FINITE GROUPS WHOSE SYLOW 2-SUBGROUPS ARE THE DIRECT PRODUCT OF A DIHEDRAL AND A SEMI-DIHEDRAL GROUP

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1. Introduction

The purpose of this paper is to classify all finite fusion-simple groups which have a Sylow 2-subgroup that is the direct product of a dihedral group with a semi-dihedral group. (We say that a group G is *fusion-simple* if $O^2(G) = G$ and $Z^*(G) = 1$. A *semi-dihedral group* is also known as a quasi-dihedral group.) Our main result is as follows:

THEOREM. *Let G be a finite fusion-simple group with a Sylow 2-subgroup that is the direct product of a dihedral group and a semi-dihedral group. Then G has a normal subgroup of odd index of the form $F_1 \times F_2$ where*

$$F_1 \cong A_7, \quad PSL(2, q_1),\ q_1 \text{ odd},\ q_1 \geq 5, \quad \text{or} \quad Z_2 \times Z_2$$

and

$$F_2 \cong M_{11}, \quad PSL(3, q_2),\ q_2 \equiv -1 (\text{mod } 4), \text{ or } PSU(3, q_2),\ q_2 \equiv 1 (\text{mod } 4).$$

The essential ideas used in proof are to be found in [6]. In particular, we assume that a group G is a minimal counter-example to our theorem. We then show that G has an involution fusion pattern compatible with the conclusion of the theorem. Next, we select an arbitrary elementary abelian subgroup A of order 16 in G. Then for suitable four-groups X and Y contained in A such that $A = X \times Y$, we establish the following assertion:

If for $a \in A^{\#}$, one sets

$$\theta(C_G(a)) = \langle C_G(a) \cap O(C_G(x)) \cap O(C_G(y)) \mid x \in X^{\#},\ y \in Y^{\#} \rangle,$$

then θ is an *A-signalizer functor* on G in the sense of Goldschmidt [4].

If θ is nontrivial, we conclude that $W_A = \langle \theta(C_G(a)) \mid a \in A^{\#} \rangle$ is a group of odd order and this allows us to show that $N_G(W_A)$ is a strongly imbedded subgroup of G. It then easily follows that θ is trivial and from this we prove that G satisfies the conclusions of our theorem. This contradiction then proves our theorem.

We use the following definitions which are slight restrictions of some definitions in [2]:

(i) A finite group G is said to be an *SD-group* if a Sylow 2-subgroup of G is a semi-dihedral group and G contains one conjugacy class of involutions and one conjugacy class of elements of order 4.

(ii) A finite group G is said to be a *Q-group* if a Sylow 2-subgroup of G

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is a semi-dihedral group and if G has two conjugacy classes of involutions and one conjugacy class of elements of order 4.

(iii) A finite group G is said to be a *D-group* if a Sylow 2-subgroup of G is a semi-dihedral group and G contains one conjugacy class of involutions and two conjugacy classes of elements of order 4, or if a Sylow 2-subgroup of G is a dihedral group and G contains at most two conjugacy classes of involutions.

(iv) Let H be a group in which $O_r(H) \neq 1$, r an odd prime and let R be an r-subgroup of H such that:

(a) $R \cap O_{r',r}(H)$ is a Sylow r-subgroup of $O_{r',r}(H)$;

(b) either R is normal in a Sylow r-subgroup of H or RK/K contains $O_r(H/K)$ for every normal subgroup K of H.

Under these conditions we say that H is *r-stable with respect to R* provided for any nontrivial subgroup P of R such that $O_{r'}(H)\cdot P$ is normal in H, we have

$$AC_H(P)/C_H(P) \subseteq O_r(N_H(P)/C_H(P))$$

for every subgroup A of R such that $[P, A, A] = 1$.

We now list some properties of simple SD-groups which are a consequence of results in [2] or [9]. If M is a simple SD-group, then by the main result in [2], $M \cong M_{11}$, $L_3(q)$, $q \equiv -1 \pmod 4$, or $U_3(q)$, $q \equiv 1 \pmod 4$. If Y is a four-group contained in M, then $N_M(Y)$ contains a group $S \cong S_4$. Let D be some dihedral group of order 8 in S. We then have the following properties:

(i) $M \cong M_{11}$.

(a) If $y \in Y^\#$, then $C_M(y) \cong GL(2, 3)$.

(b) If P is a maximal nontrivial Y-invariant p-subgroup of M, p odd, then P is a Sylow 3-subgroup of M and any two Y-invariant Sylow 3-subgroups of M are conjugate in $N_M(Y)$.

(ii) $M \cong L_3(q)$.

(a) if $y \in Y^\#$, then $C_M(y) \cong GL(2, q)/Z$ where Z is a subgroup of order $d = (3, q-1)$ in the center of $GL(2, q)$.

(b) If p is an odd prime and p does not divide $q-1$, then any two maximal Y-invariant p-subgroups of M are conjugate in $N_M(Y)$; if p does not divide q, then any two maximal D-invariant p-subgroups of M are conjugate in $N_M(D)$; if p divides $q-1$, if P and Q are two maximal Y-invariant p-subgroups of M, and if $[P, Y] \neq 1$, $[Q, Y] \neq 1$, then $P \sim Q$ in $N_M(Y)$. There is a unique maximal Y-invariant p-subgroup P such that $[P, Y] = 1$.

(iii) $M \cong U_3(q)$.

(a) If $y \in Y^\#$, then $C_M(y) \cong GU(2, q)/Z$ where Z is a subgroup of order $d = (3, q+1)$ in the center of $GU(2, q)$.

(b) Let p be an odd prime. If p divides q, then Y does not normalize any nontrivial p-subgroup of M. If p divides $q-1$ and if P and Q are maximal Y-invariant p-subgroups of M, then $P \sim Q$ in $N_M(Y)$. If p divides $q+1$ an dif P and Q are maximal Y-invariant p-subgroups of M such that $[P, Y] \neq 1$, $[Q, Y] \neq 1$, then $P \sim Q$ in $N_M(Y)$. There is a unique maximal Y-invariant

p-subgroup P such that $[P, Y] = 1$. Finally, any two D-invariant p-subgroups which are maximal are conjugate in $N_M(D)$.

Our notation is standard (see [5]) and includes the "bar" convention for homomorphic images.

2. Preliminary lemmas

We now prove some results concerning the structures of SD-groups, Q-groups, and D-groups.

LEMMA 2.1. *Let H be a group in which $O(H) = 1$ and which contains a normal simple SD-group M of odd index. Let Y be a four-group contained in M and let p be an odd prime. Then the following statements are true:*

(i) *If P is a maximal Y-invariant p-subgroup of H, then $P \cap M$ is a maximal Y-invariant p-subgroup of M.*

(ii) *If p does not divide the order of $C_M(Y)$ and if P_1 and P_2 are maximal Y-invariant p-subgroups of H, then $P_1 \sim P_2$ in $N_H(Y)$.*

(iii) *If p divides the order of $C_M(Y)$ and if P_1 and P_2 are maximal Y-invariant p-subgroups such that $[P_1, Y] \neq 1, [P_2, Y] \neq 1$, then $P_1 \sim P_2$ in $N_H(Y)$.*

Proof. We may assume that there exist nontrivial Y-invariant p-subgroups in H, or else the lemma is trivially true. Set $Q = P \cap M$ and suppose, by way of contradiction, that Q is properly contained in a Y-invariant p-subgroup of M. Since $[P, Y] \subseteq M$, $P = QC_P(Y)$.

We first consider the case that $Q = 1$. Then $P = C_P(Y)$ and for $y \in Y^{\#}, P$ normalizes $C_M(y)$. If $M \cong M_{11}$, then $p = 3$ and $C_D(y)$ contains exactly two Y-invariant 3-subgroups of order 3, each of which must be normalized by P. However this contradicts the maximality of P, and so we can assume that $M \not\cong M_{11}$.

If p divides the order of $C_M(Y)$, then a Sylow p-subgroup U of $C_M(Y)$ is nontrivial and characteristic. Then P normalizes U and again we have a contradiction. Thus we can assume that p does not divide the order of $C_M(Y)$.

Then p does not divide the order of $C_M(y) \cap N_M(Y)$, $y \in Y^{\#}$ and so P normalizes a Sylow 2-subgroup D of $C_M(y) \cap N_M(Y)$. Since $D \cong D_8$, P centralizes D. Let v be an element of order 4 in D and suppose that p divides the order of $C_M(v)$. Then a Sylow p-subgroup of $C_M(v)$ is nontrivial and characteristic and this leads to a contradiction as above. The remaining possibility is that $M \cong L_3(q)$ and p divides q. Then $C_M(y)$ contains exactly two Y-invariant subgroups of order q and so each is normalized by P, a contradiction. Thus we can assume that $Q \neq 1$.

Set $K = N_M(Q)$ and $J = C_M(Q)$. If $Y \subseteq J$, then Q is properly contained in a Sylow p-subgroup of $C_M(Y)$ which is normalized by $R = C_P(Y)$, a contradiction since $P = QR$. Suppose p divides the order of $C_M(Y)$. Then Q is abelian, Q is a Sylow p-subgroup of $O(J)$, but not of J. Also Q is centralized by a four-group and since $Y \not\subseteq J$, a Sylow 2-subgroup of J is a dihedral group. Since $Z(J)$ contains an involution, J has a normal 2-complement and this is a contradiction. Thus p does not divide the order of $C_M(Y)$. Since Q must be a Sylow p-subgroup of both $O(K)$ and $O(JQ)$, this forces $M \cong L_3(q)$ and p divides q. Since a Sylow 2-subgroup of J is cyclic, Q is a Sylow p-subgroup of JQ. In particular, $Z(Q) \supseteq Z(U)$ for some Sylow p-subgroup U of M. It follows from the structure of U that $Q \lhd U$ and so $U \subseteq K$. Since $U \ntriangleleft K$, there is a second Sylow p-subgroup V of M contained in K and by the structure of M, $Z(U) \cap Z(V) = 1$. It follows that $Z(Q)$ contains an abelian subgroup of order q^2. Such subgroups of M are self-centralizing and so $Q = Z(Q)$ and has order q^2. This now forces $K/Q \cong C_M(y)$, $y \in Y^{\#}$. In particular, K contains exactly two Y-invariant Sylow p-subgroups and this leads to a contradiction since each is normalized by P. This completes the proof of (i). We now prove (ii) and (iii).

Set $Q_i = P_i \cap M$ and $R_i = C_{P_i}(Y)$, $i = 1, 2$. It follows by (i) that Q_i is a maximal Y-invariant p-subgroup of M, and by the maximality of P_i, we have that R_i is a Sylow p-subgroup of $N_M(Q_i) \cap C_H(Y)$, $i = 1, 2$. If Y does not centralize P_i, then Y does not centralize Q_i. Hence, by the properties of simple SD-groups listed in the introduction, $Q_1^m = Q_2$ for some $m \in N_M(Y)$. Then R_1^m is a Sylow p-subgroup of $N_M(Q_2) \cap C_M(Y)$ and so for some $h \in N_M(Q_2) \cap C_M(Y)$, $R_1^{mh} = R_2$. Then $P_1^{mh} = P_2$ and $mh \in N_M(Y)$ and this proves (ii) and (iii).

LEMMA 2.2. *Let H be a Q-group in which $O(H) = 1$. Set $L = O^{2'}(H)$ and let Y be a four-group in L. If p is an odd prime, then the following statements hold:*

(i) *If P is a maximal Y-invariant p-subgroup of H, then $P \cap L$ is a maximal Y-invariant p-subgroup of L.*

(ii) *If p does not divide the order of $C_L(Y)$ and if P_1, P_2 are maximal Y-invariant p-subgroups of H, then $P_1 \sim P_2$ in $N_H(Y)$.*

(iii) *If p divides the order of $C_L(Y)$ and if P_1 and P_2 are maximal Y-invariant p-subgroups of H such that $[P_1, Y] \neq 1$, $[P_2, Y] \neq 1$, then $P_1 \sim P_2$ in $N_H(Y)$.*

Proof. Since a Sylow 2-subgroup of L is a semi-dihedral group, it follows by the results in Chapter 2 of [2] that $L \cong SL^{\pm}(2, q)$, $q \equiv -1 \pmod 4$ or $SU^{\pm}(2, q)$, $q \equiv 1 \pmod 4$. The proof of this lemma is then similar in nature to that of the preceding lemma; only in this case it is easier and it is omitted

LEMMA 2.3. *Let H be a D-group in which $O(H) = 1$ and let Y be a four-group in H. If P_1 and P_2 are maximal Y-invariant p subgroups of H for some odd prime p, then $P_1 \sim P_2$ in $N_H(Y)$.*

Proof. If $L = O^{2'}(H)$, then $L \cong A_7$, $PSL(2, q)$, $PGL(2, q)$, or $\mathrm{PGL}^*(2, q)$, q odd (where $\mathrm{PGL}^*(2, q)$ is a group with semi-dihedral Sylow 2-subgroups and is described in Chapter 2 of [2]). A maximal Y-invariant p-subgroup of

L is a Sylow p-subgroup of $C_L(y)$ for some $y \in Y^{\#}$ and it is characteristic. The lemma follows easily from these facts.

LEMMA 2.4. *Let M be a group and assume that $M \cong L_3(q)$, $q \equiv -1 \pmod 4$, $U_3(q)$, $q \equiv 1 \pmod 4$, or M_{11}. If Y is a four-group in M, then*

$$M = \langle Y, C_M(y)' \mid y \in Y^{\#}\rangle.$$

Proof. Set $M_0 = \langle Y, C_M(y)' \mid y \in Y^{\#}\rangle$. Select $y \in Y^{\#}$. Then $Y \cdot C\pi(y)$ contains a Sylow 2-subgroup of $C_M(y)$ and hence, $Y \cdot C_M(y)$ contains a Sylow 2-subgroup of M. We conclude that M_0 is an SD-group. Since no proper section of M_{11} contains an SD-group, we have that $M_0 = M$ if $M \cong M_{11}$. Thus we can assume that this is not the case. Similarly, we can assume that $M \not\cong L_3(3)$. We then have that $C_M(y)' \cong SL(2, q)$, $q \geq 5$ and so $C_M(y)'$ is perfect. Set C $= C_M(y)'$.

If $\bar{M}_0 = M_0/O(M_0)$, then $\bar{M}_0$ contains a normal subgroup $\bar{L}$ of odd index where $\bar{L}$ is a simple SD-group. Since $O(C) = 1$ and $O^{2'}(C) = C$, we have that $\bar{C} \cong C$ and $\bar{C} \subseteq \bar{L}$. We then conclude that $C_{\bar{L}}(\bar{y})' = \bar{C}$. It follows now by the results in [2] that $\bar{L} \cong M$ and hence, $M = M_0$.

LEMMA 2.5. *Let the group M be isomorphic to M_{11} and $p = 3$ or let M be isomorphic to $L_3(q)$, $q \equiv -1 \pmod 4$ and p be the odd prime that divides q. Let Y be a four-group in M and let $S \subseteq N_M(Y)$ with $S \cong S_4$. If P is a maximal Y invariant p subgroup of M, then $M = \langle P, S\rangle$.*

Proof. Set $L = \langle P, S\rangle$. Choose $y \in Y^{\#}$ and let $C = C_M(y)'$. We shall show that $C \subseteq L$. Conjugating P by a suitable element in S if necessary, we can assume that $Q = C_P(y) \neq 1$. If $M \cong M_{11}$, then $\langle Q, D\rangle = C_M(y)$ where D is a Sylow 2-subgroup of $C_S(y)$. It follows in this case that $C \subseteq L$. If $M = L_3(3)$, a similar argument shows that $C \subseteq L$. We can assume that $M \cong L_3(q)$, $q \geq 7$ next. Now by a classical result of L. Dickson (Theorem 2.8.4 of [5]), we conclude that $\langle Q, D\rangle = C$. Thus in all cases $C \subseteq L$. Since the involutions in $Y^{\#}$ are all conjugate in S, we can apply the preceding lemma to obtain our result.

LEMMA 2.6. *Let H be an SD-group in which $O_P(H) \neq 1$, p an odd prime, and let $\bar{M}$ be the normal simple SD-group of odd index in $\bar{H} = H/O(H)$. Assume that the following conditions are satisfied:*

(i) *$H = RM$ where M is the preimage in H of $\bar{M}$ and R is a maximal Y-invariant p-subgroup of H for some four-group Y in M.*

(ii) *Y does not centralize any Sylow p-subgroup of $O(H)$.*

(iii) *If $M \cong M_{11}$, then $p \neq 3$ and if $M \cong L_3(q)$, then p does not divide q.*
Then H is p-stable with respect to R and $H = N_H(Z(J(R)))O(H)$.

Proof. By [2] we have that $M \cong M_{11}$, $L_3(q)$, $q \equiv -1 \pmod 4$, or $U_3(q)$, $q \equiv 1 \pmod 4$. We proceed to verify conditions (a) and (b) in the definition of relative p-stability given in the introduction.

