BLOCK EXTENSIONS

BY

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Authors often tell us that their fictitious characters have wills of their own, and that they can grow and develop, during the writing of a long novel, in ways altogether unforeseen when the work was begun. The present article has some of the characteristics of these stubbornly independent literary creations. It started out as a simple observation—now quite buried in Theorem 10.20 below—that Brauer's First Main Theorem about blocks led to an isomorphism between certain group extensions associated with those blocks, an isomorphism which could be used as a reduction technique in the study of outer automorphisms of finite groups. During the initial write-up of this observation it developed that these group extensions behaved as if they were Clifford extensions $H[B]^*$ for blocks B of normal subgroups K of finite groups H, in the sense that the blocks of H lying over B could be computed from those of the twisted group algebra of $H[B]^*$. Furthermore, the original isomorphism became only a step in a reduction process paralleling Brauer's well-known analysis of blocks [1], a process yielding a reasonably simple formula for the Clifford extension $H[B]^*$ for the block B in terms of an ordinary Clifford extension for any of the conjugacy class of irreducible characters corresponding to B in Brauer's theory. Obviously one couldn't discuss either blocks or Brauer's analysis without a thorough study of defect groups, culminating in a method for computing the defect groups of a block of H lying over B from the defect groups of a corresponding block of the Clifford extension $H[B]^*$. Finally, the whole theory had to be put in suitable abstract settings (as in [3]) for the sake of possible generalizations as well as to clarify the actual content of the various theorems. Thus, from minor revisions to complete rewritings, from small improvements to whole new sections, the paper grew and expanded into a fullblown theory of block extensions in which the original observation is all but lost and any connection with outer automorphisms has completely disappeared.

Some of the maladjustments inherent in the manner in which this article grew are still visible in the final result, particularly in the choice of abstract settings. The axioms (2.1) used in the definition of the Clifford extension for a block and in the construction of the associated Clifford correspondence are quite suitable for the purpose, based as they are on the developed theory of [3] and [4]. However, when it came to defect groups and Brauer's analysis, no satisfactory fixed set of axioms was found. Indeed, throughout the part of the paper (§§4–9) devoted to these subjects the hypotheses change from section to section—sometimes even from theorem to theorem—in a most

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disconcerting manner, and the reader will have to pay attention in order to known just which assumptions imply which conclusions.

Perhaps a more detailed discussion of the various settings will help explain some of the problems here. The axioms (2.1) define what is, in the language of [3] and [4], a graded Clifford system $\mathfrak{O}$, $\{\mathfrak{O}_\sigma \mid \sigma \in G\}$ over a valuation ring $\mathfrak{R}$ in an algebraically closed field, with the additional hypothesis that $\mathfrak{O}$ (and hence each $\mathfrak{O}_\sigma$) be a finitely-generated free $\mathfrak{R}$-module. Of course one obtains the original situation by letting $\mathfrak{O}$ be the group ring $\mathfrak{R}H$ of H over $\mathfrak{R}$, by letting G be the factor group H/K, and by letting $\mathfrak{O}_\sigma$ be $\sum_{\tau\in\sigma} \mathfrak{R}\tau$, for every coset $\sigma \in G = H/K$.

To study defect groups we start in §4 with one of Green's theories [7] which defines them in a very simple setting consisting only of a ring $\mathfrak{D}$ with identity and a finite group E of automorphisms of $\mathfrak{D}$. His idea that defect groups belong properly to an "outside" operator group E, and not necessarily to any of the groups G, H or K involved with the ring $\mathfrak{O}$, is very useful and we employ it throughout the paper.

Having chosen to define defect groups in an operator group E, we must then decide on which of our rings and groups E is to act, and how its action is to be connected with the other structures of the theory. One obvious idea would be to make E act as $\mathfrak{R}$-automorphisms of $\mathfrak{O}$ in such a way as to permute the direct summands $\mathfrak{O}_\sigma$, $\sigma \in G$, among themselves. The permutations of the $\mathfrak{O}_\sigma$, which modules correspond one-to-one to the elements σ of G, would then define an action of E as automorphisms of the group G. Furthermore, all the other objects used to define Clifford extensions for blocks in §2 would receive actions of appropriate subgroups of E in the correct manner. The problem with this idea is that there is no way to apply it to the usual special case $G = E$. The group $G = H/K$ does not normally act as automorphisms of the ring $\mathfrak{O} = \mathfrak{R}H$! However, we know from §2 of [3] that G does act naturally as automorphisms of the centralizer $\mathfrak{C} = C(\mathfrak{O}_1 \text{ in } \mathfrak{O})$ of $\mathfrak{O}_1$ in $\mathfrak{O}$, a subring which plays an important role in this paper. Furthermore, in view of (3.4) below, all the important steps in the Clifford theory for blocks depend only on the structure of $\mathfrak{C}$ as a G-graded algebra over $\mathfrak{R}$ and on this action of G (so that one *could* dispense altogether with $\mathfrak{O}$ and the $\mathfrak{O}_\sigma$ if one wished to etherealize the subject a bit more). So the "correct" ring for E to act upon is $\mathfrak{C}$ and not $\mathfrak{O}$.

The natural hypothesis tying the action of E on $\mathfrak{C}$ to the rest of the structures would be to make E act as automorphisms of the group G in such a way that the action of G on $\mathfrak{C}$ as well as the G-grading of $\mathfrak{C}$ remain E-invariant. It turned out, however, that the latter condition alone was sufficient to imply many useful results, including generalizations of the usual method for computing defect groups by means of Sylow subgroups of centralizers of elements (in Proposition 5.10) and, more importantly, of Brauer's First Main Theorem (Theorem 8.7 below). Following the general principle that one shouldn't assume more than one needs to prove one's theorems, we stick to the weaker hypothesis that the G-grading of $\mathfrak{C}$ be E-invariant in the axioms (5.1) for §5 and §8, without worrying about the relations between the actions of G and E on $\mathfrak{C}$.

In other parts of the theory, however, the E-invariance of the action of G on $\mathfrak{C}$ is definitely needed. We introduce it into the axioms (6.1) for §6 by making the actions of E on G and of G on $\mathfrak{C}$ come from an embedding of G as a normal subgroup of E. Then we can prove a general result (Theorem 6.5) which, when specialized to the case $G = E$ in Corollary 6.6, tells us that the G-defect groups of a block $\tilde{B}$ of $\mathfrak{O}$ lying over the block B of $\mathfrak{O}_1$ are simply the G-conjugates of the defect groups in G_B (the stabilizer of B in G) of the G_B-orbit of blocks of the Clifford extension $G[B]^*$ (called $H[B]^*$ above) corresponding to $\tilde{B}$ under the Clifford theory for blocks. The additional axioms (6.1) are also used in §9 to carry out the rest of the Brauer analysis of blocks in our abstract setting, although one almost unrecognizable form of his theory (Theorem 9.5) can be proven under a weaker assumption (9.4).

