ON REGULAR FUNCTIONS ON RIEMANN SURFACES[1]

BY

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1. Introduction

If an annulus $a < |z| < b$ is mapped analytically into another annulus $a' < |z| < b'$ in such a way that the index of the images of concentric circles is equal to n ($n \neq 0$), then its module $(2\pi)^{-1} \log b/a$ is dominated by $1/|n|$-th of the module $(2\pi)^{-1} \log b'/a'$ of the image annulus. This was proved by Schiffer [8]. A number of proofs and extensions of this result were obtained by many authors [3], [4], [6]. In particular the first author of the present paper weakened the assumption about the image domain and gave an extension by the method of the extremal metric [4]. Another interesting method of proof was given by Landau and Osserman, comparing the fluxes of harmonic functions [6].

The purpose of the present paper is to extend the above result for regular functions on arbitrary Riemann surfaces which have more than one boundary component. Let us mention the result for functions $w = f(z)$ regular on the closed annulus $a \leqq |z| \leqq b$ and such that $f'(z) \neq 0$ on the boundary. The image curves $f(ae^{2\pi it})$ and $f(be^{2\pi it})$ $(0 \leqq t \leqq 1)$ divide the w-plane into several open sets on which the indices of the image curves are constant.

Let P_m and Q_n be the respective open sets on which the index of $f(ae^{2\pi it})$ is not less than m and that of $f(be^{2\pi it})$ is not greater than n. If $P_m \neq \emptyset$ and $Q_n \neq \emptyset$ for $m > n$, then the complement of the union of the closure of P_m and Q_n consists of a finite number of domains and the module of the family of curves separating $\bar{P}_m$ and $\bar{Q}_n$ dominates $(m - n)(2\pi)^{-1} \log b/a$.

Our result will be stated for regular functions on an open Riemann surface and a regular partition of its boundary. The sets corresponding to $\bar{P}_m$ and $\bar{Q}_n$ are defined in terms of exhaustions. We will give two proofs one of which is based on the method of the extremal metric and the other is based on the comparison of the fluxes of harmonic functions. It is interesting that two different methods produce the same result.

The result has many applications. For example Hayman-Kubo's estimation of the capacity of the set of omitted values [2, 5] is generalized to Riemann surfaces. As to other applications the readers are referred to the first author's paper [4].

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2. Index

Let Ω be an open Riemann surface with more than one boundary component. Let (A, B) be a regular partition of the boundary of Ω with non-void A, B [1]. Let γ be a dividing cycle of Ω with the following properties:

(i) γ consists of a finite number of analytic Jordan curves;

(ii) γ divides Ω into two sets of non-compact subdomains of Ω, each of which bears subsets of either A or B and

(iii) γ is positively oriented with respect to the domains bearing A.

For regular functions $w = f(z)$ we define the *index* of a point $w \notin f(\gamma)$ with respect to $f(\gamma)$ by

$$I(\gamma; w) = \frac{1}{2\pi i}\int_\gamma \frac{df}{f(z) - w}.$$

Let γ' be a similar cycle contained in the subdomains divided by γ and bearing B. Then γ and γ' bound a finite number of regular domains (regular region [1]). We have for $w \notin f(\gamma) \cup f(\gamma')$

$$I(\gamma; w) \leqq I(\gamma'; w). \tag{1}$$

This follows from the argument principle for regular functions f in the respective domains.

We now define two kinds of sets:

$$P_m(\gamma) = \{w \mid I(\gamma; w) \geqq m, w \notin f(\gamma)\}$$

and

$$Q_n(\gamma) = \{w \mid I(\gamma; w) \leqq n, w \notin f(\gamma)\},$$

where m, n are integers. Under the same situation as in the above paragraph we have from (1)

$$\bar{P}_m(\gamma) \subset \bar{P}_m(\gamma') \quad \text{and} \quad \bar{Q}_n(\gamma) \supset \bar{Q}_n(\gamma'), \tag{2}$$

where $\bar{P}_m$ etc. stand for the closures in the extended plane. Considering normals with respect to γ, we can deduce a more precise conclusion that the interior of $\bar{P}_m(\gamma')$ or $\bar{Q}_n(\gamma)$ contains $\bar{P}_m(\gamma)$ or $\bar{Q}_n(\gamma')$ respectively.