By the maximality of R, $R \cap O(H)$ is a Sylow p-subgroup of $O(H)$ and by our assumptions, Y does not centralize $R \cap O(H)$. If $O_{p'}(H) \nsubseteq O(H)$, then $Y \subseteq O_{p'}(H)$ because H has only one conjugacy class of involutions. But then we have $[R \cap O(H), Y] \subseteq R \cap O_{p'}(H) = 1$, a contradiction. Thus we see that $O_{p'}(H) \subseteq O(H)$ and by the maximality of R, $R \cap O_{p',p}(H)$ is a Sylow p-subgroup of $O_{p',p}(H)$. This verifies condition (a).

Now suppose that K is a normal subgroup of H. If $K \subseteq O(H)$, then $O_p(H/K) \subseteq O(H)/K$ and so $O_p(H/K) \subseteq RK/K$ because $R \cap O(H)$ is a Sylow p-subgroup of $O(H)$. If $K \nsubseteq O(H)$, then K covers M and so $H = RKO(H)$. In this case RK/K is a Sylow p-subgroup of H/K and it follows that $O_p(H/K) \subseteq RK/K$. This verifies condition (b).

Next, we show that H is p-constrained. Set $R_1 = R \cap O_{p',p}(H)$. We must show that $C_H(R_1) \subseteq O_{p',p}(H)$. If $C_H(R_1) \subseteq O(H)$, this follows because $O(H)$ is p-constrained. If $C_H(R_1) \nsubseteq O(H)$, then $C_H(R_1)O_{p'}(H)$ is a normal subgroup of even order in H and so contains Y. It follows that Y centralizes R_1 and so Y acts nontrivially on $C_R(R_1) \cap O(H)$ which is contained in R_1, a contradiction. Therefore H is p-constrained.

Since H will be p-stable with respect to R if and only if $H/O_{p'}(H)$ is p-stable with respect to $RO_{p'}(H)/O_{p'}(H)$, we can assume to begin with that $O_{p'}(H) = 1$.

Let P be a nontrivial normal subgroup of H contained in R. Suppose that A is a subgroup of R such that $[P, A, A] = 1$, but that

$$AC_H(P)/C_H(P) \nsubseteq O_p(H/C_H(P)).$$

Then as in the proof of Proposition 2.6.1 of [2], we can find an H-invariant section P_i of P which is an elementary abelian p-group on which H acts irreducibly and $A \nsubseteq C_H(P_i)$.

If $\tilde{H} = H/C_H(P_i)$, then $O_p(\tilde{H}) = 1$. Since $[P_i, A, A] = 1$, We have that $\tilde{H}$ involves $SL(2, p)$ by Theorem 3.8.3 of [5]. It follows that $C_H(P_i)$ is of odd order and so $\tilde{H} = \tilde{R}\tilde{M}$ where $\tilde{R}$ is a maximal $\tilde{Y}$-invariant p-subgroup of $\tilde{H}$. Also we see that $\tilde{M}/O(\tilde{M}) = \bar{M}$. Since we are interested in the action of $\tilde{H}$ on P_i, we shall drop the "~" for convenience. Also we shall consider $V = P_i$ as a vector space over $GF(p)$ on which H acts faithfully and irreducibly. Thus we shall obtain a contradiction to the following situation:

(i) $H = RM$ where $M/O(H)$ is a simple SD-group, Y is a four-group in M and R is a maximal Y-invariant p-subgroup of H.

(ii) $O_p(H) = 1$ and H acts faithfully and irreducibly on the vector space V over $GF(p)$.

(iii) A is a nontrivial subgroup of R and $[V, A, A] = 1$.

(iv) If $M/O(H) \cong L_3(q)$, then p does not divide q.

(v) $O_{p'}(H) \subseteq O(H)$.

By the proof of Theorem 3.8.3 of [5], if $a \in A^{\#}$, $b \sim a$ in H, and $F = \langle a, b\rangle$ is not a p-group, then F has a normal subgroup F_0 such that $F/F_0 = SL(2, p^m)$ or $p = 3$ and $F/F_0 = SL(2, 5)$.

We must have $A \cap O(H) = 1$. Else we can find $a \in A^{\#} \cap O(H)$ and $b \sim a$ in H such that $\langle a, b\rangle$ is not a p-group and is contained in $O(H)$, because $O_p(H) = 1$, a contradiction.

Set $\bar{H} = H/O(H)$. By our restrictions on p, R is centralized by an involution $\bar{y}$ in $\bar{Y}$. Also $\bar{M} \not\cong L_3(3)$, otherwise the centralizers of involutions would be solvable, a contradiction because $p \neq 3$.

Also $\bar{H} \not\cong M_{11}$ or $U_3(5)$. Otherwise $p = 3$, $\bar{A}$ is of order 3 and every subgroup of order 3 in $\bar{H}$ is conjugate to $\bar{A}$. Since $\bar{H}$ contains a subgroup isomorphic to A_4, the alternating group on 4 letters, we have a contradiction. If $\bar{M} \cong U_3(5)$, then we can assume $\bar{H} = (MA)^{-} \cong PGU(3, 5^2)$. In this case all subgroups of order 3 in $\bar{R} - (R \cap M)^{-}$ are conjugate and so $\bar{A}$ normalizes but does not centralize a Sylow 5-subgroup of $\bar{M}$, a contradiction.

Let $\bar{E} = O^{2'}(C_{\bar{M}}(\bar{y}))$ so that $\bar{E} \cong SL^{\pm}(2, q)$ if $\bar{M} \cong L_3(q)$ or $\bar{E} \cong SU^{\pm}(2, q)$ if $\bar{M} \cong U_3(q)$ and we also have in either case that $q > 5$. Let E be the preimage in H of $\bar{E}$, set $C = RE$, and let K be the semi-direct product of C and V where the action of C on V is a restriction of the action of H on V to C. Then RV is a maximal Y-invariant p-subgroup of K and K is a Q-group. We also have that $C_K(V) = V$ and $O(K) \cap C \subseteq RO(H)$.

If p does not divide q, then RV is a Sylow p-subgroup of K. If p divides q, then our assumptions force $\bar{E} \cong SU^{\pm}(2, q)$, $\bar{M} \cong U_3(q)$, and $\bar{Y}$ centralizes $\bar{R}$. In this case $\bar{R}$ centralizes a dihedral group $\bar{D}$ of order 8 in $N_{\bar{E}}(\bar{Y})$. Thus we can find a dihedral group of order 8 in E which normalizes RV and if this group is denoted by D^*, we have that RV is a maximal D^*-invariant p-subgroup of K. Since $q > 5$, we are now in a position to apply Proposition 2.6.1 of [2]. By this result we have that K is p-stable with respect to RV. It then follows that

$$AC_K(V)/C_K(V) \subseteq O_p(K/C_K(V))$$

and so $A \subseteq O(K) \cap C \subseteq RO(H)$. We see that $[\bar{E}, \bar{A}]$ is a normal subgroup of odd order in $\bar{E}$ and so $\bar{A}$ centralizes $\bar{E}$ and in particular, $\bar{A}$ centralizes $\bar{Y}$.

Choose $z \in Y - \langle y\rangle$ and let F be the preimage in H of $O^{2'}(C_{\bar{M}}(\bar{z}))$. Working in AFV and using an argument similar to that in the preceding two paragraphs, we conclude that $\bar{A}$ centralizes $\bar{F}$. Now by Lemma 2.4 we can conclude that $\bar{A}$ centralizes $\bar{M}$, and so $\bar{A} = 1$ since A is a group of odd order. Since A is not contained in $O(H)$, we have a contradiction. This completes the first part of the lemma and it remains to show that $H = N_H(Z(J(R)))O(H)$. But this is a direct consequence of the extended form of Glauberman's ZJ-Theorem (Theorem 2.7.2 of [2]).

LEMMA 2.7. *Let L be a simple SD-group and let Y be a four-group contained in L. If W is a subgroup of L of odd order such that $N_L(Y) \subseteq N_L(W)$, then $W \subseteq C_L(Y)$.*

Proof. Let $S \subseteq N_L(Y)$ such that $S \cong S_4$. Since $O(S) = 1$, we have $S \cap W = 1$. Set $X = WS$.

If $L \cong M_{11}$, then the order of W is 1 or 3. Then $Y \subseteq C_S(W)$ and so we can assume that $L \ncong M_{11}$.

Suppose that $O_2(X) = 1$. Then $F(X) = F(W)$. Let R be a Sylow p-subgroup of $F(X)$ and assume that $[R, Y] \neq 1$. If p divides the order of the centralizer of Y in L, then R is abelian. Set $Q = \Omega_1(R)$. Then S acts faithfully on $[Q, Y]$ which is cyclic, a contradiction. Next, suppose that p divides q where $L \cong L_3(q)$ or $U_3(q)$ and let $Q = \Omega_1(Z(R))$. Denote the involutions in Y by y_1, y_2, and y_3. Since $C_Q(Y) = 1$, we have that $Q = C_Q(y_1) \times C_Q(y_2)$. Since the involutions in Y are conjugate in S, we have that $C_Q(y_i) \neq 1$, $i = 1, 2, 3$. Let D be a Sylow 2-subgroup of $C_S(y_1)$. From the structure of $C_L(y_1)$ we have that $D/Z(D)$ acts regularly on $C_Q(y_1)$, a contradiction. Thus we can assume that p does not divide q and this forces R to be cyclic. Since S acts faithfully on R, we have a contradiction again. We have shown that $O_2(X) \neq 1$. Then $Y \subseteq O_2(X)$ and so $[W, Y] \subseteq W \cap O_2(X) = 1$. This completes our proof.

We shall now state two results of [6] on which our proof relies heavily. But first we introduce a definition. Let A be an elementary abelian group of order 16 acting on a group K of odd order. Suppose that $A = X \times Y$ where X and Y are four-groups. We then say that K is (X, Y)*-generated* if

$$K = \langle K_{x,y} \mid x \in X^{\#}, y \in Y^{\#} \rangle$$

where $K_{x,y}$ is a normal subgroup in $C_K(\langle x, y \rangle)$ for $x \in X^{\#}$, $y \in Y^{\#}$. An A-invariant subgroup F of K will said to be (X, Y)*-generated if*

$$F = \langle F \cap K_{x,y} \mid x \in X^{\#}, y \in Y^{\#} \rangle.$$

We now have the following result which gives sufficient conditions for every A-invariant subgroup of K to be (X, Y)-generated.

PROPOSITION (2.1 of [6]). *Suppose that A and K are given as above and assume that the following conditions hold for all $x, x' \in X^{\#}$, $y, y' \in Y^{\#}$:*

(a) *$C_{K_{x,y}}(x') \subseteq K_{x',y}$ and $C_{K_{x,y}}(y') \subseteq K_{x,y'}$;*

(b) *every element in $C_K(\langle x, y \rangle)$ inverted by the involutions in both $X - \langle x \rangle$* and *$Y - \langle y \rangle$ lies in $K_{x,y}$;*

(c) *every element in $[C_K(x), Y]' \cap C_K(y)$ inverted by the involutions in $Y - \langle y \rangle$ lies in $K_{x,y}$ and every element in $[C_K(y), X]' \cap C_K(x)$ inverted by the involutions in $X - \langle x \rangle$ lies in $K_{x,y}$;*

(d) *if P is an (X, Y)-generated p-subgroup of K where p is an odd prime then every A-invariant subgroup of P is (X, Y)-generated. Then under these conditions every A-invariant subgroup of K is (X, Y)-generated.*

We also have the following main result of [6].

THEOREM A*. *Let G be a group with a nonabelian Sylow 2-subgroup which is the direct product of two dihedral groups. If G is fusion-simple then:*

(i) $G' = L_1 \times L_2$ *where* $L_1 \cong A_7$ or $L_2(q_1)$ *with* q_1 *odd and* $q_1 \geq 5$ *and* $L_2 \cong Z_2 \times Z_2$, A_7, *or* $L_2(q_2)$ *with* q_2 *odd and* $q_2 \geq 5$;

(ii) G/G' *is of odd order and of rank at most* 2.

3. Fusion of involutions

In this and in all succeeding sections we shall assume that G is a minimal counter-example to our theorem. We let $S = S_1 \times S_2$ be a Sylow 2-subgroup of G where S_1 is a dihedral group and S_2 is a semi-dihedral group. We let z_i be an involution in the center of S_i, $i = 1, 2$. We also let r_1 and s_1 be two involutions which generate S_1 and if S_1 is abelian, we set $z_1 = r_1$. We let s_2 be an involution in $S_2 - Z(S_2)$ and we choose v_2 to be an element of maximal order in S_2 such that $v_2^{s_2} = v_2^{-1} z_2$ and hence, $S_2 = \langle s_2, v_2 \rangle$.

If we set

$$S_1^* = \langle r_1 e_1, s_1 e_2 \mid e_1, e_2 \in Z(S_2) \rangle, \qquad S_2^* = \langle s_2 e_3, \quad v_2 e_4 \mid e_3, e_4 \in Z(S_1) \rangle,$$

then $S = S_1^* \times S_2^*$ and $S_i^* \cong S_i$ for $i = 1, 2$. Also every decomposition of S as a direct product of a dihedral group with a semi-dihedral group is of this form for suitable e_i, $i = 1, 2, 3, 4$.

We have that S_2 has one conjugacy class of four-groups and that S_1 has one class if it is abelian and two otherwise. If A is an elementary abelian subgroup of S of order 16, then $A = (A \cap S_1) \times (A \cap S_2)$ and $A \supseteq Z(S)$. also S has one or two conjugacy classes of elementary abelian subgroups of order 16, according as S_1 is abelian or nonabelian.

We shall say that G has *product fusion* if it is possible to choose the factors S_1^*, S_2^* in such a way that the following conditions hold:

(a) the involutions in S_i^* are conjugate in G for $i = 1, 2$;
(b) the involutions in $S - (S_1^* \cup S_2^*)$ are conjugate in G;
(c) the elements of order four in S_2^* are conjugate in G;
(d) G has exactly three conjugacy classes of involutions.

Since G satisfies the hypotheses of our theorem, we have that $O^2(G) = G$ and $Z^*(G) = 1$. Our first goal in this section will be to show that G must have product fusion.

LEMMA 3.1. *If S_1 is nonabelian, then $N_G(S) = SC_G(S)$. If S_1 is abelian, then there is a 3-element in $N_G(S)$ which acts nontrivially on $Z(S)$ and $[N_G(S){:}SC_G(S)] = 3$.*

Proof. We first assume that S_1 is nonabelian. Then $\Omega_1(S)$ is the direct product of two nonabelian dihedral groups and is of index 2 in S. It follows that every element of odd order in $N_G(S)$ stabilizes the chain $S \supseteq \Omega_1(S) \supseteq 1$ and hence, every element of odd order must centralize S. This proves the first part of the lemma.

Next, assume that S_1 is abelian. Then $S/Z(S)$ is a dihedral group and by considering the chain $S \supseteq Z(S) \supseteq Z(S) \cap S' \supseteq 1$, we see that S admits a single nontrivial odd order automorphism which is of order 3. If $[N_G(S): SC_G(S)] = 1$, then no element in G acts nontrivially on $Z(S)$. In this case Glauberman's Z^*-Theorem gives a contradiction. The second part of the lemma now follows directly from this.

LEMMA 3.2. *Suppose that S_1 is nonabelian and let A and B be representatives of the two conjugacy classes of elementary abelian subgroups of order* 16 *in S. We than have that A is not conjugate to B in G.*

Proof. Suppose, by way of contradiction, that A is conjugate to B in G. Then by Alperin's Fusion Theorem [1] we can find C and D in S such that $C \sim A$, $D \sim B$ in S and such that C and D are contained in a Sylow 2-subgroup T of G and $N_S(S \cap T)$ is a Sylow 2-subgroup of $N_G(S \cap T)$ with $C^y = D$ for some $y \in N_G(S \cap T)$. Let W be the normal closure of C in $N_G(S \cap T)$. Since

$$C = (C \cap S_1) \times (C \cap S_2),$$

we have $W = (W \cap S_1) \times (W \cap S_2)$ where $W \cap S_i$ is a dihedral group for $i = 1, 2$. If g is of odd order in $N_G(S \cap T)$, we have from the structure of W that $C^g = C$. It follows that

$$N_G(S \cap T) = N_{N_G(S \cap T)}(C) N_S(S \cap T).$$

But then $D = C^y$ is conjugate to C in S, a contradiction. This proves the lemma.

LEMMA 3.3. *If S is nonabelian, then, relabeling if necessary, we have:*

(i) *The involutions in S_2 are conjugate in $C_G(z_1)$.*

(ii) *The involutions in S_1 are conjugate in $N_G(\Omega_1(S_2)')$.*

(iii) *The elements of order four in S_2 are conjugate in $C_G(z_1)$.*

(iv) *If A is an elementary abelian subgroup of order* 16 *in S and if $X = A \cap S_1$, $Y = A \cap S_2$, then $N_G(A)/C_G(A) = S_3 \times S_3$ (where S_3 is the symmetric group on* 3 *letters], both X and Y are normal in $N_G(A)$, and the involutions in X, in Y, and in $A - (X \cup Y)$ are conjugate in $N_G(A)$.*

Proof. By Burnside's result and by Lemma 3.1 we have that the involutions in $Z(S)$ are mutually non-conjugate in G.