Even these two sets of axioms are not enough for everything. If you think a bit about the original situation in which $\mathfrak{O} = \mathfrak{R}H$ and $G = H/K$, it becomes clear that what is needed is not a defect group in G of the block $\tilde{B}$ but a defect group in H of that block. So, after having carefully removed H and K from our axioms, leaving only the group ring $\mathfrak{O} = \mathfrak{R}H$ and the factor group $G = H/K$, we must go back and put them in again! This is done in (7.1) and (7.2). Even these axioms, which describe a twisted group algebra $\mathfrak{D}$ of H over a local ring $\mathfrak{D}_1$, together with an embedding of H as a normal subgroup of an operator group E on $\mathfrak{D}$, are slightly more general than those of a group ring. The resulting Theorem 7.3 then tells us how to compute defect groups in this E of E-orbits of blocks of $\mathfrak{O}$ in terms of defect groups of B in E_B and of defect groups in E_B/K of the corresponding E_B-orbit of blocks of the Clifford extension $G[B]^*$, where B is any block of $\mathfrak{O}_1$ lying under some block of the original E-orbit.

Because of the importance of the last result, it is worthwhile explaining in some detail what it becomes in the usual case in which $E = H$ and $\mathfrak{O} = \mathfrak{R}H$. The Clifford extension $G[B]^* = H[B]^*$ for a block B of $\mathfrak{O}_1 = \mathfrak{R}K$ is then a central extension of the multiplicative group $\bar{F}$ of the residue class field $\bar{\mathfrak{F}}$ of the valuation ring $\mathfrak{R}$ by a certain normal subgroup $G[B]$ of $G_B = H_B/K$ (see §2). The stabilizer G_B of B acts naturally as automorphisms of the Clifford extension $G[B]^*$, centralizing $\bar{F}$ and compatible with the projection of $G[B]^*$ onto $G[B] \trianglelefteq G_B$. The Clifford correspondence (Theorem 3.7) is then one-to-one between the blocks $\tilde{B}$ of $\mathfrak{O} = \mathfrak{R}H$ lying over the block B of $\mathfrak{O}_1 = \mathfrak{R}K$ and the G_B-conjugacy classes T of blocks of the twisted group algebra $\bar{\mathfrak{F}}[G[B]^*]$ of the Clifford extension $G[B]^*$.

In the above situation Green's theory gives us defect groups C of the block B in H_B coming from the conjugation action of H_B as automorphisms of the group ring $\mathfrak{O}_1 = \mathfrak{R}K$. It also gives us defect groups $\bar{D}$ in G_B of the G_B-conjugacy class T coming from the action of $G_B = H_B/K$ on the twisted group ring

$\bar{\mathfrak{F}}[G[B]^*]$. In fact, a general result of his theory (Proposition 4.9 below) is that $\bar{D}$ can be chosen to be a defect group in $G_{B,\bar{B}} = (G_B)_{\bar{B}}$ of any block $\bar{B} \in T$. We wish, of course, to compute a defect group $\tilde{D}$ in H of the block $\tilde{B}$. The result, given in Theorem 7.7 below, is the following:

(0.1a) *The factor group CK/K is a p-Sylow subgroup of $H_B/K = G_B$.*

(0.1b) *If C is chosen so that the p-Sylow subgroup CK/K contains the p-subgroup $\bar{D}$ of G_B, then the inverse image $\tilde{D}$ in C of $\bar{D} \leq CK/K$ is a defect group of $\tilde{B}$ in H.*

(0.1c) *Finally, $\tilde{D} \cap K = C \cap K$ is a defect group of B in K.*

Here, of course, p is the characteristic of the residue class field $\bar{\mathfrak{F}}$, which is supposed to be a prime.

As mentioned above, the Brauer analysis of blocks in terms of a conjugacy class of characters of the centralizers of their defect groups can be generalized into a method for computing the Clifford extension for a block in terms of some ordinary Clifford extensions of the corresponding characters. This is discussed in the long Sections 10, 11 and 12 below in the context of the "twisted group algebra over a local ring" of the axioms (7.1). The resulting Theorem 12.3 is also important enough to warrant a detailed explanation here, at least for the usual special case in which $\mathfrak{O} = \mathfrak{R}H$ and $G = H/K$.

Fix a defect group D in K of the block B of $\mathfrak{O} = \mathfrak{R}K$. Brauer's analysis [1] gives us a unique $N_K(D)$-conjugacy class of blocks b of defect zero in $DC_K(D)/D$ corresponding to B. Any such block b contains a unique modular irreducible character φ, which we can regard as an irreducible $\bar{\mathfrak{F}}$-character of $C_K(D)$. Then φ has, as in [3], a Clifford extension $C_H(D)\langle\varphi\rangle$, which is a central extension of $\bar{F}$ by the stabilizer $C_H(D)_\varphi/C_K(D)$ of φ in $C_H(D)/C_K(D)$. Furthermore, the stabilizer $N_H(D)_\varphi$ of φ in the normalizer $N_H(D)$ acts by conjugation on $C_H(D)\langle\varphi\rangle$ centralizing $\bar{F}$ and leaving invariant the projection of $C_H(D)\langle\varphi\rangle$ onto $C_H(D)_\varphi/C_K(D)$.