Let $\{\Omega_\nu\}_{\nu=0}^\infty$ with $\gamma \subset \Omega_0$ be a canonical exhaustion [1] of Ω such that every boundary contour of Ω_ν is an analytic Jordan curve. Let α_ν and β_ν be the cycles on the boundary of Ω_ν which are homologous to γ and are the relative boundaries of the complementary subdomains of Ω_ν bearing A and B respectively. From the precise conclusion above mentioned we can deduce that if $m > n$, $\bar{P}_m(\alpha_\nu) \cap \bar{Q}_n(\beta_\nu) = \emptyset$. Furthermore $\bar{P}_m(\alpha_\nu)$ and $\bar{Q}_n(\beta_\nu)$ are decreasing with ν increasing. If $P_m(\alpha_\nu) \neq \emptyset$ for all ν, we have a non-void compact set $\bar{P}_m(A) = \bigcap_\nu \bar{P}_m(\alpha_\nu)$ and similarly a set $\bar{Q}_n(B) = \bigcap_\nu \bar{Q}_n(\beta_\nu)$ if $Q_n(\beta_\nu) \neq \emptyset$ for all ν. Clearly $\bar{P}_m(A)$ and $\bar{Q}_n(B)$ are disjoint.

3. Modules

Let Γ be a family of locally rectifiable curves c. We denote by $P(\Gamma)$ the class of non-negative metrics $\rho \mid dz \mid$ with Borel measurable ρ on Ω satisfying

$$\int_c \rho \mid dz \mid \geqq 1, \quad c \in \Gamma. \tag{3}$$

The module $M(\Gamma)$ of Γ is given by

$$M(\Gamma) = \inf_{\rho \in P(\Gamma)} \iint_\Omega \rho^2 \, dx \, dy.$$

Suppose that Ω, (A, B) and γ are the same as in Section 2. Then the module of the family $\Gamma(A, B)$ of collections of a finite number of rectifiable closed curves separating A from B is equal to the reciprocal of the flux

$$h(\gamma) = \int_\gamma d\omega_B^*, \qquad d\omega_B^* = -(\partial/\partial y)\omega_B \, dx + (\partial/\partial x)\omega_B \, dy$$

of the harmonic measure ω_B of B [7]. We note that $h(\gamma)$ is equal to the square of the Dirichlet norm $\| d\omega_B \|_\Omega$.

If Ω is a bordered Riemann surface of finite genus, the metric

$$\rho_0 \mid dz \mid = (h(\gamma))^{-1} \mid \operatorname{grad} \omega_B \| dz \mid$$

is the unique extremal metric up to a set of Borel measure zero which satisfies (3) and attains the value $M(\Gamma(A, B))$.

4. Statement of the result

We are now in a position to state our result.

THEOREM 1. *Let (A, B) with non-void A, B be a regular partition of the boundary of the Riemann surface Ω. Let $w = f(z)$ be a function regular on Ω. Suppose that $\bar{P}_m(A) \neq \emptyset$ and $\bar{Q}_n(B) \neq \emptyset$ for $m > n$. Then the module of the family Γ_{mn} of curves separating $\bar{P}_m(A)$ from $\bar{Q}_n(B)$ in the w-plane is not less than $(m - n)\, M(\Gamma(A, B))$.*

Our two proofs will be given in Sections 5 and 6.

5. Proof of Theorem 1, method of the extremal metric

We can take a canonical exhaustion $\{\Omega_\nu\}$ as in Section 2 and such that $df \neq 0$ on the boundary of Ω_ν for every ν. We write Ω', α and β for Ω_ν, α_ν, and β_ν, where α_ν and β_ν are defined in Section 2. Then $f(\alpha)$ and $f(\beta)$ consist of regular analytic closed curves with senses endowed by f, i.e. analytic and with non-vanishing derivatives. Hence they have at most a finite number of intersections. In fact if two components of them have infinitely many intersections, both curves become one dimensional coverings of the same curve. For self-intersections the same is true. Therefore there are at most a finite number of components of $\bar{P}_m(\alpha)$ and $\bar{Q}_n(\beta)$ and the complement of $\bar{P}_m(\alpha) \cup \bar{Q}_n(\beta)$ consists of a finite number of domains.

We list those domains $\{\Delta_j\}_{j=1}^k$ each of whose closure has non-void intersection with both $\bar{P}_m(\alpha)$ and $\bar{Q}_n(\beta)$. Since $\bar{P}_m(\alpha) \cap \bar{Q}_n(\beta) = \emptyset$ there is at least one such Δ_j. For the module problem of Γ_{mn} in the theorem, it is enough to consider the subfamily of curves contained in $\bigcup_{j=1}^k \Delta_j$.