Let y be an involution in $S_2 - Z(S_2)$. By Thompson's lemma, y is conjugate in G to some involution t in $S_1\langle v_2\rangle$. Choose t such that a Sylow 2-subgroup of $C_G(t)$ has maximal order. Then $C_S(y)$ is a Sylow 2-subgroup of $C_G(y)$ or $C_S(t)$ is a Sylow 2-subgroup of $C_G(t)$. Suppose that t is not contained in $Z(S)$. Then for some $g \in G$, we have $C_S(t)^g \subseteq C_S(y)$ or $C_S(y)^g \subseteq C_S(t)$. In either case it follows that $z \sim z_2$, a contradiction. Thus we have that y is conjugate to an involution in $Z(S)$. But then $\langle y, z_1\rangle$ is the center of some Sylow 2-subgroup of G and so either $y \sim z_2$ or $yz \sim z_2$.

Replacing S by $\langle s_2 z_1, v_2\rangle$ if necessary, we have the involutions in S_2 are conjugate in $C_G(z_1)$.

Next, let x be an involution in $S_1 - Z(S_1)$. Again, by Thompson's lemma we have that x is conjugate to an involution in $\langle r_1 s_1\rangle S_2$. In particular, x is conjugate to an involution in $Z(S)$. But then $\langle u, z_2\rangle$ is the center of some Sylow 2-subgroup of G and so $u \sim z_1$ or $uz_2 \sim z_1$. Replacing S_1 by $\langle r_1 e_1, s_1 e_2\rangle$ for suitable e_1, e_2 in $Z(S_2)$ if necessary, we have that the involutions in S_1 are conjugate in $C_G(z_2)$.

Now let $A = X \times Y$ be as in (iv). If $a, b \in A$ and $a \sim b$ in G, then by Lemma 3.2 it follows that $a \sim b$ in $N_G(A)$. If $xy \in A$ with $x \in X^{\#}$, $y \in Y^{\#}$, then by Thompson's lemma it follows that xy is conjugate to an involution in $Z(S)$ if $xy \notin Z(S)$. We have already shown that the involutions in X, in Y, and in $Xz_2 \cup Yz_1$ are conjugate in G and hence, in $N_G(A)$. Since the involutions in $Z(S)$ are mutually non-conjugate and since $N_G(A)/C_G(A)$ is isomorphic to a subgroup of $GL(4, 2)$, it follows that (iv) holds.

By the preceding paragraph we conclude that no involution in S_2 is conjugate to an involution in $S - S_2$. Again let u be an involution in $S_1 - Z(S_1)$ and let T be a Sylow 2-subgroup of $C_G(u)$ containing $C_S(u) = \langle u, z_1\rangle \times S_2$. Then for some $g \in G$, we have that $\Omega_1(S_2)^g \subseteq T$. Since no involution in S_2 is conjugate to an involution in $S - S_2$, it follows that $g \in N_G(\Omega_1(S_2)')$. To complete the proof of the lemma we need to show (iii). Let $\langle w\rangle$ be the cyclic group of order 4 in $\Omega_1(S_2)'$ and let v be an element of order 4 in $S_2 - \langle v_2\rangle$. By Harada's Extended Transfer Theorem we have that v is conjugate to an element of order 4 in $S_1 \cdot \Omega_1(S_2)$. It follows that $v \sim wu$ where $u \in S_1$ and $u^2 = 1$. If $u = 1$, then we are done and so we can assume that this is not the case. By the preceding paragraph we can assume that $u = z_1$, since $\langle w\rangle$ char $\Omega_1(S_2)'$. If T is a Sylow 2-subgroup of $C_G(v)$ containing $C_S(v)$, we have that $\langle z_1\rangle = Z(T) \cap T'$. It follows that $v \sim wz_1$ in $C_G(z_1)$. Thus we have that $vz_1 \sim w$ in $C_G(z_1)$. Replacing S_2 by $\langle s_2, v_2 z_1\rangle$ if necessary, we conclude that the elements of order 4 in S_2 are conjugate in $C_G(z_1)$. This completes the proof of the lemma.

LEMMA 3.4. *If S_1 is abelian, then, relabeling if necessary, we have:*

(i) *The involutions in S_1 are conjugate in $C_G(S_2)$.*

(ii) *The involutions in S_2 are conjugate in $C_G(S_1)$.*

(iii) *The elements of order* 4 *in S_2 are conjugate in $C_G(S_1)$.*

(iv) *If A is an elementary abelian subgroup of order* 16 *in S and if $X = A \cap S_1$, $Y = A \cap S_2$, then $N_G(A)/C_G(A) \cong S_3 \times Z_3$, both X and Y are normal in $N_G(A)$, and the involutions in X, in Y, and in $A - (X \cup Y)$ are conjugate in $N_G(A)$.*

Proof. Let g be a 3-element in $N_G(S)$ which acts nontrivially on $Z(S)$ and which exists by Lemma 3.1. We may then relabel so that $S_1 = [Z(S), g]$ and $S_2 = C_s(g)$ and so (i) holds.

By Thompson's lemma every involution in S is conjugate to an involution in $Z(S)$. Also by Burnside's lemma we have that z_1, z_2, and $z_1 z_2$ are mutually non-conjugate in G. Let $A = X \times Y$ be as in (iv). If $a, b \in A^{\#}$ and $a \sim b$ in G, then $a \sim b$ in $N_G(A)$ since S has but one conjugacy class of elementary abelian subgroups of order 16 when S_1 is abelian. We also have that $g \in N_G(A)$ and that $Y = C_A(g)$ and since $N_G(A)/C_G(A)$ is isomorphic to a subgroup of $GL(4, 2)$, we conclude that the involutions in Y are conjugate in $C_{N_G(A)}(X)$ and that both (ii) and (iv) hold.

Next, let v be an element of order 4 in $S_2 - \Omega_1(S_2)$. By Harada's theorem $v \sim wx$ where w is an element of order 4 in $\Omega_1(S_2)$ and $x \in S_1$. By the above we can find a 3-element g in $C_G(v)$ which acts nontrivially on $C_S(v)$. It follows that there exists a 3-element in $C_G(wx)$ which acts nontrivially on $S_1 \times \langle wx \rangle$. This forces x to be 1 and we have $v \sim w$ in $N_G(S_1)$. Since $C_{N_G(S_1)}(S_2)$ covers $N_G(S_1)/C_G(S_1)$, we conclude that $v \sim w$ in $C_G(S_1)$. This completes the proof of the lemma.

PROPOSITION 3.5. *The group G has product fusion. The involutions in S_1 are conjugate in $N_G(\Omega_1(S_2)')$, and hence, in $C_G(Z(S_2))$. The involutions in S_1 are conjugate in $C_G(Z(S_1))$ and the elements of order 4 in S_2 are conjugate in $C_G(Z(S_1))$.*

Proof. This lemma is a direct consequence of Lemmas 3.3 and 3.4.

Our next goal is to determine the structures of the centralizers of involutions in G. We first prove

LEMMA 3.6. *Let $C = C_G(z_1)$. If $\bar{C} = C/O(C)$, then $\bar{C} = \bar{S}_1 \times \bar{C}_1$ where $\bar{C}_1$ has a normal subgroup $\bar{C}_0$ of odd index such that $\bar{S}_2 \subseteq \bar{C}_0$ and $\bar{C}_0 \cong M_{11}$, $L_3(q_2)$, $q_2 \equiv -1 \pmod 4$, or $U_3(q_2)$, $q_2 \equiv 1 \pmod 4$.*

Proof. Set $\bar{C}_1 = O^2(\bar{C})$. We claim that $\bar{S}_2$ is a Sylow 2-subgroup of $\bar{C}_1$. It follows by Proposition 3.5 that $\bar{S}_2 \subseteq \bar{C}_1$. Set $\bar{T}_1 = \bar{S}_1 \cap \bar{C}_1$. Then $\bar{T} = \bar{T}_1 \times \bar{S}_2$ is a Sylow 2-subgroup of $\bar{C}_1$. Suppose that $\bar{T}_1$ is non-cyclic and let $\bar{t}$ be an involution in $\bar{T}_1 - \langle \bar{z}_1 \rangle$. By Thompson's lemma $\bar{t}$ is conjugate in $\bar{C}_1$ to an involution in $\langle \bar{z}_1 \rangle \bar{S}_2$. It follows that $\bar{t}$ is conjugate in C to $\bar{z}_1$, a contradiction. Next suppose that $\bar{T}_1$ is cyclic and nontrivial. Let $\langle \bar{t} \rangle = \bar{T}_1$. By Harada's theorem $\bar{t}$ is conjugate to an element in $\langle \bar{t}^2 \rangle \bar{S}_2$. But this forces z_1 to be conjugate to an involution in $Z(S) - \langle z_1 \rangle$, a contradiction. Therefore S_2 is a Sylow 2-subgroup of $\bar{C}_1$ as asserted.

If we now set $\bar{C}_0 = O^{2'}(\bar{C}_1)$, then $\bar{C}_0$ is a simple SD-group. The lemma is now a direct consequence of the main result of [2], once we have shown that $\bar{S}_1$ centralizes $\bar{C}_1$. To see this let t be an arbitrary involution in S_1 and let C_1 be the preimage of $\bar{C}_1$ in C. Then $\langle t \rangle \times S_2$ is a Sylow 2-subgroup of $\langle t \rangle C_1$ and since G has product fusion, we have that t is isolated in $\langle t \rangle C_1$. Now Glauberman's theorem yields that t centralizes $\bar{C}_1$. Since $S_1 = \Omega_1(S_1)$, we conclude that S_1 centralizes $\bar{C}_1$.

We shall retain the notation of this lemma. Henceforth, we let $C = C_G(z_1)$ and we let C_i denote the preimage in C of $\bar{C}_i$, $i = 0, 1$.

LEMMA 3.7. *If* $D = C_G(z_2)$ *and* $\bar{D} = D/O(D)$, *then* $\bar{D}$ *has a normal subgroup* $\bar{D}_0$ *of odd index of the form* $\bar{D}_1 \times \bar{D}_2$ *where* $\bar{D}_1$ *and* $\bar{D}_2$ *have the following structures:*

(i) $\bar{S}_1 \subseteq \bar{D}_1$ *and* $\bar{D}_1 \cong A_7$, $PSL(2, q_1)$, q_1 *odd,* $q_1 \geq 5$, *or* $Z_2 \times Z_2$*:*

(ii) $\bar{S}_2 \subseteq \bar{D}_2$ *and* $\bar{D}_2 \cong SL^{\pm}(2, 3)$ *if* $C_0/O(C) \cong M_{11}$, $SL^{\pm}(2, q_2)$ *if* $C_0 \cong L_3(q_2)$, *or* $SU^{\pm}(2, q_2)$ *if* $C_0 \cong U_3(q_2)$.

Also both $\bar{D}_1$ *and* $\bar{D}_2$ *are normal in* $\bar{D}$.

Proof. *Set* $V = \langle s_2 v_2, v_2 \rangle$, so that the index of V in S_2 equals 2 and V is a generalized quaternion group. Then we have that s_2 is not conjugate in D to any involution in $S_1 \times V$. From the structure of C_0 the elements of order 4 in V are conjugate in $C_0 \cap D$. By Proposition 3.5 the involutions in S_1 are conjugate in D. It follows that D contains a subgroup E of index 2 such that $S_1 \times V$ is a Sylow 2-subgroup of E.

Set $\tilde{D} = D/Z^*(D)$. Then $\tilde{E}$ is a fusion-simple group and $\tilde{S}_1 \times \tilde{V}$ is a direct product of two dihedral groups. Furthermore, $\tilde{V}$ is nonabelian and thus we can apply Theorem A* of [6] to conclude that $\tilde{E}$ has a normal subgroup of odd index of the form $\tilde{L}_1 \times \tilde{L}_2$ where

(i) $\tilde{S}_1 \subseteq \tilde{L}_1$ and $\tilde{L}_1 \cong A_7$, $L_2(q_1)$, q_1 odd, $q_1 \geq 5$, or $Z_2 \times Z_2$;

(ii) $\tilde{V} \subseteq \tilde{L}_2$ and $\tilde{L}_2 \cong A_7$, $L_2(q_2')$, q_2 odd, $q_2' \geq 5$.

Also both $\tilde{L}_1$ and $\tilde{L}_2$ are normal in $\tilde{E}$. By considering the preimage in D of $\tilde{L}_2\langle\tilde{s}_2\rangle$, we see that $\tilde{L}_2 \not\cong A_7$.

Now let $\bar{L}_1$ and $\bar{L}_2$ be the preimages in $\bar{D}$ of $\tilde{L}_1$ and $\tilde{L}_2$ respectively. We have that $\bar{S}_1 \times \langle\bar{z}_2\rangle$ is a Sylow 2-subgroup of $\bar{L}_1$ and so $\bar{L}_1$ has a normal subgroup $\bar{D}_1$ of index 2 such that $\bar{S}_1 \subseteq \bar{D}_1$. If $\bar{S}_1 = \bar{D}_1$, then S_1 is a four-group and since G has product fusion, $\bar{D}_1 \lhd \bar{D}$. If $\bar{D}_1$ is simple, then $\bar{D}_1 \cong \tilde{L}_1$ and $\bar{D}_1$ char $\bar{L}_1$. Again, we have $\bar{D}_1 \lhd D$. By a result of Schur [8] we have that $\bar{L}_2 = SL(2, q_2')$. Moreover, $\bar{z}_1$ centralizes $\bar{L}_2$ and so $\bar{L}_2 = (L_2 \cap C_0)^-$. From this it follows that $q_2' = 3$ if $C_0/O(C) \cong M_{11}$ or $q_2' = q_2$ if $C_0/O(C) = L_3(q_2)$ or $U_3(q_2)$.

We have $\bar{S}_1 \times \langle\bar{s}_2\rangle$ is a Sylow 2-subgroup of $\langle\bar{s}_2\rangle\bar{D}_1$ and since G has product fusion, $\bar{s}_2$ is isolated in $\langle\bar{s}_2\rangle\bar{D}_1$. It follows by Glauberman's theorem that $\bar{s}_2$ centralizes $\bar{D}_1$. Set $\bar{D}_2 = \langle\bar{s}_2\rangle\bar{L}_2$. Then $\bar{D}_2$ centralizes $\bar{D}_1$ and $\bar{D}_1 \cap \bar{D}_2 = 1$. Also $\bar{D}_2 \cong SL^{\pm}(2, 3)$ if $C_0/O(C) \cong M_{11}$, $SL^{\pm}(2,q_2)$ if $C_0/O(C) \cong L_3(q_2)$, or $SU^{\pm}(2, q_2)$ if $C_0/O(C) \cong U_3(q_2)$. Moreover, $\bar{D}_2$ char $C_{\bar{D}}(\bar{D}_1)$ and so $\bar{D}_2 \lhd \bar{D}$. Set $\bar{D}_0 = \bar{D}_1 \times \bar{D}_2$. This completes the proof of the lemma.

Henceforth we let $D = C_G(z_1)$ and we let D_i denote the preimage in D of $\bar{D}_i$, $i = 0, 1, 2$. We also find it convenient to fix some further notation. We let A denote a fixed elementary abelian subgroup of order 16 in S. Set $X = A \cap S_1$ and $Y = A \cap S_2$. Also let x_i denote the involutions in X and y_i denote the involutions in Y, $i = 1, 2, 3$. Finally, we let x_1 and z_1 and $y_1 = z_2$.

We now have

PROPOSITION 3.8. *If*

$$(C_G(X))^- = C_G(X)/O(C_G(X)), \qquad (C_G(Y))^- = C_G(Y)/O(C_G(Y)),$$

$M = O^2(C_G(X))$, *and* $N = O^2(C_G(Y))$, *then* $(C_G(X))^- = \bar{X} \times \bar{M}$, $(C_G(Y))^- = \bar{Y} \times \bar{N}$, *and* $\bar{M}$ *and* $\bar{N}$ *contain characteristic subgroups of odd index*, $\bar{M}_0$ *and* $\bar{N}_0$ *respectively such that*

(i) $\bar{S}_2 \subseteq \bar{M}_0 \cong C_0/O(C)$;

(ii) $\bar{S}_1 \subseteq \bar{N}_0$ *and* $\bar{N}_0 \cong D_1/O(D)$.

Proof. This proposition is a direct consequence of Lemmas 3.6 and 3.7.

We shall retain the notation of this proposition and also we shall let M_0 and N_0 denote the preimages in $C_G(X)$ and $C_G(Y)$ of $\bar{M}_0$ and $\bar{N}_0$ respectively. We note that $O(C_G(X)) = O(M) = O(M_0)$ and $O(C_G(Y)) = O(N) = O(N_0)$.