At this point a curious complication arises due to the fact that the normal subgroup $N_K(D)_\varphi$ of $N_H(D)_\varphi$ need not centralize the Clifford extension $C_H(D)\langle\varphi\rangle$. Of course it must centralize both $\bar{F}$ and the factor group

$$C_H(D)\langle\varphi\rangle/\bar{F} \simeq C_H(D)_\varphi/C_K(D) = C_H(D)_\varphi/(N_K(D)_\varphi \cap C_H(D)_\varphi).$$

But it can still act non-trivially on $C_H(D)\langle\varphi\rangle$, i.e., the bilinear form

$$\omega: N_K(D)_\varphi \times [C_H(D)_\varphi/C_K(D)] \to \bar{F}$$

defined by

(0.2) $(y_\tau)^\sigma = \omega(\sigma, \tau)y_\tau$, *for all* $\sigma \in N_K(D)_\varphi$, $\tau \in C_H(D)_\varphi/C_K(D)$, *and* $y_\tau \in C_H(D)\langle\varphi\rangle$ *such that* τ *is the projection of* y_τ,

can be non-trivial. It turns out that the important subgroup is the "right kernel"

$$C_H(D)_\omega/C_K(D) = \{\tau \in C_H(D)_\varphi/C_K(D) \mid \omega(\sigma, \tau) = 1, \quad \textit{for all} \quad \sigma \in N_K(D)_\varphi\}$$

of this form. Its inverse image $C_H(D)_\omega$ in $C_H(D)_\varphi$ is a normal subgroup of $N_H(D)_\varphi$, and the factor group $C_H(D)_\varphi/C_H(D)_\omega$ is abelian of order relatively prime to the characteristic p of $\bar{\mathfrak{F}}$ (see (11.13) below). Its other inverse image $C_H(D)\langle\varphi\rangle_\omega$ in $C_H(D)\langle\varphi\rangle$ is canonically isomorphic to the Clifford extension $C_H(D)_\omega\langle\varphi\rangle$ of φ in $C_H(D)_\omega$ by §16 of [3]. Furthermore, $C_H(D)_\omega\langle\varphi\rangle$ is normalized by $N_H(D)_\varphi$ and centralized by $N_K(D)_\varphi$. So it is acted upon naturally by the factor group $N_H(D)_\varphi/N_K(D)_\varphi$.

Now we can state the final result. It is obvious from the description of the Clifford correspondence for blocks preceding (0.1) above (or from §3 below) that we need to compute the stabilizer G_B of B in $G = H/K$, its normal subgroup $G[B]$, and its action on the Clifford extension $G[B]^*$ as well as that Clifford extension itself. All of these things are given in terms of the above objects by Corollary 12.6 below which states that:

(0.3a) $G_B = N_H(D)_\varphi K/K$.

(0.3b) $G[B] = C_H(D)_\omega K/K$.

(0.3c) *$G[B]^*$ is naturally isomorphic to $C_H(D)_\omega\langle\varphi\rangle$ as extensions of $\bar{F}$, compatibly with the natural isomorphism of*

$$G[B] = C_H(D)_\omega K/K \quad \text{onto} \quad C_H(D)_\omega/C_K(D) = C_H(D)_\omega/[C_H(D)_\omega \cap K].$$

(0.3d) *The isomorphism in* (0.3c) *and the natural isomorphism of*

$$G_B = N_H(D)_\varphi K/K \quad \text{onto} \quad N_H(D)_\varphi/N_K(D)_\varphi = N_H(D)_\varphi/[N_H(D)_\varphi \cap K]$$

carry the action of G_B on $G[B]^*$ onto the action of $N_H(D)_\varphi/N_K(D)_\varphi$ on $C_H(D)_\omega\langle\varphi\rangle$.

Thus the Clifford extension for blocks and its associated structures are more or less effectively computable.

The rings used in this paper are usually orders over the valuation ring $\mathfrak{R}$ in an algebraically closed field $\mathfrak{F}$ (when they are not algebras over the residue class field $\bar{\mathfrak{F}}$ of $\mathfrak{R}$). We include in §1 a quick theory of such orders $\mathfrak{O}$, showing that they enjoy all the good properties of p-adic orders (such as the ability to lift idempotents and blocks from factor rings, the fact that $1 + J(\mathfrak{O})$ behaves like a p-group, the Krull-Schmidt Theorem for finitely-generated indecomposable modules, etc, etc.) as well as those stemming from the algebraic closure of both $\bar{\mathfrak{F}}$ and $\mathfrak{F}$.

Finally, in §13 we collect various miscellaneous results about the connections between the Clifford extension for a block and the Clifford extensions for the ordinary and modular characters in that block, and about the relations between the Clifford extensions for corresponding blocks of the $\mathfrak{R}$-order $\mathfrak{O}_1$ and its residue class $\bar{\mathfrak{F}}$-algebra $\bar{\mathfrak{O}}_1$.

We have already said that the present article was much rewritten during its gestation. The rewriting, in fact, has continued even after the paper was accepted by the Illinois Journal. Thus the present introduction is newly written in June, 1972, some two years after the rest of the work. At the same time the numbering of statements and theorems has been almost completely changed as a result of a new division into 13 sections in place of the original 6. In a further (probably vain) attempt to improve its readability, an index of symbols and definitions has been added to the end of the paper.

One notational convention must be mentioned immediately. The two articles [3] and [4], on which we depend heavily, are denoted in the text by their initials [CCT] and [ICE], respectively. Thus a reference to [CCT, §2] is to section 2 of [3], while [ICE, 1.11] sends the reader to numbered statement 1.11 (in this case a proposition) of [4].

It is obvious from the above descriptions that this whole work is but a minor generalization of that of Professor Richard Brauer, in particular of [1]. It was therefore appropriate that the results in it were first announced at a symposium honoring his 70th birthday (see [5], where the reader will find a further description of the contents below). I wish to thank publicly Professor Brauer for his help and inspiration over the many years of our acquaintance. He has been an excellent teacher and colleague, as well, of course, as the discoverer of practically all the basic results in the theory of blocks. My only regret in our relationship is that I had no interest at all in group theory when I was his student, and hence could only profit partially from his teaching! For these reasons this paper is dedicated to him.

1. The orders

When dealing with projective representations of finite groups, it is very convenient to use only algebraically closed fields. Otherwise one is forever making finite extensions of the ground field to simplify factor sets, obtain absolutely indecomposable modules, or what have you. So we begin by choosing a valuation ring $\mathfrak{R}$ subject only to the condition that its field of fractions $\mathfrak{F}$ be algebraically closed. We denote by $\mathfrak{p}$ the unique maximal ideal of $\mathfrak{R}$ and by $\bar{\mathfrak{F}}$ the residue class field $\mathfrak{R}/\mathfrak{p}$. The fact that $\mathfrak{F}$ is algebraically closed clearly implies:

(1.1) *The residue class field $\bar{\mathfrak{F}}$ is algebraically closed.*

Incidentally we do not exclude the useful case in which $\mathfrak{p} = 0$, and $\mathfrak{R} = \mathfrak{F} = \bar{\mathfrak{F}}$.

