)$. Let $S(E)$ be provided with a polar topology. Then the following are equivalent:*

(1) $S(E)' = S(E)^*$,
(2) *If* $R_i \uparrow R$ *then* $f|_{R_i} \to f|_R$.

If, in addition, Z is second countable and E is separable, the above are equivalent to:
(3) $S(E)$ is separable.

Proof. (1) $\Rightarrow$ (2). This follows immediately from Proposition 3.3(2) and the Mackey-Arens theorem.

(2) $\Rightarrow$ (1). Clearly $S(E)^* \subseteq S(E)'$. In the proof of Proposition 3.4(1) it was shown that $A(K)$ is bounded and so the result follows from Proposition 3.4(2).

Now we suppose that Z is second countable and E is separable.

(1) $\Rightarrow$ (3). It is sufficient to show that $S(E)$ is $\sigma(S(E), T(E'))$ separable. Since $\Gamma(E)$ separates points of $T(E')$ it is weakly dense in $S(E)$. Let X be a countable dense set in E and let $\mathcal{K}$ be the countable collection of compact sets constructed in [9, Corollary 6.3]. Set

$$A = \{f \in \Gamma(E) : f = \sum c(K_j)x_j \text{ with } K_j \in \mathcal{K} \text{ and } x_j \in X\}$$

and

$$A' = \{f \in \Gamma(E) : f = \sum c(R_j)x_j \text{ with } x_j \in X\}.$$

We shall show that A is weakly dense in A' and A' is weakly dense in $\Gamma(E)$, finishing this part of the proof since A is a countable set.

Let $f = \sum c(R_j)x_j \in A'$ be given and choose, for each j and n a $K_{nj} \in \mathcal{K}$ with $\pi(R_j - K_{nj}) + \pi(K_{nj} - R_j) < n^{-1}$. This can be done because of the way $\mathcal{K}$ was constructed and because of the regularity of the measure. Set $f_n = \sum_j c(K_{nj})x_j$. Then $f_n \in A$ and a simple calculation shows $f_n \to f$ weakly in $S(E)$.

Now let $f = \sum c(R_j)x_j \in \Gamma(E)$ be given and for each j, pick a net $(x_{j,\alpha})$ such that $x_{j,\alpha} \in X$ and $x_{j,\alpha} \to x_j$. We may index each of the nets $(x_{j,\alpha})$ with the same index set, a base of neighborhoods in E. Set $f_\alpha = \sum c(R_j)x_{j,\alpha} \in A'$. The map $V : S(E)$ given by $Vx = c(R)x$ has, by [9, Proposition 6.1(2)], an adjoint $V^* : T(E') \to E'$ given by $V^*g = \int_R g\, d\pi$. Thus V is weakly continuous and it follows that $f_\alpha \to f$ weakly in $S(E)$.

(3) $\Rightarrow$ (2) Let $S(E)$ have the topology of uniform convergence on a set of absolutely convex sets $\mathcal{B}$. We first show that if $B \in \mathcal{B}$ and if (g_j) is any sequence in B, then there is a subsequence (g_{j_k}) such that for any $f \in S(E)$, $\lim \langle f, g_{j_k}\rangle$ exists in the topology $\sigma(L^1, L^\infty)$. Let $\{f_n\}$ be dense in $S(E)$. Since

for a fixed n the sequence $(\int\langle f_n, g_j\rangle\, d\pi)$ is bounded, we may, by a diagonal procedure, pick a subsequence, which we again denote (g_j), such that $\lim \int\langle f_n, g_j\rangle\, d\pi$ exists for each n. Let $f \in S(E)$ and $\varepsilon > 0$ be given. Since

$$\{g_p - g_q : p, q = 1, 2, \cdots\} \subseteq 2B,$$

we can find a $f_{n'}$ in $\{f_n\}$ so that $|\int\langle f - f_{n'}, g_p - g_q\rangle\, d\pi| < \varepsilon$ for all p and q. Since $\lim_{j\to\infty} \int\langle f_{n'}, g_j\rangle\, d\pi$ exists, we may find an index N such that if $p, q \geq N$, then $|\int\langle f_{n'}, g_p - g_q\rangle\, d\pi| < \varepsilon$. Combining the two inequalities gives

$$\left|\int \langle f, g_p - g_q\rangle\, d\pi\right| < 2\varepsilon \quad \text{for} \quad p, q \geq N,$$

i.e., $\lim_{j\to\infty} \int\langle f, g_j\rangle\, d\pi$ exists. Since $S(E)$ is solid this implies that for any $a \in L^\infty$. $\lim_{j\to\infty} \int a\langle f, g_j\rangle\, d\pi$ exists, i.e., $\langle f, g_j\rangle$ is weakly Cauchy in L^1. Since L^1 is weakly sequentially complete [4, p. 92], $\lim_{j\to\infty} \langle f, g_j\rangle$ exists in the topology $\sigma(L^1, L^\infty)$.