LEMMA 3.9. *If* $B = C_G(z_1 z_2)$ *and* $\bar{B} = B/O(B)$, *then* $\bar{B} = \bar{S}_1 \times \bar{B}_1$ *where* $\bar{B}_1$ *has a normal subgroup* $\bar{B}_0$ *of odd index such that* $\bar{S}_2 \subseteq \bar{B}_0$ *and* $\bar{B}_0 \cong D_2/O(D)$.

Proof. We first show that z_1 is isolated in B. Suppose, on the contrary, that $z_1 \sim t$ in B where $t \in S - \langle z_1 \rangle$. Since G has product fusion, we have $t \in S_1$. But then $z_2 = z_1 z_1 z_2 \sim t z_1 z_2 \sim z_1 z_2$, a contradiction. It follows that $\bar{B} = (C_B(z_1))^- = (C_B(z_2))^-$ and this lemma is now a direct consequence of Lemmas 3.6 and 3.7.

Henceforth, we shall let $B = C_G(z_1 z_2)$ and B_i shall denote the preimage in B of $\bar{B}_i$, $i = 0, 1$.

4. Subgroup structure of G

In this section we study the subgroup structure of G to the extent needed to enable us to construct a suitable signalizer functor on G. In this section H will denote a proper subgroup of G. Moreover, since we are primarily concerned with the subgroups of G which contain $A = X \times Y$, we shall assume that $A \subseteq H$. In order to study the abstract structure of H, we can assume without loss of generality that $H \cap S$ is a Sylow 2-subgroup of H.

We first prove

LEMMA 4.1. *If* H *has an isolated involution, then either* $C_H(x)$ *or* $C_H(y)$ *covers* $H/O(H)$ *for some* $x \in X^{\#}$ *or* $y \in Y^{\#}$.

Proof. Set $T = H \cap S$. Then we must have $Z(T) \subseteq A$. Thus if z is an isolated involution in H, we can assume that $z \in A$. By Glauberman's theorem $C_H(z)$ covers $H/O(H)$. Suppose that $z \notin X \cup Y$. Then $z = xy$ with $x \in X^{\#}$, $y \in Y^{\#}$. By Lemma 3.9 both x and y are isolated in $C_G(z)$. In particular, they are isolated in $C_H(z)$ and it follows that both $C_H(x)$ and $C_H(y)$ cover $H/O(H)$ in this case.

LEMMA 4.2. *If* H *contains no isolated involution and* $\bar{H} = H/O(H)$, *then* $O^2(\bar{H})$ *has a normal subgroup of odd index of the form* $\bar{L}_1 \times \bar{L}_2$ *where both* $\bar{L}_1$ *and* $\bar{L}_2$ *are normal in* H *and have the following structures:*

(i) $\bar{S}_1 \cap \bar{L}_1$ *is a Sylow 2-subgroup of* $\bar{L}_1$ *and* $\bar{L}_1 \cong A_7$, $L_2(r_1)$, r_1 *odd*, $r_1 \geq 5$, *or* $Z_2 \times Z_2$.

(ii) $\bar{S}_2 \cap \bar{L}_2$ *is a Sylow 2-subgroup of* $\bar{L}_2$ *and* $\bar{L}_2 \cong M_{11}$, $L_3(r_2)$, $r_2 \equiv -1 \pmod 4$, $U_3(r_2)$, $r_2 \equiv 1 \pmod 4$, A_7, $L_2(r_2)$, r_2 *odd*, $r_2 \geq 5$, *or* $Z_2 \times Z_2$.

Proof. Set $K = O^2(H)$ and $T = S \cap K$ so that $\bar{K} = O^2(\bar{H})$ and T is a Sylow 2-subgroup of K. Also set $T_i = T \cap S_i$, $i = 1, 2$.

Suppose first that $T_1 \cap X = 1$. Then we must have $T_1 = 1$. But $H \cap S$ covers H/K and it follows that z_1 is isolated in H, contrary to our assumptions. Thus we have $T_1 \cap X \neq 1$. A similar argument gives that $T_2 \cap Y \neq 1$.

If T_i is cyclic, $i = 1, 2$, or if T_2 is generalized quaternion, then it follows that K contains an isolated involution which is not the case. Thus T_1 is a dihedral group and T_2 is a dihedral or a semi-dihedral group.

Suppose that $T \neq T_1 \times T_2$. Since $T_2 \lhd T$ and $[N_S(T_2):T_2 C_S(T_2)] \leq 2$, we conclude that $T_1 \times T_2 = T_2 C_T(T_2)$ is of index 1 or 2 in T. Thus this index must be 2.

If there are involutions in $T - T_1 \times T_2$, let $t = t_1 t_2$ be one where t_i is an involution in S_i, $i = 1, 2$. If there are none, let $t = t_1 t_2$ be an element of order 4 in $T - T_1 \times T_2$ where t_1 is an involution in S_1 and t_2 is an element of order 4 in S_2. Either by Thompson's lemma or by Harada's theorem t is conjugate in K to some element u in $T_1 T_2$.

We have that $E = C_T(t)$ is abelian of type $(2, 2, 2)$ or $(2, 4)$ and $C_T(u)$ contains an abelian subgroup of type $(2, 2, 2, 2)$ or $(2, 2, 4)$. It follows that E is contained in abelian subgroup of K of type $(2, 2, 2, 2)$ or $(2, 2, 4)$. We can then conclude that for some $k \in C_K(\langle z_1, z_2 \rangle)$, we have t^k is contained in $T_1 T_2$. By Section 3 we see that $C_K(\langle z_1, z_2 \rangle)$ has a normal subgroup of index 2 which contains $T_1 T_2$ and this is a contradiction. Therefore $T = T_1 \times T_2$ is a Sylow 2-subgroup of K.

If T is abelian, then the structure of K is determined by [10]. Since G has product fusion, $K \ncong L_2(16)$ and consequently, $\bar{K}'$ is of odd order and has the asserted structure. Suppose that T is nonabelian. Then we are in a position to apply either Theorem A* of [6] or we utilize the fact that we are working in a minimal counter-example to our theorem. In either case $\bar{K}$ is fusion-simple and our lemma is proved.

LEMMA 4.3. *If H contains no isolated involutions, if $J = O^2(H)A$, and if $\bar{H} = H/O(H)$, then $\bar{J}$ contains a normal subgroup of odd index of the form $\bar{F}_1 \times \bar{F}_2$ where $\bar{F}_1$ and $\bar{F}_2$ have the following structures:*

(i) $\bar{S}_1 \cap \bar{F}_1$ *is a Sylow 2-subgroup of* $\bar{F}_1$ *and* $\bar{F}_1 \cong A_7$, $L_2(r_1)$, $PGL(2, r_1)$, r_1 *odd, or* $Z_2 \times Z_2$.

(ii) $\bar{S}_2 \cap \bar{F}_2$ *is a Sylow 2-subgroup of* $\bar{F}_2$ *and* $\bar{F}_2 \cong M_{11}$, $L_3(r_2)$, $r_2 \equiv -1 \pmod 4$, $U_3(r_2)$, $r_2 \equiv 1 \pmod 4$, A_7, $L_2(r_2)$, $PGL(2, r_2)$, r_2 *odd, or* $Z_2 \times Z_2$.

Also both $\bar{F}_1$ and $\bar{F}_2$ are normal in $\bar{J}$.

Proof. By the preceding lemma $\bar{K} = O^2(\bar{H})$ has a normal subgroup of odd index of the form $\bar{L}_1 \times \bar{L}_2$ where $\bar{S}_i \cap \bar{L}_i$ is a Sylow 2-subgroup of $\bar{L}_i$ and

$\bar{L}_i$ has the structure specified in that lemma, $i = 1, 2$. Since G has product fusion, Glauberman's theorem yields that $\bar{X}$ centralizes $\bar{L}_2$ and $\bar{Y}$ centralizes $\bar{L}_1$. Set $\bar{F}_1 = \bar{L}_1 \bar{X}$ and $\bar{F}_2 = \bar{L}_2 \bar{Y}$. Then $\bar{F}_i$ has the structure asserted in the conclusion of this lemma and $\bar{S}_i \cap \bar{F}_i$ is a Sylow 2-subgroup of $\bar{F}_i$, $i = 1, 2$. Moreover, $\bar{F}_1$ and $\bar{F}_2$ centralize each other and $\bar{F}_i \cap \bar{F}_2 = 1$. In order to obtain the conclusion of the lemma it is sufficient to show that $\bar{F}_i \lhd \bar{J}$, $i = 1, 2$. If $\bar{X} \subseteq \bar{L}_1$ and $\bar{Y} \subseteq \bar{L}_2$, then this follows from the fact that $\bar{L}_i \lhd \bar{J}$, $i = 1, 2$.

Suppose then that $\bar{X} \nsubseteq \bar{L}_1$. If $\bar{L}_2 \cong Z_2 \times Z_2$, then $W = K \cap S_2$ is a four-group and $\bar{W} = \bar{L}_2$. It follows that $\bar{J} = (N_J(W))^-$. From Proposition 3.8 we conclude that $\bar{F}_1$ char $C_J(\bar{W}) \lhd \bar{J}$ and so $\bar{F}_1 \lhd \bar{J}$. If $\bar{L}_2$ is a simple group, then $C_J(\bar{L}_2) \cap \bar{L}_2 = 1$. Since $C_J(\bar{L}_2)$ is a D-group, we have $\bar{F}_1$ char $C_J(\bar{L}_2)$ and that $\bar{F}_1 \lhd \bar{J}$.

Suppose now that $\bar{Y} \nsubseteq \bar{L}_2$. Then $\bar{S}_2 \cap \bar{L}_2$ is a dihedral group and so $\bar{Y}(\bar{S}_2 \cap \bar{L}_2)$ is also a dihedral group. Arguing as in the preceding paragraph we can conclude that $\bar{F}_2 \lhd \bar{J}$ as well. This completes the proof.

We now find it convenient to fix some more notation. We set $p_2 = 3$ if $M_0/O(M) \cong M_{11}$, we set p_2 to be the prime which divides q_2 if $M_0/O(M) \cong L_3(q_2)$, and we do not define p_2 if $M_0/O(M) \cong U_3(q_2)$. Also we let $\mathcal{S}_2$ denote the set of all odd primes which divide the order of $C_{M_0/O(M)}(Y)$.

Our next goal is to prove some results on the transitivity of maximal A-invariant p-subgroups under conjugation by the elements in $N_G(A)$ where p is an odd prime.

LEMMA 4.4. *Suppose that p is an odd prime and that $p \notin \mathcal{S}_2$. If P_1 and P_2 are maximal A-invariant p-subgroups of H, then $P_1 \sim P_2$ in $N_H(A)$.*

Proof. Any maximal A-invariant p-subgroup of H contains a Sylow p-subgroup of $O(H)$ and any two A-invariant Sylow p-subgroups of $O(H)$ are conjugate in $C_{O(H)}(A)$. It is immediate from this that it will suffice to prove that $\bar{P}_1 \sim \bar{P}_2$ in $N_{\bar{H}}(\bar{A})$ where $\bar{H} = H/O(H)$.

If H has no isolated involution and if $J = O^2(H)A$, then by the preceding lemma $\bar{J}$ has a normal subgroup of odd index of the form $\bar{F}_1 \times \bar{F}_2$ where $\bar{S}_i \cap \bar{F}_i$ is a Sylow 2-subgroup of $\bar{F}_i$ and $\bar{F}_i$ has the structure asserted in the conclusion of that lemma, $i = 1, 2$.

If F_i denotes the preimage in J of $\bar{F}_i$, $i = 1, 2$, then $\bar{F}_i = (F_1 \cap N_0)^-$ and $\bar{F}_2 = (F_2 \cap M_0)^-$. Since $p \notin \mathcal{S}_2$, we have that p does not divide the order of $C_{\bar{F}_2}(\bar{Y})$.

Set $\bar{U}_i = \bar{P}_i \cap \bar{F}_1$, $\bar{V}_i = \bar{P}_i \cap \bar{F}_2$, and $\bar{R}_i = C_{\bar{P}_i}(\bar{A})$. Then by Lemmas 2.1, 2.2, or 2.3 and by the structures of $\bar{F}_1$ and $\bar{F}_2$, $U_1 V_1 \sim U_2 V_2$ in $N_{\bar{F}_1 \bar{F}_2}(\bar{A})$. Also by the maximality of P_i we have that $\bar{R}_i$ is a Sylow p-subgroup of $N_J(\bar{U}_i \bar{V}_i) \cap C_J(\bar{A})$, $i = 1, 2$. Since $\bar{P}_i = \bar{U}_i \bar{V}_i \bar{R}_i$, $i = 1, 2$, we can conclude as in the proof of Lemma 2.1 that $\bar{P}_1 \sim \bar{P}_2$ in $N_J(\bar{A})$.

Next we consider the case that H contains an isolated involution. Suppose first that some $x \in X^{\#}$ is isolated. For definiteness, let $x = x_1$. Then we have $\bar{J} = \bar{X} \times (J \cap C_1)^-$. Set $\bar{F} \times O^{2'}(J \cap C_1)^-$. Then $\bar{S}_2 \cap \bar{F}$ is a Sylow 2-subgroup of $\bar{F}$ and since $p \notin \mathcal{S}_2$, p does not divide the order of $C_{\bar{F}}(\bar{Y})$. Since $\bar{F}$ is an SD-group, a Q-group, a D-group, or a 2-group, we have by Lemmas 2.1, 2.2, and 2.3 that $\bar{P}_1 \sim \bar{P}_2$ in $N_{(J\cap S_1)^-}(\bar{Y})$ and hence, in $N_{\bar{J}}(\bar{A})$.

If no involution in X is isolated in H, then by Lemma 4.1 we have that some $y \in Y^{\#}$ is isolated in H. For definiteness, let $y = y_1$. Set $J_1 = J \cap D_1$, $J_2 = J \cap D_2$, and $E = O(C_J(A))$. Since $\bar{J} = (J \cap D)^-$, we have $\bar{J}_1$ and $\bar{J}_2$ are normal in $\bar{J}$. Also $J_1 \cap J_2$ has odd order and so $\bar{J}_1\bar{J}_2 = \bar{J}_1 \times \bar{J}_2$. Now let $K = J_1 J_2 EO(H)$ and $\bar{K} = K/O(K)$. Also set $\bar{F}_i = O^{2'}(\bar{J}_i)$, $i = 1, 2$. As above we have that p does not divide $C_{\bar{F}_2}(\bar{Y})$. Since

$$\bar{P}_i = (\bar{P}_i \cap \bar{F}_1 \times \bar{P}_i \cap F_2)(\bar{P}_i \cap \bar{E}), \quad i = 1, 2,$$

we conclude that $\bar{P}_1 \sim \bar{P}_2$ in $N_{\bar{K}}(\bar{A})$. This completes the proof of the lemma.

PROPOSITION 4.5. *Suppose that p is an odd prime and that $p \notin \mathcal{S}_2$. If P_1 and P_2 are maximal A-invariant p-subgroups of G, then $P_1 \sim P_2$ in $N_G(A)$.*

Proof. Suppose that the proposition is false and choose P_1 and P_2 such that they violate it and such that the order of $R = P_1 \cap P_2$ is maximal subject to this. Set $K = N_G(R)$. Without loss we can assume that $K \cap S$ is a Sylow S-subgroup of K. If $R \neq 1$, then K is a proper subgroup of G and the preceding lemma is applicable. This leads to a contradiction by a standard argument.

In any case, we have $P_i \neq 1$, $i = 1, 2$. Considering the action of A on P_i, we see then that $C_{P_i}(T_i) \neq 1$ for some maximal subgroup T_i of A, $i = 1, 2$. Setting $H = C_G(T)$ where $T = T_1 \cap T_2 \neq 1$ and hence, $C_{P_i}(T) \neq 1, i = 1, 2$, we can assume without loss that $S \cap H$ is a Sylow 2-subgroup of H. Again, the preceding lemma is applicable. We let Q_i be a maximal A-invariant p-subgroup of G containing $C_{P_i}(T)$ as well as a maximal A-invariant p-subgroup R_i of H, $i = 1, 2$. Then $Q_i \cap P_i \neq 1$, $i = 1, 2$ and Q_1^u contains R_2 for some $u \in N_H(A)$ by the preceding lemma. These conditions together with our maximal choice of $P_1 \cap P_2 = R$ now force $R \neq 1$ and the lemma is proved.

LEMMA 4.6. *If $p \in \mathcal{S}_2$ and if P is a maximal A-invariant p-subgroup of G, then $P \cap M_0$ covers a maximal Y-invariant p-subgroup of $M_0/O(M)$.*

Proof. Let Γ be the set of all maximal A-invariant p-subgroups P^* of G such that P^* covers a maximal Y-invariant p-subgroup of $M_0/O(M)$. Suppose, by way of contradiction, that there exist maximal A-invariant p-subgroups of G not contained in Γ. Among those not in Γ, select the subset $\mathfrak{Z}$ consisting of all subgroups P_1 such that the order of $P_1 \cap M_0$ is maximal. For each group $P_1 \in \mathfrak{Z}$, let $\Gamma(P_1)$ denote the subset of Γ which consists of all groups P_2 such that $P_2 \supseteq P_1 \cap M_0$. It is clear that $\Gamma(P_1) \neq \emptyset$ for each $P_1 \in \mathfrak{Z}$. We now consider the set of all pairs of groups (P_1, P_2) where $P_1 \in \mathfrak{Z}$ and $P_2 \in \Gamma(P_1)$ and among these we choose a pair (P_1, P_2) such that the order of $R = P_1 \cap P_2$ is maximal.