Any finitely-generated torsion-free module over the valuation ring $\mathfrak{R}$ is free of finite rank. As usual, we call such modules $\mathfrak{R}$-*lattices*. Evidently any finitely-generated $\mathfrak{R}$-submodule of an $\mathfrak{R}$-lattice $\mathfrak{L}$ is again an $\mathfrak{R}$-lattice, as is any torsion-free factor $\mathfrak{R}$-module $\mathfrak{L}/\mathfrak{K}$. In the latter case, $\mathfrak{K}$ is called a *pure* $\mathfrak{R}$-submodule of $\mathfrak{L}$. In fact, such an $\mathfrak{K}$ is an $\mathfrak{R}$-direct summand of $\mathfrak{L}$ (since $\mathfrak{L}/\mathfrak{K}$ is free), and hence is itself an $\mathfrak{R}$-lattice. The *residue class module* $\bar{\mathfrak{L}} = \mathfrak{L}/\mathfrak{p}\mathfrak{L}$ is a vector space over $\bar{\mathfrak{F}}$ whose finite dimension equals the $\mathfrak{R}$-rank of $\mathfrak{L}$.

An $\mathfrak{R}$-*order* (or simply an *order*) $\mathfrak{O}$ is an associative $\mathfrak{R}$-algebra with identity $1 = 1_{\mathfrak{O}}$ which is an $\mathfrak{R}$-lattice when considered an an $\mathfrak{R}$-module. Then $\mathfrak{p}\mathfrak{O}$ is a two-sided ideal of $\mathfrak{O}$, and the *residue class algebra* $\bar{\mathfrak{O}} = \mathfrak{O}/\mathfrak{p}\mathfrak{O}$ is a finite-dimensional associative algebra with identity over $\bar{\mathfrak{F}}$. The Nakayama Lemma for the finitely-generated $\mathfrak{R}$-module $\mathfrak{O}$ implies that any maximal left ideal of $\mathfrak{O}$ contains $\mathfrak{p}\mathfrak{O}$. It follows that the Jacobson radicals $J(\mathfrak{O})$ and $J(\bar{\mathfrak{O}})$ of $\mathfrak{O}$ and $\bar{\mathfrak{O}}$, respectively, are related by:

(1.2) *$J(\mathfrak{O})$ is the inverse image in $\mathfrak{O}$ of $J(\bar{\mathfrak{O}})$.*

In particular, $\mathfrak{O}/J(\mathfrak{O}) \simeq \bar{\mathfrak{O}}/J(\bar{\mathfrak{O}})$ is a finite-dimensional semi-simple algebra over $\bar{\mathfrak{F}}$. So the family Max $(\mathfrak{O})$ of all maximal two-sided ideals of $\mathfrak{O}$ (which is the inverse image of the corresponding family Max $(\mathfrak{O}/J(\mathfrak{O}))$ for $\mathfrak{O}/J(\mathfrak{O})$) is finite.

Since $\mathfrak{R}$ is not noetherian (unless $\mathfrak{R} = \mathfrak{F}$), not every $\mathfrak{R}$-subalgebra with identity of an $\mathfrak{R}$-order $\mathfrak{O}$ is a suborder of $\mathfrak{O}$. However, most of the subalgebras which occur in the usual constructions are suborders. For example, if S is a subset of $\mathfrak{O}$, then the *centralizer of S in $\mathfrak{O}$*, the set $C(S \text{ in } \mathfrak{O})$ of all $x \in \mathfrak{O}$ such that $xs = sx$, for all $s \in S$, is clearly an $\mathfrak{R}$-subalgebra of $\mathfrak{O}$ containing $1_{\mathfrak{O}}$. Using the fact that $\mathfrak{O}$ is a torsion-free $\mathfrak{R}$-module, one easily sees that $C(S \text{ in } \mathfrak{O})$ is a pure $\mathfrak{R}$-submodule of $\mathfrak{O}$, and hence is an $\mathfrak{R}$-lattice. Therefore:

(1.3) $C(S \text{ in } \mathfrak{O})$ *is a suborder of $\mathfrak{O}$, for any subset S of $\mathfrak{O}$.*

Taking $S = \mathfrak{O}$, we obtain:

(1.4) *The center $Z(\mathfrak{O})$ of $\mathfrak{O}$ is a suborder of $\mathfrak{O}$.*

If e is an idempotent of $\mathfrak{O}$, then $e\mathfrak{O}e$ is an $\mathfrak{R}$-subalgebra with e as its identity. The Peirce decomposition implies that $e\mathfrak{O}e$ is an $\mathfrak{R}$-direct summand of $\mathfrak{O}$, and hence that:

(1.5) *$e\mathfrak{O}e$ is a suborder of $\mathfrak{O}$, for any idempotent e of $\mathfrak{O}$.*

Another trick is to use the $\mathfrak{R}$-linear transformation $T : y \rightarrow xy$ of $\mathfrak{O}$ defined by an element $x \in \mathfrak{O}$. Since $\mathfrak{O}$ is a free $\mathfrak{R}$-module of finite rank, we can define the characteristic polynomial

$$f(X) = \det (X1 - T) = X^n + a_1 X^{n-1} + \cdots + a_n$$

of T in the usual way. Its coefficients $a_1, \cdots, a_n$ all lie in $\mathfrak{R}$. From the Hamilton-Cayley equation $f(T) = 0$, we deduce:

(1.6) $x^n + a_1 x^{n-1} + \cdots + a_n = f(x) = f(x) \cdot 1_{\mathfrak{O}} = [f(T)](1_{\mathfrak{O}}) = 0.$

It follows that the $\mathfrak{R}$-subalgebra $\mathfrak{R}[x]$ of all $\mathfrak{R}$-polynomials in x is generated as an $\mathfrak{R}$-module by $1, x, x^2, \cdots, x^{n-1}$. Hence:

(1.7) *$\mathfrak{R}[x]$ is a suborder of $\mathfrak{O}$, for any $x \in \mathfrak{O}$.*

Equation (1.6) can also be used to compute inverses of units. If x is a unit of $\mathfrak{O}$, then T is an invertible $\mathfrak{R}$-linear transformation of the free $\mathfrak{R}$-module $\mathfrak{O}$ of finite rank. It follows that the determinant $(-1)^n a_n$ of T is a unit of

$\mathfrak{R}$. Writing (1.6) as

$$x(x^{n-1} + a_1 x^{n-2} + \cdots + a_{n-1}) = -a_n ,$$

we obtain

$$x^{-1} = (-a_n)^{-1}(x^{n-1} + a_1 x^{n-2} + \cdots + a_{n-1}) \in \mathfrak{R}[x].$$