Let ω_j be the harmonic measure of $\partial\Delta_j \cap \bar{Q}_n(\beta)$ with respect to Δ_j. Then the extremal metric $\rho_0 \mid dw \mid$ of the above module problem is given by

$$\begin{aligned} \rho_0 \mid dw \mid &= \left(\textstyle\sum_{j=1}^k \| d\omega_j \|_{\Delta_j}^2\right)^{-1} \mid \operatorname{grad} \omega_j \mid \mid dw \mid && \text{in } \Delta_j \\ &= 0 && \text{elsewhere.} \end{aligned}$$

We shall construct a metric $\rho_1 \mid dz \mid \in \mathrm{P}(\Gamma(\alpha, \beta))$ so that

$$\iint_{\Omega'} \rho_1^2 \, dx\, dy \leqq \frac{1}{m-n} \iint_{|w|<\infty} \rho_0^2 \, du\, dv.$$

where $\Gamma(\alpha, \beta)$ is the corresponding curve family in Ω' separating α and β.

Orthogonal trajectories κ of level curves of ω_j in Δ_j are analytic Jordan arcs joining $\partial\Delta_j \cap \bar{P}_m(\alpha)$ and $\partial\Delta_j \cap \bar{Q}_n(\beta)$ except for a finite number of trajectories ending at critical points of ω_j. We will prove that such a simple κ contains at least $m - n$ subarcs whose inverse images are Jordan arcs joining α and β unless it meets a branch point of $f^{-1}(w)$. To see this, let $\tilde{\kappa}$ be an extension of κ which is the union of κ and two infinite half rays towards the outer normals of $\partial\Delta_j$ at the end points of κ. At a corner with positive angle we take the bisector of two tangents as the normal. The trajectories ending at cusp points are excluded. We parametrize $\tilde{\kappa} : \tilde{\kappa}(t)$ $(-\infty < t < \infty)$ so that $t = \omega_j(\tilde{\kappa}(t))$ on κ. Take two points $w_0 = \tilde{\kappa}(-\varepsilon) \in P_m(\alpha)$ and $w_1 = \tilde{\kappa}(1 + \varepsilon) \in Q_n(\beta)$ for sufficiently small $\varepsilon > 0$. We investigate contributions of the image surface to $I(\alpha; w_0)$ and $I(\beta; w_1)$. For sufficiently large t_0, $I(\alpha; \tilde{\kappa}(t_0)) = 0$, since f is regular. When $\tilde{\kappa}(t)$ crosses $f(\alpha)$, $I(\alpha; \tilde{\kappa}(t))$ increases by $\tilde{\kappa} \times f(\alpha)$. Here $\tilde{\kappa} \times f(\alpha)$ denotes the intersection number [1]. We consider the lifts of maximal subarcs of $\tilde{\kappa}$ on the image domain regarded as a covering surface. They are curves joining either two points of $f(\alpha)$ and $f(\beta)$ or two points of $f(\alpha)$ or $f(\beta)$. In the first case the arc between two intersections has inverse image which is an arc joining α and β. Note that $\tilde{\kappa} \times f(\alpha) = \tilde{\kappa} \times f(\beta)$ at the respective end points. Let t_α and t_β be the parameters of those points. Then $t_\alpha < t_\beta$ or $t_\alpha > t_\beta$ according as $\tilde{\kappa} \times f(\alpha) = \mp 1$. We enumerate the cases where contributions occur:

(I)	$t_\alpha < 0$,	$t_\beta > 1$	occurs l_1 times
(II)	$t_\alpha \geqq 0$,	$t_\alpha < t_\beta \leqq 1$	" l_2 "
(III)	$t_\alpha \geqq 0$,	$t_\alpha < t_\beta$, $t_\beta > 1$	" l_3 "
(I′)	$t_\beta < 0$,	$t_\alpha \geqq 0$	" l_1' "
(II′)	$0 \leqq t_\beta < 1$,	$t_\beta < t_\alpha \leqq 1$	" l_2' "
(III′)	$0 \leqq t_\beta \leqq 1$,	$t_\alpha > 1$	" l_3' "
(IV′)	$t_\beta > 1$,	$t_\beta < t_\alpha$	" l_4' "

In the second case a contribution to $I(\alpha; w_0)$ occurs only when the lift joins two points of $f(\alpha)$ and its projection contains w_0. It is always -1 and we denote the amount by $L_\alpha \leqq 0$. Similarly the amount of contributions to $I(\beta; w_1)$ of the lifts joining two points of $f(\beta)$ is given by $L_\beta \geqq 0$. We have

$$I(\alpha; w_0) = l_2 + l_3 - l_1' - l_2' - l_3' - l_4' + L_\alpha \geqq m$$

and

$$I(\beta; w_1) = l_1 + l_3 - l_4' + L_\beta \leqq n.$$

Thus we get

$$l_2 + l_2' \geqq l_2 - l_1 - l_1' - l_2' - l_3' + L_\alpha - L_\beta \geqq m - n$$

which is the desired result.