Suppose that $R = 1$. Since $P_1 \cap M_0 \subseteq R$, $P_1 \cap M_0 = 1$. Since $P_1 \neq 1$,

$C_{P_1}(x) \neq 1$ for some $x \in X^{\#}$. But a maximal A-invariant p-subgroup of $C_G(x)$ covers a maximal Y-invariant p-subgroup of $M_0/O(M)$ and so we can find some $P_3 \in \Gamma$ containing $C_{P_1}(x)$. Since $P_1 \cap M_0 = 1$, we have $P_3 \in \Gamma(P_1)$. Since $P_1 \cap P_3 \neq 1$, we have contradicted the maximality of R. Thus we have that $R \neq 1$.

Set $H = N_G(R)$. Without loss we can assume $S \cap H$ is a Sylow 2-subgroup of H. Also set $K = O^2(H)A$. Now let $Q_i = N_{P_i}(R)$ and let V_i be a maximal A-invariant p-subgroup of G containing Q_i and a maximal A-invariant p-subgroup U_i of H, $i = 1, 2$.

Suppose that $V_1 \in \Gamma$; since $V_1 \supseteq R \supseteq P_1 \cap M_0$, we have $V_1 \in \Gamma(P_1)$. Since $V_1 \cap P_1 \supseteq Q_1 \supset R$, we have a contradiction. It follows that $V_1 \notin \Gamma$. Since $V_1 \cap M_0 \supseteq P_1 \cap M_0$, we see that $V_1 \in \mathfrak{J}$.

Suppose next that $V_2 \notin \Gamma$. Since $V_2 \supseteq R \supseteq P_1 \cap M_0$, it follows that $V_1 \in \mathfrak{J}$ and also that $V_2 \cap M_0 = P_1 \cap M_0$. From this it follows that $P_2 \in \Gamma(V_2)$ and since $P_2 \cap V_2 \supset R$, we again have a contradiction. Thus we have that $V_2 \in \Gamma$.

Now suppose that H has no isolated involution and set $K = O^2(H)A$ and $\bar{H} = H/O(H)$. Also let $\bar{F} = \bar{F}_1 \times \bar{F}_2$ be the normal subgroup of odd index in $\bar{K}$ which satisfies the conclusion of Lemma 4.3 and retain the notation of that lemma. Finally, let F_i denote the preimage in H of $\bar{F}_i$, $i = 1, 2$. Then we see that $\bar{F}_2 = (F_2 \cap M_0)^-$.

By the maximality of U_1 we have $(U_1 \cap F_2)^-$ is a maximal Y-invariant p-subgroup of $\bar{F}_2$. Since $U_1 \cap M_0 \supseteq R \cap M_0$ and the order of $U_1 \cap M_0$ equals the order of $P_1 \cap M_0$, we conclude that

$$U_1 \cap M_0 \subseteq R \subseteq O(H).$$

Since $(U_1 \cap F_2)^- = (U_1 \cap F_2 \cap M_0)^-$, it follows that $(U_1 \cap F_2)^- = 1$. But now the structure of $\bar{F}_2$ forces $(U_2 \cap F_2)^- = 1$ also. It follows then that $\bar{U}_i = (U_i \cap F_1)^- C_{\bar{U}_i}(\bar{A})$ for $i = 1, 2$ and we also have that $(U_i \cap F_1)^-$ is a maximal X-invariant p-subgroup of $\bar{F}_1$. As in previous arguments we then have that $U_2^k = U_1$ for some $k \in N_K(A)$. Since normalizes M_0 and $V_2^k \supseteq R \supseteq P_1 \cap M_0$, it follows that $V_2^k \in \Gamma(P_1)$. This again contradicts the maximality of R, since $V_2^k \cap P_1 \supset R$.

We can assume then that H contains an isolated involution. First suppose that x is isolated in H for some $x \in X^{\#}$. Then we have that $\bar{K} = \bar{X} \times (K \cap M)^-$ and that $\bar{U}_i$ is a maximal y-invariant p-subgroup of $(K \cap M)^-$, $i = 1, 2$. Since $(K \cap M_0)^- \lhd (K \cap M)^-$, we see that

$$\bar{U}_i = (U_i \cap K \cap M_0)^- C_{\bar{U}_i}(\bar{A}), \quad i = 1, 2.$$

But as above we conclude that $(U_1 \cap K \cap M_0)^- = 1$ and hence, that $(U_2 \cap K \cap M_0)^- = 1$ also. Then $\bar{U}_i$ is a Sylow p-subgroup of $C_{\bar{K}}(\bar{A})$, $i = 1, 2$ and it follows easily that $U_1 \sim U_2$ in $N_K(A)$. But this leads to the same contradiction as above.

Finally, we consider the case that an involution y in Y is isolated in H.

For definiteness, we let $y = y_1$. Set $K_i = K \cap D_i$, $i = 1, 2$ and let $E = O(C_K(A))$. Also set $L = K_1 K_2 E O(H)$ and let $\bar{L} = L/O(L)$. Then we have that $U_i \subseteq L$ and that $\bar{L} = (\bar{K}_1 \times \bar{K}_2)\bar{E}$. Finally, let $\bar{F}_i = O^{2'}(\bar{K}_i)$ and let F_i denote the preimage in L of $\bar{F}_i$, $i = 1, 2$. Then $\bar{F}_2 = (F_2 \cap M_0)^-$ and we conclude as above that $(U_i \cap F_2)^- = 1$, $i = 1, 2$. Again, we see that $U_1 \sim U_2$ in $N_L(A)$ and this leads to the same contradiction as above. This however forces our lemma to be true.

LEMMA 4.7. *Suppose that $p \in \mathcal{S}_2$ and that P_i is a maximal A-invariant p-subgroup of G, $i = 1, 2$. If $P_1 \cap P_2 \cap M_0$ covers a maximal Y-invariant p-subgroup of $M_0/O(M)$, then $P_1 \sim P_2$ in $N_G(A)$.*

Proof. Suppose that the lemma is false and among all pairs of subgroups violating the lemma choose P_1 and P_2 such that the order of $R = P_1 \cap P_2$ is maximal. Then we see that R covers a maximal Y-invariant p-subgroup of $M_0/O(M)$ and in particular, $R \neq 1$.

Set $H = N_G(R)$ and let U_i be a maximal A-invariant p-subgroup of H containing $N_{P_i}(R)$, $i = 1, 2$. Since R covers a maximal Y-invariant p-subgroup of $M_0/O(M)$ and argueing as in the previous lemma, we can conclude that $U_1 \sim U_2$ in $N_H(A)$. This leads to a contradiction by a standard argument and proves the lemma.

PROPOSITION 4.8. *Suppose that $p \in \mathcal{S}_2$. Then there exist maximal A-invariant p-subgroups P_1 and P_2 of G such that $P_i \cap M_0$ is a maximal A-invariant p-subgroup of M_0 and $[P_1 \cap M_0, Y] \subseteq O(M)$ and $[P_2 \cap M_0, Y] \not\subseteq O(M)$. Let P be any maximal A-invariant p-subgroup of G. If $[P \cap M_0, Y] \subseteq O(M)$, then $P \sim P_1$ and if $[P \cap M_0, Y] \not\subseteq O(M)$, then $P \sim P_2$ in $N_G(A)$.*

Proof. Let Q_i be a maximal A-invariant p-subgroup of M_0, $i = 1, 2$ such that $[Q_1, Y] \equiv O(M)$ and $[Q_2, Y] \not\subseteq O(M)$ and let P_i be a maximal A-invariant p-subgroup of G containing Q_i, $i = 1, 2$.

Suppose that P is a maximal A-invariant p-subgroup of G and that

$$[P \cap M_0, Y] \subseteq O(M).$$

Since $P \cap M_0$ covers a maximal Y-invariant p-subgroup of $M_0/O(M)$ by Lemma 4.6, we have

$$(P \cap M_0)^m \subseteq P_1 \cap M_0 \quad \text{for some } m \in N_{M_0}(A).$$

Then $P^m \cap P_1 \cap M_0 = (P \cap M_0)^m$ covers a maximal Y-invariant p-subgroup of $M_0/O(M)$. By the preceding lemma $P^m \sim P_1$ in $N_G(A)$ and hence, $P \sim P_1$ in $N_G(A)$.

If $[P \cap M_0, Y] \not\subseteq O(M)$, we can apply a similar argument to conclude that $P \sim P_2$ in $N_G(A)$.

We shall say that a proper subgroup H of G *covers* $M_0/O(M)$ if $H \cap M_0$ covers $M_0/O(M)$. Similarly, we shall say that H *covers* $N_0/O(N)$ if $H \cap N_0$

covers $N_0/O(N)$. We now prove some results concerning p-local subgroups of G which cover $M_0/O(M)$ and $N_0/O(N)$.

LEMMA 4.9. *Suppose that a Sylow p-subgroup of $O(C)$ is nontrivial. If P is a maximal A-invariant p-subgroup of G, then there is a p-local subgroup K of G containing PA and covering $M_0/O(M)$.*

Proof. Let Q be an A-invariant Sylow p-subgroup of $O(C)$. Then $N_C(Q)$ contains A and covers $C_0/O(C)$ and hence, covers $M_0/O(M)$. Let R be a maximal A-invariant p-subgroup of $N_C(Q)$ so that R covers a maximal Y-invariant p-subgroup of $M_0/O(M)$. If $p \in \mathcal{S}_2$, we choose R such that $\bar{R} \sim (M_0 \cap P)^-$ in $N_{\bar{M}_0}(\bar{Y})$ where $\bar{M}_0 = M_0/O(M)$.

Now among all p-local subgroups of G containing RA and covering $M_0/O(M)$ choose K such that an A-invariant p-subgroup U of K has maximal order subject to containing R. Suppose that U is not a maximal A-invariant p-subgroup of G. Then there exists an A-invariant p-subgroup of G which properly contains and normalizes U. We denote this subgroup by V.

By Lemmas 4.1, 4.2, and 4.3, $O^2(K)A$ has a normal subgroup L containing $O(K)$ such that $\bar{L} = (L \cap M_0)^- \cong M_0/O(M)$ in $\bar{K} = K/O(K)$. Set $F = LUA$ and let $\bar{F} = F/O(F)$. By the maximality of U we have that $V_0 = U \cap O(F)$ is a Sylow p-subgroup of $O(F)$. Also $\bar{F} = (L \cap M_0)^- \bar{U} \times \bar{X}$ and $N_{L \cap M_0}(V_0)$ covers $L \cap M_0$ by the Frattini argument and hence, covers $M_0/O(M)$.

Now $C_V(x) \not\subseteq U$ for some $x \in X^{\#}$. Suppose x centralizes V_0. Then $C_V(x) \supset U$ and we can find a maximal A-invariant p-subgroup of $C_G(x)$ containing $C_V(x)$ such that this subgroup and A are contained in a p-local subgroup of $C_G(x)$ which covers $M_0/O(M)$. However, this contradicts the maximality of U and our choice of K. Thus we can assume that $[V_0, x] \neq 1$. Set $V_1 = [V_0, x]$. Then V_1 is normalized by $UC_V(x)$, by A, and by $N_{L \cap M_0}(V_0)$. But then $N_G(V_1)$ contradicts our choice of K. This contradiction then proves our lemma since U must be a maximal A-invariant p-subgroup of G and $U \sim P$ in $N_G(A)$.

LEMMA 4.10 *Suppose that a Sylow p-subgroup of $O(D)$ is nontrivial. Then there exists a p-local subgroup H of G containing A and a maximal A-invariant p-subgroup P of G such that H covers $N_0/O(N)$.*

Proof. Let V be an A-invariant Sylow p-subgroup of $O(D)$. Then $N_D(V)$ contains A and covers $D_1/O(D)$ and hence, covers $N_0/O(N)$. Now among all p-local subgroups of G containing A and covering $N_0/O(N)$ choose H such that an A-invariant p-subgroup of H has maximal order. Suppose that P is not a maximal A-invariant p-subgroup of G. Then there is an A-invariant p-subgroup U of G containing P properly and normalizing P. Then $C_U(y) \not\subseteq P$ for some $y \in Y^{\#}$. We can now argue as in the preceding lemma to obtain a contradiction to our choice of H. It follows that P is a maximal A-invariant p-subgroup of G and the lemma is proved.

We conclude this section with a result needed in the next.

LEMMA 4.11. *Suppose $p \in \mathcal{S}_2$ and R is an A-invariant p-subgroup such that*

$$R = \langle R \cap O(C_G(x)) \mid x \in X^{\#} \rangle.$$

Then R is contained in maximal A-invariant p-subgroups P and Q of G such that

$$[P \cap M_0, Y] \subseteq O(M) \quad \text{and} \quad [Q \cap M_0, Y] \nsubseteq O(M).$$

Proof. Let U be a maximal A-invariant p-subgroup of G containing R. By the preceding Lemma 4.9 there is a p-local subgroup K of G containing UA and covering $M_0/O(M)$. Then $O^2(K)A$ has a normal subgroup L containing $O(K)$ such that

$$\bar{L} = (L \cap M_0)^- \cong M_0/O(M) \quad \text{in } \bar{K} = K/O(K).$$

Let $J = LUA$ and set $\bar{J} = J/O(J)$. Then $\bar{J} = (L \cap M_0)^- \bar{U} \times \bar{X}$ and for $x \in X^{\#}$ we have that

$$[(R \cap O(C_G(x))),^- (L \cap M_0)^-]$$

is a normal subgroup of odd order in $\bar{L} = (L \cap M_0)^-$. It follows that $(R \cap O (C_G(x)))^-$ centralizes $\bar{L}$ and so we conclude that $R \cap O\ (C_G(x)) \subseteq O(J)$. It follows that $R \subseteq O(J)$. Since J covers $M_0/O(M)$, we have that maximal A-invariant p-subgroups of J cover maximal Y-invariant p-subgroups of $M_0/O(M)$. Moreover, we can find maximal A-invariant p-subgroups Q_1 and Q_2 of J such that $O(J) \cap U \subseteq Q_1 \cap Q_2$ and such that

$$[Q1 \cap M_0, Y] \subseteq O(M) \quad \text{and} \quad [Q_2 \cap M_0, Y] \nsubseteq O(M).$$

It is now only necessary to choose P_i to be a maximal A-invariant p-subgroup of G containing Q_i, $i = 1, 2$ in order to obtain the conclusion of our lemma.

5. An A-signalizer functor

Our main goal in this section is to show that if for $a \in A^{\#}$ we set

$$\theta(C_G(a)) = \langle C_G(a) \cap (C_G(x)) \cap O(C_G(y)) \mid x \in X^{\#}, y \in Y^{\#} \rangle,$$

then θ is an A-signalizer functor on G. In order to do this we must show that $\theta(C_G(a))$ has odd order for all $a \in A^{\#}$ and that θ satisfies the *balance condition*

$$\theta(C_G(a)) \cap O\ C_G(b) \subseteq \theta(C_G(b)), \quad a, b \in A^{\#}.$$

We shall use Proposition 2.1 of [6] to show this.

We are retaining the following notation of the preceding sections: B, C, D, B_1, C_1, D_0, D_1, D_2, M, M_0, N, N_0, C_0, B_0.

We first prove the following useful lemma.

LEMMA 5.1. *The following conditions hold for all* $x, x' \in X^{\#}$, $y, y' \in Y^{\#}$:

(i) $C_G(x) \cap O(C_G(x')) \subseteq O(C_G(x))$.

(ii) $C_G(y) \cap O(C_G(x)) \cap O(C_G(y')) \subseteq O(C_G(y))$.

(iii) $C_G(xy) \cap O(C_G(x')) \subseteq O(C_G(xy))$.

(iv) $[D \cap O(C_G(x)), D_2] \subseteq O(D)$ *and in particular,* $(D \cap O(C_G(x)))^{-}$ *centralizes* $\bar{Y}$ *in* $\bar{D} = D/O(D)$.

Proof. Choose $x \in X - \langle x_1 \rangle$ and set $R = C \cap O(C_G(x))$. If $\bar{C} = C/O(C)$, then $[\bar{R}, \bar{C}_1]$ is a normal subgroup of odd order in $\bar{C}_1$ since $\bar{C}_1 = (C_1 \cap M)^{-}$. It follows that $\bar{R}$ centralizes $\bar{C}_1$ and so $R \subseteq O(C)$. Now (i) follows easily from this.