Therefore:

(1.8) *If x is a unit of $\mathfrak{O}$, then x^{-1} lies in $\mathfrak{R}[x]$.*

The property (1.8) has the following useful consequence:

PROPOSITION 1.9. *If $\mathfrak{O}'$ is a suborder of $\mathfrak{O}$ (with or without the same identity), then $J(\mathfrak{O}) \cap \mathfrak{O}' \subseteq J(\mathfrak{O}')$.*

Proof. The identity e of $\mathfrak{O}'$ is an idempotent of $\mathfrak{O}$. By (1.5), $e\mathfrak{O}e$ is a suborder of $\mathfrak{O}$ containing $\mathfrak{O}'$. We know that $J(e\mathfrak{O}e) = eJ(\mathfrak{O})e = J(\mathfrak{O}) \cap e\mathfrak{O}e$. So it suffices to prove the proposition for $e\mathfrak{O}e$ and $\mathfrak{O}'$, i.e., we can assume that $1 = 1_{\mathfrak{O}} \in \mathfrak{O}'$.

Evidently $J(\mathfrak{O}) \cap \mathfrak{O}'$ is a two-sided ideal of $\mathfrak{O}'$. So it suffices to show that $1 + y$ is a unit of $\mathfrak{O}'$, for every $y \in J(\mathfrak{O}) \cap \mathfrak{O}'$. But $1 + y$ is a unit of $\mathfrak{O}$, since $y \in J(\mathfrak{O})$, and $(1 + y)^{-1} \in \mathfrak{R}[1 + y] \subseteq \mathfrak{O}'$, by (1.8). Therefore $1 + y$ is a unit of $\mathfrak{O}'$, and the proposition is proved.

The critical property of our orders is that idempotents of the residue class algebra $\bar{\mathfrak{O}}$ can be "lifted" to idempotents of $\mathfrak{O}$. It is convenient to prove this in a mildly stronger form.

PROPOSITION 1.10. *If $\mathfrak{J}$ is a two-sided ideal of $\mathfrak{O}$, then every idempotent e of the factor ring $\mathfrak{O}/\mathfrak{J}$ is the image of an idempotent of $\mathfrak{O}$.*

Proof, First we simplify $\mathfrak{O}$. Let x be any element of $\mathfrak{O}$ mapping onto e. Form the polynomial $f(X) = X^n + a_1 X^{n-1} + \cdots + a_n$ of (1.6). Let $\mathfrak{R}[X]$ be the polynomial ring in the variable X over $\mathfrak{R}$, and y be the image of X in the factor ring $\mathfrak{R}[y] = \mathfrak{R}[X]/f(X)\mathfrak{R}[X]$. Evidently $\mathfrak{R}[y]$ is an $\mathfrak{R}$-algebra with identity, and is a free $\mathfrak{R}$-module with $1, y, y^2, \cdots, y^{n-1}$ as a basis. So it is an order. By (1.6) there is an $\mathfrak{R}$-algebra epimorphism of $\mathfrak{R}[y]$ onto $\mathfrak{R}[x]$ sending y onto x. Composing this with the natural map of $\mathfrak{R}[x]$ into $\mathfrak{O}/\mathfrak{J}$, we obtain an $\mathfrak{R}$-algebra epimorphism of $\mathfrak{R}[y]$ onto the subalgebra $\mathfrak{R}1 + \mathfrak{R}e$ of $\mathfrak{O}/\mathfrak{J}$, sending y onto e. If we can find an idempotent $e' \in \mathfrak{R}[y]$ mapping onto e, then its image $e'' \in \mathfrak{R}[x]$ will be an idempotent of $\mathfrak{O}$ mapping onto e. Hence we can assume that $\mathfrak{O} = \mathfrak{R}[y]$ and $\mathfrak{O}/\mathfrak{J} = \mathfrak{R}1 + \mathfrak{R}e$.

Because $\mathfrak{F}$ is algebraically closed and $\mathfrak{R}$ is integrally closed in $\mathfrak{F}$, the polynomial $f(X)$ factorizes completely in $\mathfrak{R}[X]$:

$$f(X) = (X - c_1) \cdots (X - c_n) \quad \textit{for some} \quad c_1, \cdots, c_n \in \mathfrak{R}.$$

Let $\bar{c}_1, \cdots, \bar{c}_n$ be the images of $c_1, \cdots, c_n$, respectively, in $\bar{\mathfrak{F}}$, and let $d_1, \cdots, d_m$ be the distinct members of the set $\{\bar{c}_1, \cdots, \bar{c}_n\}$. For each $j = 1$,

$\cdots, m$, define $f_j(X)$ to be the product of those $X - c_i$, $i = 1, \cdots, n$, for which $\bar{c}_i = d_j$.

If $j, k = 1, \cdots, m$ and $j \neq k$, then the images in $\bar{\mathfrak{F}}[X] = \mathfrak{R}[X]/\mathfrak{p}\mathfrak{R}[X]$ of $f_j(X)$ and $f_k(X)$ are powers of $X - d_j$ and $X - d_k$, respectively. Since $d_j \neq d_k$, they are relatively prime and their resolvant is a non-zero element of $\bar{\mathfrak{F}}$. It follows that the resolvant of $f_j(X)$ and $f_k(X)$ is a unit of $\mathfrak{R}$ and hence that

$$\mathfrak{R}[X] = f_j(X)\mathfrak{R}[X] + f_k(X)\mathfrak{R}[X], \quad \textit{for all} \quad j, k = 1, \cdots, m \quad \textit{with} \quad j \neq k.$$

Because $f(X) = f_1(X) \cdots f_m(X)$, this implies that $\mathfrak{R}[y]$ has the ring decomposition

$$\begin{aligned} \mathfrak{R}[y] &= \mathfrak{R}[X]/f(X)\mathfrak{R}[X] \\ &\simeq \mathfrak{R}[X]/f_1(X)\mathfrak{R}[X] \oplus \cdots \oplus \mathfrak{R}[X]/f_m(X)\mathfrak{R}[X] \qquad (1.11) \\ &= \mathfrak{O}_1 \oplus \cdots \oplus \mathfrak{O}_m, \end{aligned}$$

where $\mathfrak{O}_j$ is the image in $\mathfrak{R}[y]$ of $\mathfrak{R}[X]/f_j(X)\mathfrak{R}[X]$, for each $j = 1, \cdots, m$.