Following [4] we transplant $\rho_0 \,|\, dw \,|$ along $m - n$ arcs which are inverse images of orthogonal trajectories between two boundary components of Δ_j, $j = 1, 2, \cdots k$. We can take those inverse images so that they fill a finite number of strips S_l between α and β (cf. [4]). Set

$$\begin{aligned} \rho_1 \,|\, dz \,| &= (1/(m - n))\rho_0(f^{-1}(w)) \,|\, df^{-1}(w) \,|, \quad && z = f^{-1}(w) \in S_l \\ &= 0 && \text{elsewhere.} \end{aligned}$$

Since every $c \in \Gamma(\alpha, \beta)$ meets all the inverse images, $\rho_1 \,|\, dz \,|$ satisfies (3) and we have

$$\frac{1}{m-n} \iint_{|w|<\infty} \rho_0^2 \, du \, dv \geq \iint_{\Omega'} \rho_1^2 \, dx \, dy \geq M(\Gamma(\alpha, \beta))$$

which implies the assertion for Ω'.

For general Ω, on letting $\nu \to \infty$, each sequence of harmonic measures which define extremal metrics in Ω_ν and the w-plane converges and so does the corresponding sequence of modules.

It should be pointed out that the proof for Ω', with $df \neq 0$ on $\partial\Omega'$ provides us information for equality of the theorem in this case. Indeed, if equality holds, then either $n = 0$ or $m = 0$ and f maps Ω' onto a precisely m or $-n$ sheeted unbounded (branched) covering of a plane domain Δ. This is easily verified by checking extremal metrics.

6. Alternative proof, comparison of fluxes

It suffices to show the result for the surface Ω' in Section 5. From (2), $P_m(\alpha) \cap Q_n(\beta) = \emptyset$ and $P_m(\alpha) \cup Q_n(\beta) - (f(\alpha) \cup f(\beta))$ consists of a finite number of connected components. We take a regular subdomain of each component bounded by a finite number of piece-wise analytic curves. Let P_m' and Q_n' be the totality of the subdomains contained in $P_m(\alpha)$ and $Q_n(\beta)$ respectively. We define a continuous function $u(w)$ on the extended plane as follows.

$$\begin{aligned} u(w) &= 0 && \text{on } \bar{P}_m' \\ &= 1 && \text{on } \bar{Q}_n' \\ &= \text{harmonic} && \text{in } (\bar{P}_m' \cup \bar{Q}_n')^c. \end{aligned}$$

The function $v(z) = u(f(z))$ is continuous on $\bar{\Omega}'$ and sectionally harmonic in Ω'. v is harmonic in some neighborhood of each component of α and β.

Let E and F be the inverse images of $\bar{P}'_m$ and $\bar{Q}'_n$ under f respectively, which consist of a finite number of closed subdomains of Ω bounded by piece-wise analytic curves. $v = 0$ on E and $v = 1$ on F. Let $v_1(z)$ be a function continuous on $\bar{\Omega}'$ and such that

$$\begin{aligned} v_1(z) &= v(z) && \text{on } \alpha \cup F \\ &= 1 && \text{on } \beta \\ &= \text{harmonic} && \text{in } \Omega' - F. \end{aligned}$$

Since $v_1 - v \geqq 0$ and $v_1 - v = 0$ on $\alpha \cup \partial F$ on considering inner normal derivatives of $v_1 - v$ with respect to $\Omega' - F$, we have

$$\int_{\alpha \cup \partial F} d(v_1 - v)^* \geqq 0,$$

where ∂F is taken so that $\alpha \cup \partial F$ is homologous to β and $d(v_1 - v)^* = \partial/\partial n(v_1 - v)\, ds \geqq 0$. Hence we get

$$\int_\beta dv_1^* \geqq \int_{\alpha \cup \partial F} dv^*.$$

Let ω_β denote the harmonic measure of β with respect to Ω'. Similarly we have

$$\int_\beta d\omega_\beta^* \geqq \int_\beta dv_1^* \geqq \int_{\alpha \cup \partial F} dv^* = \int_{f(\alpha) \cup f(\partial F)} du^*.$$