With this established we prove (2). Suppose that $R_n \uparrow R$ but there is an $f \in S(E)$ such that $f|_{R_n} \to f|_R$ is false. Set $S_n = R - R_n$. Then there is a $B \in \mathfrak{B}$, an $\varepsilon > 0$, and a sequence $(g_n) \subseteq B$ such that for all n, $|\int_{S_n}\langle f, g_n\rangle\, d\pi| \geq \varepsilon$. By the above we may assume by taking a subsequence, that $\lim\langle f, g_n\rangle$ exists and so the set $\{\langle f, g_n\rangle\}$ is weakly relatively compact in L^1. But then by Proposition 3.2, $\lim \int_{S_n} |\langle f, g_n\rangle|\, d\pi = 0$, which is a contradiction. ▌

We are now able to show that a conjecture of Các [3, p. 609] is true.

Corollary 3.6. *Let E be a Banach space. Let $(S(E), T(E'))$ be a dual pair of solid v.f.s.'s with $T(E') = S(E)^*$ and $S(E) \supseteq \bar{\Phi}(E)$. Then the absolutely convex hull of the solid hull of a $\sigma(T(E'), S(E))$ compact set is again weakly relatively compact.*

Proof. Let $S(E)$ be given the topology ξ of uniform convergence on the solid hulls of the weakly compact sets in $T(E')$. By Proposition 3.3 and Theorem 3.5, $(S(E), \xi)' = T(E')$. Now let B be the solid hull of any weakly compact set of $T(E')$. Then B^0 is a ξ-neighborhood and so B^{00}, which contains the absolutely convex hull of B, is weakly compact. ▌

Proposition 3.7. *Let E be a Banach space. Let $(S(E), T(E'))$ be a dual pair of solid v.f.s.'s with $T(E') = S(E)^*$ and $S(E) \supseteq \Phi(E)$. Let $S(E)$ be given a topology of the dual pair and $T(E')$ the topology of uniform convergence on the precompact sets in $S(E)$. Then $T(E')$ is quasicomplete.*

Proof. Let (g_α) be a Cauchy net in $T(E')$ contained in a bounded set B. Let G be the strong dual of $S(E)$ and let G be given the topology of uniform convergence on the precompact sets in $S(E)$. Let $\bar{B}$ be the closure of B in G. Then $\bar{B}$ is compact in the topology $\sigma(G, S(E))$, being contained in the polar of a strong neighborhood. Thus $\bar{B}$ is compact in the topology of precompact convergence [11, p. 106]. Thus $g_\alpha \to \phi \in G$. We shall show, using [9, Theorem 7.1], that $\phi \in T(E')$, completing the proof.

For condition (1) of that theorem let $f \in S(E)$ and $\varepsilon > 0$ be given and suppose that $R_j \uparrow R$. Set $S_j = R - R_j$. Then by Theorem 3.5, we have $f|_{S_j} \rightarrow 0$ and so $\{f|_{S_j}\}$ is precompact. Thus there is an α_0 such that we have

$$|\langle f|_{S_j}, \phi - g_{\alpha_0}\rangle| \leq \varepsilon/2 \quad \text{for all } j.$$

Now choose j_0 so that $|\langle f|_{S_j}, g_{\alpha_0}\rangle \leq \varepsilon/2$ for all $j \geq j_0$. Then if $j \geq j_0$,

$$|\phi(f|_{S_j})| \leq |\langle f|_{S_j}, \phi - g_{\alpha_0}\rangle| + |\langle f|_{S_j}, g_{\alpha_0}\rangle| \leq \varepsilon.$$

Thus $\phi(f|_{S_j}) \rightarrow 0$, i.e., $\phi(f|_{R_j}) \rightarrow \phi(f|_R)$. For condition (2) note that by the proof of Proposition 3.4 (1) we have $A(K)$ bounded in the strong topology and so $\phi(A(K))$ is bounded. ∎

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