Choose $y \in Y - \langle y_1 \rangle$ and set $Q = D \cap O(C) \cap O(C_G(y))$. If $\bar{D} = D/O(D)$, then $[\bar{Q}, \bar{D}_i]$ is a normal subgroup of odd order in $\bar{D}_i$, $i = 1, 2$ since $\bar{D}_1 = (D_1 \cap N)^{-}$ and $\bar{D}_2 = (D_2 \cap M)^{-}$. From this it follows that $\bar{Q}$ centralizes both $\bar{D}_1$ and $\bar{D}_2$ and hence, $R \subseteq O(D)$. Then (ii) follows easily from this.

Next, choose x in $X - \langle x_1 \rangle$ and set $P = B \cap O(C_G(x))$. If $\bar{B} = B/O(B)$, then $[\bar{P}, \bar{B}_1]$ is a normal subgroup of odd order in $\bar{B}_1$ since $\bar{B}_1 = (B_1 \cap M)^{-}$. We conclude that $R \subseteq O(B)$ and (iii) follows easily from this.

If $\bar{D} = D/O(D)$, then $[(D \cap O(C))^{-}, \bar{D}_2]$ is a normal subgroup of odd order in $\bar{D}_2$. It follows that $(D \cap O(C))^{-}$ centralizes $\bar{D}_2$ and (iv) is a consequence of this.

LEMMA 5.2. *Let E be an A-invariant subgroup of G of odd order such that $AE \subseteq H \cap K$ where H is a proper subgroup of G covering $N_0/O(N)$ and K is a proper subgroup of G covering $M_0/O(M)$. Then $E \subseteq O(H) \cap O(K)$ if and only if*

$$E = \langle E \cap O(C_G(x)) \cap O(C_G(y)) \mid x \in X^{\#}, y \in Y^{\#} \rangle.$$

Proof. Without loss of generality we can assume that $H = O^2(H)A$ and $K = O^2(K)A$. Set $\bar{H} = H/O(H)$ and $\bar{K} = K/O(K)$. By Lemmas 4.1, 4.2, and 4.3 we have that $\bar{H}$ has a normal subgroup of odd index of the form $\bar{F}_1 \times \bar{F}_2$ where $\bar{X} \subseteq \bar{F}_1$, $\bar{Y} \subseteq \bar{F}_2$, and $\bar{F}_1 \cong N_0/O(N)$ and $\bar{K}$ has a normal subgroup of odd index of the form $\bar{L}_1 \times \bar{L}_2$ where $\bar{X} \subseteq \bar{L}_1$, $\bar{Y} \subseteq \bar{L}_2$, and $\bar{L}_2 \cong M_0/O(M)$. Let F_i be the preimage in H of $\bar{F}_i$, $i = 1, 2$ and let L_j be the preimage in K of $\bar{L}_j$, $j = 1, 2$. We then have $\bar{F}_1 = (F_1 \cap N_O)^{-}$, $\bar{F}_2 = (F_2 \cap M_0)^{-}$, $\bar{L}_1 = (L_1 \cap N_0)^{-}$, and $\bar{L}_2 = (L_2 \cap M_0)^{-}$.

First assume that $E \subseteq O(H) \cap O(K)$ and let $R = E \cap C \cap D$. Set $\bar{C} = C/O(C)$. Since $\bar{C} = (C_0 \cap M_0)^{-} = (C_0 \cap M_0 \cap L_2)^{-}$, it follows that $[\bar{R}, \bar{C}_0]$ is a normal subgroup of odd order in $\bar{C}_0$ and thus, that $\bar{R}$ centralizes $\bar{C}_0$. We conclude that $R \subseteq O(C)$. Now set $\bar{D} = D/O(D)$. Then $\bar{D}_1 = (D_1 \cap N_0 \cap F_1)^{-}$ and $\bar{D}_2 = (D_2 \cap C)^{-}$ and so we see as above that $\bar{R}$ centralizes both $\bar{D}_1$ and $\bar{D}_2$. It follows that $R \subseteq O(D)$. From this we easily conclude that

$$C_E(\langle x, y \rangle) \subseteq O(C_G(x)) \cap O(C_G(y)) \quad \text{for all } x \in X^{\#}, y \in Y^{\#}.$$

Since $E = \langle C_E(\langle x, y\rangle) \mid x \in X^{\#}, y \in Y^{\#}\rangle$, the "only if" part of the lemma is proved.

Next, assume that $E = \langle E \cap O(C_G(x)) \cap O(C_G(y))\rangle$. Set

$$R = O(C) \cap O(D) \cap E.$$

We then see that $\bar{R}$ centralizes $\bar{F}_1$ and $\bar{F}_2$ in $\bar{H}$ and that $\bar{R}$ centralizes $\bar{L}_1$ and $\bar{L}_2$ in $\bar{K}$. It follows that $R \subseteq O(H) \cap O(K)$ and easily conclude that $E \subseteq O(H) \cap O(K)$. This completes the proof of the lemma.

We now select an arbitrary $a \in A$ and set $K = \theta(C_G(a))$. By Lemma 5.1 K has odd order. If for $x \in X^{\#}$ and $y \in Y^{\#}$ we set

$$K_{x,y} = K \cap O(C_G(x)) \cap O(C_G(y)),$$

then $K_{x,y} \lhd C_K(x, y)$ and $K = \langle K_{x,y} \mid x \in X^{\#}, y \in Y^{\#}\rangle$. We shall show that every A-invariant subgroup of K is (X, Y)-generated with respect to the subgroups $K_{x,y}$.

LEMMA 5.3. *For all $x, x' \in X^{\#}$, $y, y' \in Y^{\#}$ we have*

$$C_{K_{x,y}}(x') \subseteq K_{x',y} \quad \text{and} \quad C_{K_{x,y}}(y') \subseteq K_{x,y'}.$$

Proof. By Lemma 5.1, $C_{K_{x,y}}(x'] \subseteq O(C_G(x'))$ and $C_{K_{x,y}}(y') \subseteq O(C_G(y'))$ and the lemma follows easily from this.

LEMMA 5.4 *Every element in $C_K(\langle x, y\rangle)$ inverted by the involutions in both $X - \langle x\rangle$ and $Y - \langle y\rangle$ lies in $K_{x,y}$.*

Proof. Suppose that $k \in C_K(x, y)$ and that k is inverted by the involutions in both $X - \langle x\rangle$ and $Y - \langle y\rangle$. By (i) of Lemma 5.1, $k \in O(C_G(x))$ and then by (iv) of the same lemma, $k \in O(C_G(y))$. It follows that $k \in K_{x,y}$.

LEMMA 5.5. *The elements in $[C(x_K), Y]' \cap C_K(x)$ inverted by the involutions in $Y - \langle y\rangle$ lie in $K_{x,y}$. The elements in $[C_K(y), x]' \cap C_K(x)$ inverted by the involutions in $X - \langle x\rangle$ lie in $K_{x,y}$.*

Proof. For definiteness let $x = x_1$ and $y = y_1$. We then have that

$$[(K \cap C)^-, \bar{Y}] \subseteq (C_0 \cap O(C_G(a))^- \subseteq O(C_{\bar{C}_0}(\bar{a}))$$

which is abelian in $\bar{C} = C/O(C)$. It follows that

$$[K \cap C, Y]' \subseteq O(C).$$

If $g \in [K \cap C, Y]' \cap D$ and g is inverted by y_2, then $g \in O(D)$ by (iv) of Lemma 5.1. Thus we have $g \in K_{x,y}$.

We also see that $[K \cap D)^-, \bar{X}]$ is an X-invariant subgroup of $\bar{D}_1$ of odd order and so it is abelian by the structure of $\bar{D}_1$ in $\bar{D} = D/O(D)$. It follows that $[K \cap D, X]' \subseteq O(D)$. If $g \in [K \cap D, X]' \cap C$ and g is inverted by x_2, then $g \in O(C)$ since g is of odd order and so $g \in K_{x,y}$.

LEMMA 5.6. *If R is an (X, Y)-generated p-subgroup of K for some prime p, then every A-invariant subgroup of R is (X, Y)-generated.*

Proof. We assume that $R \neq 1$, otherwise the lemma is trivial. By Lemmas 4.9, 4.10, and 4.11 we can find a maximal A-invariant p-subgroup P of G containing R and we can find p-local subgroups H and K of G containing PA such that H covers $N_0/O(N)$ and K covers $M_0/O(M)$. Now by Lemma 5.2 we have $R \subseteq O(H) \cap O(K)$ and by the same lemma we conclude that every A-invariant subgroup of R is (X, Y)-generated.

PROPOSITION 5.7. *We have that θ is an A-signalizer functor* on G *and that the group $W = \langle \theta(C_G(a)) \mid a \in A^{\#}\rangle$ is of odd order.*

Proof. Since $\theta(C_G(a))$, $a \in A^{\#}$, is of odd order, we need only verify the balance condition. Choose $a, b \in A^{\#}$ and set $K = \theta(C_G(a))$. Then Lemmas 5.3–5.6 show that K satisfies conditions (a)–(d) of Proposition 2.1 of [6]. It follows by that proposition that every A-invariant subgroup of K is (X, Y)-generated. Since $K \cap C_G(b)$ is A-invariant, we conclude that

$$K \cap C_G(b) = \langle K_{x,y} \cap C_G(b) \mid x \in X^{\#}, y \in Y^{\#}\rangle \subseteq \theta(C_G(b)).$$

Thus θ is an A-signalizer functor on G and the second part of the lemma is a consequence of the main result of Goldschmidt's paper [4].

6. A strongly imbedded subgroup

In this section we will show that $N_G(W)$ is a strongly imbedded subgroup of G if $W \neq 1$ where W is the group defined in Proposition 5.7. We retain the notation of the preceding sections and we set $G^* = N_G(W)$.

If H is a proper subgroup of G containing A and covering $N_0/O(N)$, then by Lemmas 4.1, 4.2, and 4.3 we conclude that H has a normal subgroup F containing $O(H)$ such that $X \subseteq F$ and $\bar{F} = (F \cap N_0)^- \cong N_0/O(N)$ in $\bar{H} = H/O(H)$. Similarly, if K is a proper subgroup of G containing A and covering $M_0/O(M)$, then K has a normal subgroup L containing $O(K)$ such that $Y \subseteq L$ and $\bar{L} = (L \cap M_0)^- \cong M_0/O(M)$. We shall use these facts several times in this section. We also note here that $W = \langle O(C_G(x)) \cap O(C_G(y)) \mid x \in X^{\#}, y \in Y^{\#}\rangle$.

LEMMA 6.1. *We have that*

$$\langle N_G(A), O(M), O(N), O(C_G(y)) \mid y \in Y^{\#}\rangle \subseteq G^*.$$

Proof. Since the subgroups $O(C_G(x)) \cap O(C_G(y))$, $x \in X^{\#}$ and $y \in Y^{\#}$, are permuted among themselves by $N_G(A)$, we see that $N_G(A) \subseteq G^*$.

Let E be an A-invariant Sylow p-subgroup of $O(M)$ and let $E = E_0 E_1' E_2' E_3'$ be the Y-decomposition of E. If $g \in E_1'$ then $\bar{g}$ centralizes $\bar{Y}$ in $\bar{D} = D/O(D)$. Since g is inverted by y_2, we see that $g \in O(D)$ and it follows that $g \in W$. We then see that $E_i' \subseteq G^*$, $i = 1, 2, 3$. Since $E_0 \subseteq N_G(A)$, we conclude that $E \subseteq G^*$. It follows that $O(M) \subseteq G^*$. Recalling that if $g \in O(N) \cap O(C_G(x))$

where $x \in X^{\#}$, then $g \in O(C_G(y))$ for all $y \in Y^{\#}$, we may use a similar argument to show that $O(N) \subseteq G^*$.

Now let R be an A-invariant Sylow p-subgroup of $O(D)$. This time we let $R = R_0 R_1' R_2' R_3'$ be the X-decomposition of R. Suppose that $g \in R_1'$. Since g is of odd order and is inverted by x_2, we see that $g \in O(C)$. It follows that $R_i' \subseteq W \subseteq G^*$, $i = 1, 2, 3$. Now suppose that $g \in R_0$ and is inverted by y_2. If $\bar{M} = M/O(M)$, then we see that $\bar{g} \in (M_0 \cap O(D))^- \subseteq Z(C_{\bar{M}_0}(\bar{y}_1))$. Since y_2 inverts g, we conclude that $g \in O(M)$. It now follows that $R_0 \subseteq G^*$ and thus, that $R \subseteq G^*$. It then follows that $O(C_G(y)) \subseteq G^*$, $y \in Y^{\#}$.

Now set $W_{x,y} = O(C_G(x)) \cap O(C_G(y))$, $x \in X^{\#}$, $y \in Y^{\#}$. Then

$$W = \langle W_{x,y} \mid x \in X^{\#}, y \in Y^{\#} \rangle$$

and so W is (X, Y)-generated. We then have

LEMMA 6.2. *Every A-invariant subgroup of W is (X, Y)-generated.*

Proof. We again use Proposition 2.1 of [6]. Condition (iv) follows by the proof of Lemma 5.6 and conditions (i) and (ii) follow by the proofs of Lemmas 5.3 and 5.4. Thus we need only verify condition (iii) to prove our lemma.

Suppose that $u \in [C_W(x), Y]' \cap C_G(y)$ where $x \in X^{\#}$, $y \in Y^{\#}$ and suppose that u is inverted by the involutions in $Y - \langle y \rangle$. For definiteness let $x = x_1$ and $y = y_1$. We then have $[C \cap W, Y]$ is a subgroup of odd order in C_0 which is normalized by $N_{C_0}(A)$. It follows by Lemma 2.7 that

$$[(C \cap W)^-, \bar{Y}] \subseteq C_{\bar{C}_0}(\bar{Y})$$

and so is abelian in $\bar{C} = C/O(C)$ and thus $[C \cap W, Y]' \subseteq O(C)$. Since y_2 inverts u, we have by Lemma 5.1 that $u \in O(D)$ and so $u \in W_{x,y}$.

Now suppose that $u \in [D \cap W, X]' \cap C$ and that u is inverted by the involutions in $X - \langle x_1 \rangle$. Since the order of u is odd, $u \in O(C)$. Since $[(D \cap W)^-, \bar{X}]$ is an X-invariant subgroup of $\bar{D}_1$ of odd order in $\bar{D} = D/O(D)$, it is abelian. It follows that $[D \cap W, X]' \subseteq O(D)$ and that $u \in W_{x_1, y_1}$. Thus condition (iii) of Proposition 2.1 of [6] is verified and this lemma is a direct consequence of that proposition.

We now introduce a concept defined in [6]. We then prove a result which gives a sufficient condition for the existence of a p-local subgroup J of G which covers both $N_0/O(N)$ and $M_0/O(M)$ and which contains A. Let H be a subgroup of G which contains A. We say that H is *(X, p)-constrained* if X does not centralize any Sylow p-subgroup of $O(H)$ and we say that H is *(Y, p)-constrained* if Y does not centralize any Sylow p-subgroup of $O(H)$.

We recall that p_2 divides q_2 if $M_0/O(M) \cong L_3(q_2)$. We then have

LEMMA 6.3. *Let p be an odd prime such that $p \neq p_2$. If C is (Y, p)-constrained and if D is (X, p)-constrained, then for some maximal A-invariant p-subgroup P of G we have $N_G(Z(J(P)))$ covers both $M_0/O(M)$ and $N_0/O(N)$.*

Proof. By our assumptions Sylow p-subgroups of both $O(C)$ and $O(D)$ are nontrivial. By Lemma 4.10 we can find a p-local subgroup H of G containing A and covering $N_0/O(N)$ such that H also contains a maximal A-invariant p-subgroup P of G. By the proof of that lemma we can assume that P contains an A-invariant Sylow p-subgroup R_1 of $O(D)$. Then H contains a normal subgroup F such that $XO(H) \subseteq F$ and $\bar{F} = (F \cap N_0)^-$ in $\bar{H} = H/O(H)$ and $\bar{F} \cong N_0/O(N)$. Without loss we can assume that $H = FPA$. Let $H_1 = FP$ and let $Q = P \cap O(H_1)$. Since $R_1 \subseteq Q$, we conclude that H is (X, p)-constrained and so $O_{p'}(H_1) \subseteq O(H_1)$. As in section 5 of [6], we have that H_1 is p-stable with respect to P and by the extended form of Glauberman's ZJ-theorem we have that $N_{H_1}(Z(J(P)))$ covers $H_1/O(H_1)$ and hence, covers $N_0/O(N)$.

By Lemma 4.9 we can find a p-local subgroup K of G containing PA and covering $M_0/O(M)$. Then K has a normal subgroup L containing $YO(K)$ such that $\bar{L} = (L \cap M_0)^- \cong M_0/O(M)$ in $\bar{K} = K/O(K)$. Without loss of generality we can assume that $K = LPA$. Since P is a maximal A-invariant p-subgroup of G, we can also assume that P contains an A-invariant Sylow p-subgroup R_2 of $O(C)$. Let $V = P \cap O(K) = P \cap O(LP)$. Then $R_2 \subseteq V$. Suppose that $O_{p'}(LP) \not\subseteq O(LP)$. Then $O_{p'}(LP)$ has even order and so we can assume that $Y \subseteq O_{p'}(LP)$. But then $[Y, R_2] \subseteq R_2 \cap O_{p'}(LP) = 1$ and this contradicts our assumption that C is (Y, p)-constrained. We may now apply Lemma 2.6 to conclude that $N_{LP}(Z(J(P)))$ covers $LP/O(LP)$ and hence covers $M_0/O(M)$. This then completes the proof of the lemma.