Decompose P'_m and Q'_n into connected components $\{P_m^{(j)}\}_{j=1}^{N_1}$ and $\{Q_n^{(j)}\}_{j=1}^{N_2}$ and denote the indices of $f(\alpha), f(\beta)$ on each $P_m^{(j)}, Q_n^{(j)}$ by $p_j(\alpha)$, $p_j(\beta)$, $q_j(\alpha)$, $q_j(\beta)$. Since $f(\Omega')$ covers $Q_n^{(j)}$ $(q_j(\beta) - q_j(\alpha))$ times, we have

$$\begin{aligned} \int_{f(\alpha) \cup f(\partial F)} du^* &= \textstyle\sum_{j=1}^{N_1} p_j(\alpha) \displaystyle\int_{\partial P_m^{(j)}} du^* + \textstyle\sum_{j=1}^{N2} q_j(\alpha) \displaystyle\int_{\partial Q_n^{(j)}} du^* \\ &\qquad + \textstyle\sum_{j=1}^{N_2} (q_j(\beta) - q_j(\alpha) \displaystyle\int_{\partial Q_n^{(j)}} du^* \\ &= \textstyle\sum_{j=1}^{N_1} p_j(\alpha) \displaystyle\int_{\partial P_m^{(j)}} du^* + \textstyle\sum_{j=1}^{N_2} q_j(\beta) \displaystyle\int_{\partial Q_n^{(j)}} du^*. \end{aligned}$$

Since $p_j(\alpha) \geqq m$, $q_j(\beta) \leqq n$,

$$\int_{\partial P_m^{(j)}} du^* \geqq 0, \qquad \int_{\partial Q_n^{(j)}} du^* \leqq 0,$$

$$\textstyle\sum_{j=1}^{N_1} \displaystyle\int_{\partial P_m^{(j)}} du^* = \int_{\partial P'_m} du^* = -\int_{\partial Q'_n} du^* = -\textstyle\sum_{j=1}^{N_2} \displaystyle\int_{\partial Q_n^{(j)}} du^*,$$

we obtain

$$M(\Gamma(\alpha, \beta))^{-1} = \int_\beta d\omega_\beta^* \geqq (m - n) \int_{\partial P'_m} du^* = M(\Gamma'_{mn})^{-1},$$

where Γ'_{mn} is the family of curves separating P'_m from Q'_n.

Expand P'_m and Q'_n so that they exhaust $P_m(\alpha)$ and $Q_n(\beta)$. Let Δ_j and ω_j be the regions and harmonic measures defined in Section 5. Then u tends to ω_j uniformly on every compact subset of Δ_j. Take a dividing cycle δ contained in $\bigcup \Delta_j$ and homologous to $\partial P'_m$ in $(\bar{P}'_m \cup \bar{Q}'_n)^c$. Then we have

$$\int_{\partial P'_m} du^* = \int_\delta du^* \to \sum_j \int_{\delta \cap \Delta_j} d\omega_j^* \quad \text{as} \quad u \to \omega_j$$

which implies the assertion for Ω'.

7. Applications

This theorem can be used to obtain extensions of Hayman [2]-Kubo's [5] estimation of omitted values. They used transfinite and hyperbolic transfinite diameters which have simple relations with reduced modules and modules. One such extension is given as follows.

Let Ω be a Riemann surface whose boundary consists of a finite number of contours, say B and a non-void set of the other boundary components, say A. We regard B as a boundary cycle β positively oriented with respect to Ω. Suppose that a regular function $w = f(z)$ maps Ω into the unit disc so that $f(B) = \{ |w| = 1\}$. Clearly $I(\beta; 0) = n > 0$. We denote by E the set of omitted values of f in $|w| < 1$. Let Γ be the family of collections of a finite number of rectifiable closed curves separating E from $|w| = 1$. Then we have

COROLLARY 1. *If $E \neq \emptyset$,*

$$M(\Gamma) \geqq nM(\Gamma(A, B)). \tag{4}$$

Proof. By the argument principle, E coincides with $\bar{P}_n(A)$. Hence $\Gamma = \Gamma_{n0}$ in Theorem 1.

A discussion about equality in (4) should be indicated. We can deduce that if equality occurs in (4), f is an $(n, 1)$ map of Ω onto $\{ |w| < 1\} - E$ except for a possible compact set of logarithmic capacity zero. This is verified directly by using the second author's result [9].

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