Evidently any $\bar{\mathfrak{O}}_j$ is isomorphic to $\bar{\mathfrak{F}}[X]/\bar{f}_j(X)\bar{\mathfrak{F}}[X]$, where $\bar{f}_j(X)$ is the image in $\bar{\mathfrak{F}}[X]$ of $f_j(X)$. By construction $\bar{f}_j(X)$ is a power of $X - d_j$. Hence

$$\bar{\mathfrak{O}}_j/J(\bar{\mathfrak{O}}_j) \simeq \bar{\mathfrak{F}}[X]/(X - d_j)\bar{\mathfrak{F}}[X] \simeq \bar{\mathfrak{F}}.$$

In view of (1.2), this implies that $\mathfrak{O}_j/J(\mathfrak{O}_j) \simeq \bar{\mathfrak{F}}$ is a field, and hence that $\mathfrak{O}_j$ is a local ring.

The epimorphism of $\mathfrak{R}[y]$ onto $\mathfrak{R}1 + \mathfrak{R}e$ carries the decomposition (1.11) onto the ring decomposition

$$\mathfrak{R}1 + \mathfrak{R}e = \mathfrak{O}_1' \oplus \cdots \oplus \mathfrak{O}_m'$$

where $\mathfrak{O}_j'$ is the image of $\mathfrak{O}_j$, for $j = 1, \cdots, m$. Since $\mathfrak{O}_j$ is a local ring, its image $\mathfrak{O}_j'$ is either a local ring or zero. So the identity e_j' of $\mathfrak{O}_j'$ is either a primitive idempotent of $\mathfrak{R}1 + \mathfrak{R}e$ or zero. Because $\mathfrak{R}1 + \mathfrak{R}e$ is commutative, the above decomposition implies that the non-zero e_j' are its only primitive idempotents and that any idempotent of $\mathfrak{R}1 + \mathfrak{R}e$ is a sum of certain of the e_j'. In particular, e is such a sum. So it is the image of the sum of the identities e_j of the corresponding $\mathfrak{O}_j$, which sum is an idempotent of $\mathfrak{R}[y]$ by (1.11). This completes the proof of the proposition.

An arbitrary idempotent of $\bar{\mathfrak{O}}$ can be the image of many idempotents of $\mathfrak{O}$. However, this is not true for central idempotents.

PROPOSITION 1.12. *If e is a central idempotent of $\bar{\mathfrak{O}}$, then there is a unique idempotent e^* of $\mathfrak{O}$ having e as its image. This idempotent e^* is central in $\mathfrak{O}$. The map $e \rightarrow e^*$ sends the central idempotents of $\bar{\mathfrak{O}}$ one-to-one onto those of $\mathfrak{O}$, and the primitive central idempotents of $\bar{\mathfrak{O}}$ one-to-one onto those of $\mathfrak{O}$.*

Proof. By Proposition 1.10 there is an idempotent e^* of $\mathfrak{O}$ mapping onto e. We first prove that any such e^* is central in $\mathfrak{O}$.

The Peirce decomposition of $\mathfrak{O}$ is

$$\mathfrak{O} = e^*\mathfrak{O}e^* \oplus e^*\mathfrak{O}(1 - e^*) \oplus (1 - e^*)\mathfrak{O}e^* \oplus (1 - e^*)\mathfrak{O}(1 - e^*) \quad (\textit{as } \mathfrak{R}\textit{-modules}).$$

Clearly each summand is an $\mathfrak{R}$-lattice. By the Nakayama Lemma, the summand $e^*\mathfrak{O}(1 - e^*)$ is zero if its image

$$e^*\mathfrak{O}(1 - e^*)/\mathfrak{p}e^*\mathfrak{O}(1 - e^*) \simeq e\bar{\mathfrak{O}}(1 - e)$$

is zero. But $e\bar{\mathfrak{O}}(1 - e) = e(1 - e)\bar{\mathfrak{O}} = 0$, since e is a central idempotent of $\bar{\mathfrak{O}}$. Hence $e^*\mathfrak{O}(1 - e^*) = 0$. Similarly $(1 - e^*)\mathfrak{O}e^* = 0$. This implies that e^* is a central idempotent of $\mathfrak{O}$.

Now let e_1^* be another idempotent of $\mathfrak{O}$ mapping onto e. Both e^* and e_1^* are central idempotents of $\mathfrak{O}$. Hence so are e^*e_1, $e^* - e^*e_1^*$ and $e_1^* - e^*e_1^*$. The images in $\bar{\mathfrak{O}}$ of the last two idempotents are both $e - e^2 = 0$. So (1.2) implies that $e^* - e^*e_1^*$ and $e_1^* - e^*e_1^*$ are idempotents of $\mathfrak{O}$ lying in $J(\mathfrak{O})$. Therefore they are both zero, and $e^* = e^*e_1^* = e_1^*$.

We have now proved the first two statements of the proposition. The rest of it follows directly from these.

The *blocks* of the order $\mathfrak{O}$ are defined by its primitive central idempotents. Into the block B corresponding to such an idempotent e we put the usual assortment of things which can somehow be attached to e and $\mathfrak{O}$, such as the indecomposable ring direct summands $e\mathfrak{O}$ and $eZ(\mathfrak{O})$ of $\mathfrak{O}$ and $Z(\mathfrak{O})$, respectively, or the maximal ideals $\mathfrak{M} \in \operatorname{Max}(\mathfrak{O})$ satisfying $e \equiv 1 \pmod{\mathfrak{M}}$. In view of Proposition 1.12, the map sending e into its image $\bar{e}$ in $\bar{\mathfrak{O}}$ defines a one-to-one correspondence between blocks of $\mathfrak{O}$ and those of $\bar{\mathfrak{O}}$. It is customary to put everything in a block $\bar{B}$ of $\bar{\mathfrak{O}}$ into the corresponding block B of $\mathfrak{O}$. We shall follow this custom as much as possible, but, as we shall see below (in Example 13.15), the Clifford extension for $\bar{B}$ can differ from that for B (although they are the same for group rings). So we cannot completely identify these two blocks.