The next proposition incorporates many ideas found in Section 5 of [6].

PROPOSITION 6.4. *Let p be a prime divisor of the order of W. If R is an A-invariant Sylow p-subgroup of W, then $N_G(R)$ covers $M_0/O(M)$.*

Proof. We assume, by way of contradiction, that the proposition is false. The proof is then broken into a number of steps.

By Lemma 6.2, R is (X, Y)-generated and so by Lemmas 4.9, 4.10, and 4.11 we can find a maximal A-invariant p-subgroup P of G containing R and we can find p-local subgroups H and K of G containing PA such that H covers $N_0/O(N)$ and K covers $M_0/O(M)$. As we have seen previously, H contains a normal subgroup F such that $XO(H) \subseteq F$ and $\bar{F} = (F \cap N_0)^- \cong N_0/O(N)$ in $\bar{H} = H/O(H)$ and K contains a normal subgroup L such that $YO(K) \subseteq L$ and $\bar{L} = (L \cap M_0)^- \cong M_0/O(M)$ in $\bar{K} = K/O(K)$. Without loss of generality we can assume that $H = FPA$ and that $K = LPA$. We may also choose H and K such that the orders of $O_p(H)$ and $O_p(K)$ are maximal. If $Q = P \cap O(H)$ and $V = P \cap O(K)$, then we have Q is a Sylow p-subgroup of $O(H)$ and by the maximality of $O_p(H)$, $Q \lhd H$, and we have V is a Sylow p-subgroup of $O(K)$ and also $V \lhd K$. By Lemma 5.2 we conclude that $R = Q \cap V$ is a Sylow p-subgroup of $O(H) \cap O(K)$.

(a) *We have RA is not contained in any proper subgroup J of G such that J covers both $M_0/O(M)$ and $N_0/O(N)$.*

Proof. Suppose there is such a J. Then by Lemma 5.2 we have $R \subseteq O(J)$ and $O(J) \subseteq W$ and so R is a Sylow p-subgroup of $O(J)$. It follows that $N_{J \cap M_0}(R)$ covers $(J \cap M_0)O(J)/O(J)$ by the Frattini argument and hence, covers $M_0/O(M)$, contrary to our assumption.

(b) *We have that* $p \neq p_2$.

Proof. Set $E = N_{M_0}(R)$. If $p = p_2$, then E contains a maximal Y-invariant p-subgroup $P \cap M_0$ of M_0 since $R \lhd P$. By the Frattini argument $N_{M_0}(W)W = EW$ and so in $\bar{M}_0 = M_0/O(M)$, $\bar{E}$ contains a subaroup $S \cong S_4$. By Lemma 2.5 we conclude that $\bar{M}_0 = \bar{E}$, a contradiction. This proves (b). We shall retain the notation $E = N_{M_0}(R)$.

(c) *We have that* Y *does not centralize* V.

Proof. If Y centralizes V, then $C_{L \cap M_0}(V)$ covers $L/O(K)$ and hence' covers $M_0/O(M)$. Since $R \subseteq V$, this is a contradiction.

(d) *We have that* X *centralizes* Q *and that* $V \not\subseteq Q$.

Proof. Suppose that X does not centralize Q. Then H is (X, p)-constrained and as in the proof of Lemma 6.3, we conclude that $N_H(Z(J(P)))$ covers $F/O(H)$ and hence, covers $N_0/O(N)$. Since Y does not centralize V by (c), we see that K is (Y, p)-constrained and since $p \neq p_2$ by (b), we also conclude that $N_K(Z(J(P)))$ covers $L/O(K)$ and hence, covers $M_0/O(M)$. Since $RA \subseteq N_G(Z(J(P)))$, this contradicts (a). Thus X centralizes Q.
If $V \subseteq Q$, then $R = V$ and so $R \lhd K$, a contradiction.

(e) *We have that* X *centralizes* V *and* P.

Proof. Suppose that $V_1 = [V, X] \neq 1$. Since X centralizes Q, we have that $C_H(Q)$ covers $F/O(H)$ and also that $V_1 \subseteq C_H(Q)$. Moreover, we have $V_1 \subseteq Q$. Since $C_P(X)$ covers $\bar{P}$ in $\bar{K}$, we have that $[P, X] = V_1$ and so $\bar{V}_1$ is a Sylow p-subgroup of $C_{\bar{F}}(\bar{x})$ for some $x \in X^{\#}$ in $\bar{H}$. Then $C_{V_1}(X) \subseteq Q$. Since $V_1 \lhd P$ and R centralizes X, we see that R normalizes $C_{V_1}(X)$. But $L \cap M_0$ and $C_H(Q)$ both normalize $C_{V_1}(X)$ and this contradicts (a), if $C_{V_1}(X) \neq 1$. It follows that $V_1 \cap Q = 1$.

Set $P_1 = C_p(X)$ and so $P = P_1 V_1$. Since $V_1 \neq 1$, we have $F = L_2(q_1)$, q_1 odd and $q_1 \geq 5$. It follows that $\bar{F} \cap \bar{P}_1 = 1$. If $\bar{P}_1 = 1$, then $P_1 = Q$ and $R = C_V(X)$ is normal in $L \cap M_0$, a contradiction. Thus $\bar{P}_1 \neq 1$ and so $C_{\bar{F}}(\Omega_1(P_1)) \cong L_2(q)$ where $q^p = q_1$ and $C_{\bar{P}_1}(\bar{V}_1) = 1$ by the proof of (v) of Lemma 2.4.2 of [2]. It follows that $C_{P_1}(V_1) = Q$.
Set $V^* = C_{V \cap P_1}(V_1)$. Then we have $V^* \lhd V_1 P_1 = P$. Also we have $V_1 \cong \bar{V}_1$ since $V_1 \cap Q = 1$. We claim that $V^* \neq 1$. Now Y centralizes V_1 and since $V = C\bar{V}(X)V_1$, we conclude that for all $y \in Y^{\#}$, y does not centralize $C_V(X)$. We can also find a 3-element $u \in L \cap M_0$ which permutes the involutions in Y cyclically and so $\langle u \rangle Y$ acts on $C_V(X)$. Now as in the proof

of Lemma 5.10 in [6] we conclude that $C_V(X)$ contains an elementary abelian subgroup of order p^3. Since $C_V(X)$ normalizes V_1 and V_1 is cyclic, we see that $V^* = C_{C_V(X)}(V_1) \neq 1$ as asserted. Now $L \cap M_0$ normalizes V^* and since $V^* \subseteq Q$, $C_H(Q)$ centralizes V^*, and R normalizes V^*, we have contradicted (a). This forces $V_1 = 1$ and (e) is proved.

(f) *We have* $F \cong L_2(q_1)$, q_1 *odd*, $q_1 \geq 5$ *and* $\bar{P} \cap \bar{F} = 1$ *in* $\bar{H} = H/O(H)$.

Proof. To prove (f) it is sufficient to show that $F \not\cong A_7$, since we already have that $F \not\cong Z_2 \times Z_2$. Suppose then that $F \cong A_7$. Then $\bar{V} = \bar{P}$ is of order 3 in $\bar{F}$. Without loss of generality we can assume that $S_1 \subseteq C_H(Q)$ and we also have that $S_1 \cong D_8$. It follows that S_1 normalizes $C_G(X)$ and centralizes $M/O(M)$. We can also assume that S_1 acts on $\bar{P}$ in $\bar{F}$ and thus that S_1 normalizes P. By our maximal choice of $O_p(K)$ we must have that V is a Sylow p-subgroup of $O(M)$ and so S_1 also normalizes $V = P \cap O(M)$. We then have $C_V(S_1) \subseteq Q \cap V \subseteq C_V(S_1)$ and so $R = C_V(S_1)$. Since $C_G(S_1) \cap N_{M_0}(V)$ covers $M_0/O(M)$ and normalizes R, we have a contradiction. This proves (*f*).

Now as in the proof of Lemma 5.13 of [6], we can find an A-invariant t-subgroup T^* where t is an odd prime distinct from p such that $T^* \subseteq C_F(QY\langle x\rangle)$ for some $x \in X^\#$, T^* is permutable with P, and $[T^*, X] = T^*$. For definiteness let $x = x_1$. We also have $C_{\bar{P}}(\bar{T}^*) = 1$ in $\bar{H}$ and so $C_P(T^*) \subseteq Q$. Since $T^* = [T^*, X]$, $T^* \subseteq C_H(Q)$ and so $Q = C_P(T^*)$.

As we have seen above, V is a Sylow p-subgroup of $O(M)$ and for the same reason we have V is a Sylow p-subgroup of $O(C_G(x))$ for all $x \in X^\#$. Since $T^* = [T^*, X]$, we have $T^* \subseteq O(C)$ and so we can find an A-invariant Sylow t-subgroup T of $O(C)$ containing T^* and permutable with P. Then T is also permutable with V since $VT = PT \cap O(C)$. Thus we see that VT is a Hall-subgroup of $O(C)$. We set $I = [T, X]$ and see that $I = [TV, X] \lhd TV$ and $I \neq 1$ since $T^* \subseteq I$.

(g) *We have that* $C_V(I) = 1$.

Proof. Set $V_1 = C_V(I)$ and assume that $V_1 \neq 1$. Since $N_C(VT)$ covers $C_0/O(C)$, we have $J = N_{M_0}(VT)$ covers $M_0/O(M)$. Also we have $I \lhd J$ and $V \subseteq J$ and since V is a Sylow p-subgroup of $O(M)$, V is a Sylow p-subgroup of $O(J)$. By the Frattini argument $J_1 = N_J(V)$ covers $J/O(J)$ and hence, covers $M_0/O(M)$. We also have $V_1 \lhd J_1$. Since $T^* \subseteq I$ and $Q = C_P(T^*)$, we have $V_1 \subseteq Q$ and so $C_H(V_1)$ covers $N_0/O(N)$. Since R normalizes I, this contradicts (a). Thus $V_1 = 1$ and this proves (g).

(h) *If* $I = I_0 I_1' I_2' I_3'$ *is the* Y*-decomposition of* I, *then* $I_i' \subseteq O(C_G(y_i))$ *and* X *does not centralize* I_i' *for each* $i = 1, 2, 3$.

Proof. First we show that Y does not centralize I. Set $J = IVY$. If $Y \subseteq C_J(I)$, then $[V, Y] \subseteq C_J(I)$ because $C_J(I) \lhd J$. Since $[V, Y] \neq 1$, this contradicts (g). Thus Y does not centralize I. Since $N_{M_0}(I)$ covers $M_0/O(M)$ we can find a 3-element which cyclically permutes the involutions in Y and which is contained in $N_{M_0}(I)$. This element then cyclically permutes I_i', $i = 1, 2, 3$. Since $I_i' \subseteq O(C)$, we conclude that $I_i' \subseteq O(C_G(y_i))$, $i = 1, 2, 3$ by Lemma 5.1. If X centralizes I_i', $i = 1, 2, 3$, then $[I, X] \subseteq I_0 = C_I(Y)$. Since $I = [I, X]$, this is a contradiction. It follows that X does not centralize I_i' for each $i = 1, 2, 3$.

(i) *There is a maximal A-invariant t-subgroup U of G permutable with V and containing I and there is a t-local subgroup J of G covering $M_0/O(M)$ and containing UVA. If $t \in \mathcal{S}_2$, and if U^* is any maximal A-invariant t-subgroup of G, then a conjugate of U^* by a suitable element in $N_G(A)$ has the properties of the preceding sentence.*

Proof. Let T_0 be a maximal A-invariant t-subgroup of C containing T such that T_0 is permutable with V. If $t \in \mathcal{S}_2$, we can choose T_0 such that $[T_0, Y] \subseteq O(C)$ or we can choose T_0 such that $[T_0, Y] \not\subseteq O(C)$. Sinc T_0 covers a maximal Y-invariant t-subgroup of $M_0/O(M)$, we see by Proposition 4.8 that in order to prove (i) it is sufficient to show that V is permutable with a maximal A-invariant t-subgroup U of G containing T_0 such that UVA is contained in a t-local subgroup of G which covers $M_0/O(M)$.

Now $N_C(I)$ contains T_0VA and covers $M_0/O(M)$. Among all t-local subgroups of G containing T_0VA and covering $M_0/O(M)$ choose J such that an A-invariant t-subgroup T_1 of J containing T_0 and permutable with V has maximal order and relative to this, choose J such that an A-invariant t-subgroup of J containing T_1 has maximal order.

We first show that T_1 is a maximal A-invariant t-subgroup of J. Without loss of generality we can assume that $J = O^2(J)A$. By Lemmas 4.1, 4.2, and 4.3 J has normal subgroups L_1 and L_2 such that $XO(J) \subseteq L_1$, $YO(J) \subseteq L_2$, and in $\bar{J} = J/O(J)$ we have $\bar{L}_1\bar{L}_2 = \bar{L}_1 \times \bar{L}_2$ is of odd index and $\bar{L}_2 = (L_2 \cap M_0)^- \cong M_0/O(M)$. Since $V \subseteq O(M)$, we see that $\bar{V}$ centralizes $\bar{L}_2$ and so $\bar{V} = (C_V(A))^-$. Let T_2 T_1 where T_2 is a maximal A-invariant t-subgroup of J. Since L_1 is a 2-group or $L_1 \cong L_2(q)$ or $PGL(2, q)$, q odd, we conclude that $[\bar{T}_2, \bar{X}]$ is a characteristic subgroup of $C_{\bar{L}_1}(\bar{x})$ for some $x \in X^{\#}$. It follows that $\bar{V}$ normalizes $[\bar{T}_2, \bar{X}]$ and so $V \subseteq J_0$ where

$$J_0 = L_2 T_2 O(C_J(A))A.$$

If $\bar{J}_0 = J_0/O(J_0)$, then X centralizes $O^2(\bar{J}_0)$ and since $V \subseteq O(M)$, we conclude that $V \subseteq O(J_0)$. We claim that V is a Sylow p-subgroup of $O(J_0)$. Suppose V is contained in an A-invariant p-subgroup V_1 of $O(J_0)$. Since X centralizes P, we conclude that X centralizes every A-invariant p-subgroup of G and in particular, X centralizes V_1. We then have that V_1 centralizes

$$(J_0 \cap M_0)O(M)/O(M) = M_0/O(M)$$

and hence, $V_1 \subseteq O(M)$. It follows that $V_1 = V$ and that V is a Sylow p-subgroup of $O(J_0)$. We then conclude that V is permutable with a conjugate T_2^j of T_2 containing T_1 where $j \in N_{J_0}(A)$ and by our maximal choice of T_1 we have $T_1 = T_2^j = T_2$. It follows that T_1 is a maximal A-invariant t-subgroup of J as asserted. We can now assume without loss that $J = L_2 T_1 VA$.

Suppose that T_1 is contained in an A-invariant t-subgroup U of G. We shall show that U must equal T_1 and this will then show that T_1 is a maximal A-invariant t-subgroup of G and complete the proof of (i). Assume, by way of contradiction, that T_1 is properly contained in U; we may choose U such that $T_1 \lhd U$. Then $C_U(x) \nsubseteq T_1$ for some $x \in X^{\#}$. Since $T_0 \subseteq T_1$ and T_0 is a maximal A-invariant t-subgroup of C, we conclude that $x \neq x_1$. Since X centralizes $J = J/O(J)$, we see that $I \subseteq O(J)$. Set $I_0 = T_1 \cap O(J)$ and set $I_1 = [I_0, x]$. Since $I \subseteq I_1$, we have $I_1 \neq 1$. By the Frattini argument $N_J(I_0)$ covers J and so $N_{M_0}(I_0)$ covers $M_0/O(M)$. We also have $[T_1 V, x] = [T_1, x] = [I_0, x] = I_0$ and so V normalizes I_1. Since $I_1 = [T_1 C_U(x), x]$, we have $I_1 \lhd T_1 C_U(x)$. Finally, we have $I_1 \lhd N_{M_0}(I_0)$ and since $T_1 \subset T_1 C_U(x)$, we have contradicted our original choice of J. This contradiction completes the proof of (i).

(j) *There is a maximal A-invariant t-subgroup U of G containing I and permutable with V such that $N_G(Z(J(U)))$ contains UVA and covers $N_0/O(N)$ and such that UVA is contained in a t-local subgroup J of G which covers $M_0/O(M)$.*

Proof. By (h) we see that X does not centralize any Sylow t-subgroup of $O(D)$ and so by Lemma 4.10 we can find a t-local subgroup H_0 of G containing A and a maximal A-invariant t-subgroup U_0 of G. Without loss we can assume that $H_0 = F_0 U_0 A$ where $F_0 \lhd H_0$ and $F_0 \supseteq XO(H_0)$ and $\bar{F}_0 = (F_0 \cap N_0)^- \cong N_0/O(N)$ in $\bar{H}_0 = H_0/O(H_0)$. By Propositions 4.5 and 4.8, a conjugate U of U_0 by an element in $N_G(A)$ satisfies the conclusions of (i). Without loss of generality we can assume that $U = U_0$. If $I = I_0 I_1' I_2' I_3'$ is the Y-decomposition of I, we have by (h) that $I_i' \subseteq O(C_G(y_i))$ and hence, $I_i' \subseteq O(H_0)$. Also by (h) we conclude that H_0 is (X, t)-constrained. Argueing as in Lemma 6.3, we conclude that $N_{H_0}(Z(J(U)))$ covers $\bar{H}_0$ and hence, covers $M_0/O(M)$. To complete the proof it remains to show that V normalizes $Z(J(U))$. Since $I \subseteq U$ and $C_V(I) = 1$, $O_p(UV) = 1$. By Glauberman's ZJ-theorem we have $Z(J(U))$ is normal in UV and this completes the proof.

(k) *We have $t = p_2$.*

Proof. Let U and J be as in the conclusion of (j). We assume, by way of contradiction, that $t \neq p_2$. As we have seen before, J has a normal subgroup L_2 containing $YO(J)$ such that $\bar{L}_2 = (L_2 \cap M_0)^- \cong M_0/O(M)$ in $\bar{J} = J/O(J)$. Without loss of generality we can assume that $J = L_2 UVA$. Since $I \subseteq O(C)$, we conclude that $I \subseteq O(J)$. In the proof of (h) we have seen that Y does not centralize I and it follows that J is (Y, t)-constrained. Since we are assuming that $t \neq p_2$, we can now argue as in Lemma 6.3 to conclude that $N_J(Z(J(U)))$ covers J and hence, covers $M_0/O(M)$. Since V normalizes $Z(J(U))$ and $R \subseteq V$, we have contradicted (a). This proves (k).

Again, let U and J be as in the conclusion of (j) and set

$$J^* = N_M(Z(J(U)))A$$

so that J^* covers $M_0/O(M)$. Set $Z = O(J) \cap O(J^*)$ and so by Lemma 5.2, R is a Sylow p-subgroup of Z. Let U_0 be an A-invariant Sylow t-subgroup of $N_{UZ}(R)$ so that $UZ = U_0Z$. Since U covers a maximal Y-invariant t-subgroup of $M_0/O(M)$ and $\bar{U}_0 = \bar{U}$ in $\bar{J}$, we conclude that U_0 covers a maximal Y-invariant t-subgroup of $M_0/O(M)$. Set $\bar{M}_0 = M_0/O(M)$ and recall that $E = N_{M_0}(R)$. As in the proof of (a) we see that $\bar{E}$ contains a subgroup $\bar{S} \cong S_4$. Since $t = p_2$ and $(U_0 \cap M_0)^-$ is a Y-invariant Sylow t-subgroup of $\bar{M}_0$, we have by Lemma 2.7 that $\bar{E} = \bar{M}_0$, contrary to our original assumption. This contradiction proves our proposition.

Lemma 6.5. *We have $G^* = N_G(W)$ contains $C_G(X)$.*

Proof. We have $C_G(X) = X \times M$ and $M = N_M(A)M_0$. By Lemma 6.1 we see that $N_M(A)$ and $O(M)$ are contained in G^*. Let R be an A-invariant Sylow p-subgroup of W. By the preceding proposition we have $M_0 = N_{M_0}(R)O(M)$. It follows that $R^m \subseteq W$ for all $m \in M_0$ and this implies that $M_0 \subseteq G^*$. This proves the lemma.

Lemma 6.6. *We have $O(C_G(y)) \subseteq O(G^*)O(C_G(A))$ for all $y \in Y^\#$.*

Proof. Since $N_G(A) \subseteq G^*$, it will be sufficient to show that

$$O(D) \subseteq O(G^*)O(C_G(A)).$$

Set $G_0 = O^2(G^*)A$ and $\bar{G}_0 = G_0/O(G_0)$. Then by Lemmas 4.1, 4.2, and 4.3, G_0 has normal subgroups L_1 and L_2 such that $XO(G_0) \subseteq L_1$, $YO(G_0) \subseteq L_2$, $\bar{L}_1 = (L_1 \cap N_0)^-$ and $\bar{L}_1$ is a 2-group or $\bar{L}_1 \cong A_7$, $L_2(q)$, or $PGL(2, q)$, q odd and $\bar{L}_2 = (L_2 \cap M_0)^- \cong M_0/O(M)$. Then $(O(D))^-$ centralizes $\bar{L}_1$ and $(O(D) \cap L_2)^- \subseteq Z(C_{\bar{L}_2}(\bar{y}_1))$ so that $(O(D))^-$ also centralizes $\bar{Y}$. Since $X \subseteq L_1$, we conclude that $C_{O(D)}(A)$ covers $(O(D))^-$ and the lemma follows from this.

Lemma 6.7. *If $g \in C_G(Y)$, then $W^g \subseteq O(G^*)O(C_G(A))$. Also we have $O(G^*) = WC_{O(G^*)}(Y)$.*

Proof. Since $W = \langle W \cap O(C_G(y)) \mid y \in Y^\# \rangle$, we see that

$$W^g \subseteq \langle O(C_G(y)) \mid y \in Y^\# \rangle$$

and so $W^g \subseteq O(G^*)O(C_G(A))$ by the preceding lemma.

Let E be an A-invariant Sylow p-subgroup of W. Since $M \subseteq G^*$, we have

$$C_E(x) \subseteq O(C_G(x)) \quad \text{for all } x \in X^\#.$$

Let $F = C_E(\langle x, y\rangle)$ for some $x \in X^{\#}$, $y \in Y^{\#}$. Also let F^* denote the set of elements in F which are inverted by the involutions in $Y - \langle y\rangle$. By Lemma 5.1 we have $F^* \subseteq O(C_G(y))$ and hence, $F^* \subseteq W$. We then have $F \subseteq WC_{O(G^*)}(Y)$ and it follows that $E \subseteq WC_{O(G^*)}(Y)$. The lemma follows immediately from this.

LEMMA 6.8. *If $y \in Y^{\#}$, then $C_G(y)W$ is a group and*

$$O(C_G(y)W) = O(C_G(y))W.$$

Proof. For definiteness let $y = y_1$. Using Lemma 3.7 we see that

$$D = N_D(A)(D_2 \cap M_0)O(D)(D_1 \cap N_0)$$

and so if $d \in D$, then $W^d \subseteq O(G^*)O(C_G(A))$. Thus we have

$$[W, D] \subseteq O(G^*)O(C_G(A)) = WC_{O(G^*)}(Y)O(C_G(A))$$

by the preceding lemma. It follows that $[W, D]$ is of odd order and is contained in WD. Since $W[W, D]D = WD$, we conclude that WD is a group. We then see that $W \lhd WD$ and so $W \subseteq O(WD)$. Since $O(D)$ is also contained in $O(WD)$, we have $D \cap O(WD) = O(D)$ and it follows that $O(WD) = WO(D)$. This completes the proof.

LEMMA 6.9. *If $g \in C_G(Y)$, then $W^g \subseteq O(G^*)$.*

Proof. Let G_0, $\bar{G}_0$, L_1, and L_2 be as in the proof of Lemma 6.6 and let O denote the intersection of the groups $WO(C_G(y_i))$, $i = 1, 2, 3$. Then O is of odd order in G_0 and $\bar{O}$ centralizes $(L_1 \cap N_0)^- = \bar{L}_1$.

Now $C_{L_2}(y_i)$ contains a subgroup J_i such that $\bar{J}_i \cong SL^{\pm}(2, 3)$ if $\bar{L}_2 \cong M_{11}$, $\bar{J}_i \cong SL^{\pm}(2, q_2)$ if $\bar{L}_2 \cong L_3(q_2)$, or $\bar{J}_i \cong SU^{\pm}(2, q_2)$ if $\bar{L}_2 \cong U_3(q_2)$, $i = 1, 2, 3$. We then have $[\bar{J}_i, \bar{O}]$ is a normal subgroup of odd order in $\bar{J}_i$, because $\bar{J}_i$ char $C_{\bar{L}_2}(\bar{y}_i)$ and it follows that $\bar{O}$ centralizes $\bar{J}_i$, $i = 1, 2, 3$. Since $\bar{L}_2 = \langle \bar{J}_i \mid i = 1, 2, 3\rangle$ by Lemma 2.4, we conclude that $\bar{O}$ centralizes $\bar{L}_2$. It follows that $O \subseteq O(G^*)$ and since $W^g \subseteq O$ by the preceding lemma, this lemma is proved.

LEMMA 6.10. *If $N^* = NW$, then N^* is a group and $O(N^*) = WO(N)$.*

Proof. By the previous lemma we have $[W, N] \subseteq O(G^*)$ and by Lemma 6.7, $O(G^*) = W(O(G^*) \cap N)$. This lemma then follows by a proof similar to that of Lemma 6.8.

LEMMA 6.11. *If $Z = O(N^*) \cap O(G^*)$, then Z contains W and Z is normal in N^*.*

Proof. We first show that $O(N^*) = WC_{O(N)}(X)$. Let E be an A-invariant Sylow p-subgroup of $O(N)$. If $E = E_0E_1'E_2'E_3'$ is the X-decomposition of E, then E_i' has odd order and so $E_i' \subseteq O(C_G(x_i))$, $i = 1, 2, 3$. We then see that E_i' centralizes $\bar{D}_0$ in $\bar{D} = D/O(D)$ and so $E_i' \subseteq O(D)$, $i = 1, 2, 3$.

It follows that $E_i' \subseteq W$, $i = 1, 2, 3$. Thus $O(N) \subseteq WC_{O(N)}(X)$ and so $O(N^*) = WC_{O(N)}(X)$.

Let W^* be the normal closure of W in N^* and set $\bar{N}^* = N^*/W^*$. Also set $J_0 = N_0 O(N^*)$. We then have $O(\bar{J}_0) = (O(N^*))^-$ and $J_0/O(J_0) \cong N_0/O(N)$. But $(O(N^*))^- = (WC_{O(N)}(x))^- = (C_{O(N)}(X))^-$ and so $C_{\bar{J}_0}(O(N^*))^-$ covers $J_0/O(J_0)$. It follows that $C_{\bar{J}_0}(\bar{Z})$ also covers $J_0/O(J_0)$ and since $Z \lhd O(N^*)$, we conclude that $\bar{Z} \lhd \bar{J}_0$. Since $W^* \subseteq Z$, we have $Z \lhd J_0$. Since $N^* = N_N^*(A)N_0 O(N^*)$, we have $Z \lhd N^*$ and the lemma is proved.

LEMMA 6.12. *We have $Z = W$ and so both M and N normalize W.*

Proof. Let G_0 and L_2 be as in the proof of Lemma 6.6. Set $\bar{L}_2 = L_2/W$. Then

$$O(\bar{L}_2) = (O(G^*))^- = (WC_{O(G^*)}(Y))^- = (C_{O(G^*)}(Y))^-$$

and

$$L_2/O(L_2) \cong M_0/O(M).$$

Then $C_{\bar{L}_2}(O(G^*))^-$ covers $L_2/O(L_2)$ and so $C_{\bar{L}_2}(\bar{Z})$ also covers $L_2/O(L_2)$. Since $Z \lhd O(G^*)$ and $W \subseteq Z$, we conclude that $Z \lhd L_2$. Now $N_G(Z)$ contains L_2 which covers $M_0/O(M)$, contains N^* which covers $N_0/O(N)$, and contains A and so by Lemma 5.2 we conclude that $Z \subseteq W$. Thus $Z = W$ and the lemma follows from this.

PROPOSITION 6.13. *We have $W = 1$ and so for all $x \in X^{\#}$, $y \in Y^{\#}$ we have*

$$O(C_G(x)) \cap O(C_G(y)) = 1.$$

Proof. Suppose $W \neq 1$. Since $D = N_D(A)(D_2 \cap M_0)N_0 O(D)$, we have $D \subseteq G^*$ and so $C_G(y) \subseteq G^*$ for all $y \in Y^{\#}$. We also see that $C_{O(C)}(y) \subseteq G^*$ for all $y \in Y^{\#}$ and thus, $O(C) \subseteq G^*$. Since $S \subseteq D$ and since $C = SMO(C)$, we have $C \subseteq G^*$ and it follows that $C_G(x) \subseteq G^*$ for all $x \in X^{\#}$. Acting on $O(B)$ with Y, we conclude that $O(B) \equiv G^*$ and it follows that $B \subseteq G^*$. We now see that $C_G(a) \subseteq G^*$ for all $a \in A^{\#}$ and since every involution in S is conjugate in G^* to an involution in A, we conclude that $G^* \subseteq C_G(z)$ for every involution $z \in G^*$. Thus G^* is a strongly imbedded subgroup of G and by a well known argument it follows that G has only one conjugacy class of involutions, a contradiction. Therefore $W = 1$ and the proposition is proved.

7. The proof of the main theorem

In this section we show that our minimal counter-example G satisfies the conclusion of our main theorem. This contradiction then proves that theorem. We retain the notation of the preceding sections.

LEMMA 7.1. *We have $B = C \cap D$.*

Proof. By Lemma 3.9, it is sufficient to show that $x_1 = z_1$ centralizes

$O(B)$. Suppose that this is not the case. Since

$$O(B) = \langle C_{O(B)}(\langle x, y\rangle) \mid x \in X^{\#}, y \in Y^{\#}\rangle \quad \text{for some } x \in X^{\#}, y \in Y^{\#}$$

there is an element g in $C_{O(B)}(\langle x, y\rangle)$ such that $g \neq 1$ and g is inverted by x_1 and hence, by y_1 also. Since g is of odd order, $g \in O(C_G(x))$ and so by Lemma 5.1, $g \in O(C_G(y))$ because g is inverted by y_1. This contradicts Proposition 6.13. Thus $B \subseteq C$ and it follows that $B = C \cap D$.

LEMMA 7.2. *The order of G equals the order of CD and so $G = CD$.*

Proof. For $z = x_1, y_1$, and $x_1 y_1$ let $J(z)$ be the set of all ordered pairs (u, v) such that $u \sim x_1$ and $c \sim y_1$ in G and $z \in \langle uv\rangle$. By a result of Thompson (proven in [7]) we have

$$[G{:}C][G{:}D] = [G{:}C]n(x_1) + [G{:}D]n(y_1) + [G{:}B]n(x_1 y_1]$$

where $n(z)$ denotes the order of $J(z)$, $z = x_1, y_1, x_1 y_1$.

We claim that $n(x_1) = n(y_1) = 0$. Suppose first that $u \sim x_1$, $v \sim y_1$ in G and that $x_1 \in \langle uv\rangle$. Then both u and v are contained in C. If $\bar{C} = C/O(C)$, then $\bar{C} = \bar{S}_1 \times \bar{C}_1$ where $\bar{C}_1$ is as in Lemma 3.6. We then see that $\bar{u} \in \bar{S}_1$ and $\bar{v} \in \bar{C}_1$. Since $(\bar{u}\bar{v})^k = \bar{x}_1$ for some integer k, we must have k odd and it follows that $\bar{u}\bar{v} = \bar{x}_1$. It follows that $\bar{v} \in \hat{S}_1$ and this is a contradiction. Thus $n(x_1) = 0$. Next, suppose that $u \sim x_1$ and $v \sim y_1$ in G and that $y_1 \in \langle uv\rangle$. If $\bar{D} = D/O(D)$, then $\bar{u} \in \bar{D}_1$ and $\bar{v} \in \bar{D}_2$ and it follows that $\bar{u} \in \bar{D}_2$, a contradiction. Thus $n(y_1) = 0$.

Now suppose that $u \sim x_1$ and $v \sim y_1$ in G and that $x_1 y_1 \in \langle uv\rangle$. We claim that $u = x_1$ and $v = y_1$. If $\bar{B} = B/O(B)$, then argueing as above we have $\bar{u} \in \bar{S}_1$ and $\bar{v} \in \bar{B}_1$ and it follows that $\bar{u}\bar{v} = \bar{x}_1\bar{y}_1$. We then see that $\bar{u} = \bar{x}_1$ and $\bar{v} = \bar{y}_1$. Since $B = C \cap D$, we conclude that $u = x_1$ and $v = y_1$. It follows that $n(x_1 y_1) = 1$ and that $|G| = |C||D|/|B| = |CD|$.

We are now in a position to complete the proof of our main theorem. Let F be the normal closure in G of x_1 and let L be the normal closure in G of y_1. By the preceding lemma, $F \subseteq D$ and $L \subseteq C$. It follows that $F \subseteq D_1$ and that $F/O(F) \cong D_1/O(D)$. We also have $L \subseteq C_0$ and $L/O(L) \cong C_0/O(C)$. Since $O(G) = 1$, we have $O(F) = O(L) = 1$ and since $F \cap L$ has odd order we see that $FL = F \times L$. Since the index of FL in G is odd, we conclude that G satisfies the conclusions of our theorem and this is contrary to our choice of G. This then proves our main theorem.

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