A *lattice* $\mathfrak{L}$ over the order $\mathfrak{O}$ is a unitary left $\mathfrak{O}$-module which is finitely-generated and torsion-free as an $\mathfrak{R}$-module, i.e., which is also an $\mathfrak{R}$-lattice. Obviously $\bar{\mathfrak{L}}$ is then a finite-dimensional unitary left module over the residue class algebra $\bar{\mathfrak{O}}$. The ring $\operatorname{End}_{\mathfrak{O}}(\mathfrak{L})$ of all $\mathfrak{O}$-endomorphisms of $\mathfrak{L}$ is naturally an associative $\mathfrak{R}$-algebra with identity. In fact, we have:

PROPOSITION 1.13. *For any $\mathfrak{O}$-lattice $\mathfrak{L}$, the endomorphism ring* $\operatorname{End}_{\mathfrak{O}}(\mathfrak{L})$ *is an order.*

Proof. Since $\mathfrak{L}$ is a free $\mathfrak{R}$-module of finite rank n, its $\mathfrak{R}$-endomorphism ring $\operatorname{End}_{\mathfrak{R}}(\mathfrak{L})$ is isomorphic to the ring of all $n \times n$ matrices with entries in $\mathfrak{R}$, and hence is an order. Each element $x \in \mathfrak{O}$ determines an $\mathfrak{R}$-endomorphism $l \to xl$ of $\mathfrak{L}$. Let S be the set of all such $\mathfrak{R}$-endomorphisms. Then $\operatorname{End}_{\mathfrak{O}}(\mathfrak{L})$ is clearly the subalgebra $C(S \text{ in } \operatorname{End}_{\mathfrak{R}}(\mathfrak{L}))$ of $\operatorname{End}_{\mathfrak{R}}(\mathfrak{L})$, which is an order by (1.3). This proves the proposition.

Now we can apply Proposition 1.10 to the endomorphism ring $\mathrm{End}_{\mathfrak{O}}(\mathfrak{L})$ and to its ideal $J(\mathrm{End}_{\mathfrak{O}}(\mathfrak{L}))$. We know that $\mathfrak{L}$ is an indecomposable $\mathfrak{O}$-module if and only if 0 and $1 \neq 0$ are the only idempotents of $\mathrm{End}_{\mathfrak{O}}(\mathfrak{L})$. Since the Jacobson radical $J(\mathrm{End}_{\mathfrak{O}}(\mathfrak{L}))$ can contain no non-zero idempotent, Proposition 1.10 implies that this occurs if and only if 0 and $1 \neq 0$ are the only idempotents of $\mathrm{End}_{\mathfrak{O}}(\mathfrak{L})/J(\mathrm{End}_{\mathfrak{O}}(\mathfrak{L}))$. The latter ring is a finite-dimensional semi-simple algebra over $\bar{\mathfrak{F}}$ by (1.2). Since $\bar{\mathfrak{F}}$ is algebraically closed, our condition on the idempotents is equivalent to $\mathrm{End}_{\mathfrak{O}}(\mathfrak{L})/J(\mathrm{End}_{\mathfrak{O}}(\mathfrak{L}))$ being isomorphic to $\bar{\mathfrak{F}}$. So we have:

(1.14) (Fitting's Lemma) *An $\mathfrak{O}$-lattice $\mathfrak{L}$ is indecomposable if and only if* $\mathrm{End}_{\mathfrak{O}}(\mathfrak{L})/J(\mathrm{End}_{\mathfrak{O}}(\mathfrak{L})) \simeq \bar{\mathfrak{F}}$, *i.e., if and only if* $\mathrm{End}_{\mathfrak{O}}(\mathfrak{L})$ *is a* (*non-commutative*) *local ring.*

As usual (compare the proof of Theorem (14.5) from Lemma (14.4) in [2]), this implies:

(1.15) (Krull-Schmidt Theorem) *If* $\mathfrak{L} = \mathfrak{L}_1 \oplus \cdots \oplus \mathfrak{L}_l = \mathfrak{K}_1 \oplus \cdots \oplus \mathfrak{K}_k$ *are two decompositions of an $\mathfrak{O}$-lattice $\mathfrak{L}$ as direct sums of indecomposable $\mathfrak{O}$-submodules, then $l = k$ and, after reindexing, $\mathfrak{L}_i$ is $\mathfrak{O}$-isomorphic to $\mathfrak{K}_i$, for $i = 1, \cdots, l$.*

Since the order $\mathfrak{O}$ is a ring with identity, its subset $1 + J(\mathfrak{O})$ is a subgroup of its unit group $U(\mathfrak{O})$. This subgroup has the following useful property:

PROPOSITION 1.16. *Let n be a positive integer not divisible by the characteristic of $\bar{\mathfrak{F}}$. Then the group $1 + J(\mathfrak{O})$ is exactly divisible by n, i.e., for any element $y \in 1 + J(\mathfrak{O})$, there is a unique element $y^{1/n} \in 1 + J(\mathfrak{O})$ such that $(y^{1/n})^n = y$.*

Proof. We first assume that the order $\mathfrak{O}$ is a commutative local ring. Let y be any element of $1 + J(\mathfrak{O})$. Form the polynomial ring $\mathfrak{O}[Z]$ in one variable Z over $\mathfrak{O}$. Let z be the image of Z in the quotient ring

$$\mathfrak{O}[z] = \mathfrak{O}[Z]/(Z^n - y)\mathfrak{O}[Z].$$

Then $\mathfrak{O}[z]$ is a free $\mathfrak{O}$-module of rank n with $1, z, z^2, \cdots, z^{n-1}$ as a basis. Hence it is a commutative order. Evidently $J(\mathfrak{O})\mathfrak{O}[z] = \mathfrak{O}[z]J(\mathfrak{O})$ is a two-sided ideal of $\mathfrak{O}[z]$ which is nilpotent modulo $\mathfrak{p}\mathfrak{O}[z]$, since $J(\mathfrak{O})$ is nilpotent modulo $\mathfrak{p}\mathfrak{O}$. It follows from this and (1.2) that $J(\mathfrak{O})\mathfrak{O}[z] \subseteq J(\mathfrak{O}[z])$.

Let $\bar{z}$ be the image of z in the factor ring $\mathfrak{O}[z]/J(\mathfrak{O})\mathfrak{O}[z]$. Since $\mathfrak{O}/J(\mathfrak{O}) \simeq \bar{\mathfrak{F}}$, this factor ring is the $\bar{\mathfrak{F}}$-algebra $\bar{\mathfrak{F}}[\bar{z}]$ generated by $\bar{z}$. From the construction of $\mathfrak{O}[z]$, it is clear that the natural epimorphism of the polynomial ring $\bar{\mathfrak{F}}[Z]$ onto $\bar{\mathfrak{F}}[\bar{z}]$ (sending Z into $\bar{z}$) has the ideal $(Z^n - 1)\bar{\mathfrak{F}}[Z]$ as its kernel (remember that $y \equiv 1 \pmod{J(\mathfrak{O})}$!). Because n is not divisible by the characteristic of $\bar{\mathfrak{F}}$, the algebra $\bar{\mathfrak{F}}[Z]/(Z^n - 1)\bar{\mathfrak{F}}[Z]$ is semi-simple with its n epimorphisms onto $\bar{\mathfrak{F}}$ sending the image of Z into the n distinct n^{th} roots of unity, say $\omega, \cdots, \omega^{n-1}, \omega^n = 1$, in $\bar{\mathfrak{F}}$. We conclude that

$$J(\mathfrak{O}[z]) = J(\mathfrak{O})\mathfrak{O}[z], \tag{1.17a}$$

(1.17b) $\mathfrak{O}[z]/J(\mathfrak{O}[z]) = \bar{\mathfrak{F}}[\bar{z}] = \bar{\mathfrak{F}}e_1 \oplus \cdots \oplus \bar{\mathfrak{F}}e_n$ (*as $\bar{\mathfrak{F}}$-algebras*),

(1.17c) $$\bar{z} = \omega e_1 \oplus \omega^2 e_2 \oplus \cdots \oplus \omega^n e_n ,$$

where $e_1 , \cdots , e_n$ are the primitive idempotents of $\bar{\mathfrak{F}}[\bar{z}]$ arranged in a certain order.

Proposition 1.12 tells us that the primitive idempotents $e_1^* , \cdots , e_n^*$ of the commutative order $\mathfrak{O}[z]$ have $e_1 , \cdots , e_n$, respectively, as images in $\bar{\mathfrak{F}}[\bar{z}]$. By (1.17a, b) these images generate $\mathfrak{O}[z]/J(\mathfrak{O})\mathfrak{O}[z]$ as an $\mathfrak{O}$-module. Since $\mathfrak{O}[z]$ is a finitely-generated $\mathfrak{O}$-module, the Nakayama Lemma implies that $\mathfrak{O}[z] = \mathfrak{O}e_1^* + \cdots + \mathfrak{O}e_n^*$. Because the e_i^* are the primitive central idempotents of $\mathfrak{O}[z]$, we conclude that

(1.18) $$\mathfrak{O}[z] = \mathfrak{O}e_1^* \oplus \cdots \oplus \mathfrak{O}e_n^* \quad (\textit{as } \mathfrak{R}\textit{-orders}).$$

Since $\mathfrak{O}$ is a local ring, it is an indecomposable $\mathfrak{O}$-module. So is each of its non-zero homomorphic images $\mathfrak{O}e_i^*$, $i = 1, \cdots , n$, for the same reason. By construction, $\mathfrak{O}[z]$ is a free $\mathfrak{O}$-module of rank n. This, (1.18), and the Krull-Schmidt Theorem (1.15) imply:

(1.19) *The map $x \to xe_i^*$ is an isomorphism of the order $\mathfrak{O}$ onto $\mathfrak{O}e_i^*$, for each $i = 1, \cdots , n$.*

Now there exist unique elements $z_1 , \cdots , z_n \in \mathfrak{O}$ such that

$$z = z_1 e_1^* \oplus \cdots \oplus z_n e_n^* .$$

Evidently $z^n = y = ye_1^* \oplus \cdots \oplus ye_n^*$ implies $(z_i)^n = y$, for $i = 1, \cdots , n$. On the other hand, any n^{th} root u of y in $\mathfrak{O}$ defines an epimorphism of the $\mathfrak{O}$-algebra $\mathfrak{O}[z]$ onto $\mathfrak{O}$ sending z onto u. In view of the structure (1.18–19) of $\mathfrak{O}[z]$, this epimorphism must send

$$x = x_1 e_1^* \oplus \cdots \oplus x_n e_n^* ,$$

where $x_1 , \cdots , x_n \in \mathfrak{O}$, onto x_i , for some fixed $i = 1, 2, \cdots , n$. Hence $u = z_i$. Therefore $z_1 , \cdots , z_n$ are precisely the n^{th} roots of y in $\mathfrak{O}$. By (1.17c) their respective images in $\bar{\mathfrak{O}} = \bar{\mathfrak{F}}$ are the distinct elements $\omega, \cdots , \omega^{n-1}$, $\omega^n = 1$. So $z_n = y^{1/n}$ is the unique n^{th} root of y in $1 + J(\mathfrak{O})$, which proves the proposition whenever $\mathfrak{O}$ is a commutative local ring.

Now let $\mathfrak{O}$ be an arbitrary order and y be any element of $1 + J(\mathfrak{O})$. Then $\mathfrak{R}[y]$ is a suborder of $\mathfrak{O}$ (by (1.7)) which is obviously commutative and has the same identity as $\mathfrak{O}$. Proposition 1.9 says that $J(\mathfrak{O}) \cap \mathfrak{R}[y]$ is an ideal of $\mathfrak{R}[y]$ contained in $J(\mathfrak{R}[y])$. Evidently $y \equiv 1 \pmod{J(\mathfrak{O}) \cap \mathfrak{R}[y]}$ and (by (1.2))

$$J(\mathfrak{O}) \cap \mathfrak{R}[y] \supseteq \mathfrak{p}\mathfrak{O} \cap \mathfrak{R}[y] \supseteq \mathfrak{p}\mathfrak{R}[y].$$

It follows that $\mathfrak{R}[y]/(J(\mathfrak{O}) \cap \mathfrak{R}[y]) \simeq \bar{\mathfrak{F}}$. We conclude that

$$J(\mathfrak{O}) \cap \mathfrak{R}[y] = J(\mathfrak{R}[y])$$

and that $\mathfrak{R}[y]$ is a commutative local ring. The above argument tells us that there is a unique element $y^{1/n} \in 1 + J(\mathfrak{R}[y])$ such that $(y^{1/n})^n = y$. Because $J(\mathfrak{O}) \cap \mathfrak{R}[y] = J(\mathfrak{R}[y])$, the element $y^{1/n}$ is the unique n^{th} root of y in

$$(1 + J(\mathfrak{O})) \cap \mathfrak{R}[y] = 1 + J(\mathfrak{R}[y]).$$

Suppose that $z \in 1 + J(\mathfrak{O})$ satisfies $z^n = y$. Applying the above argument to $\mathfrak{R}[z]$ in place of $\mathfrak{R}[y]$, we see that $\mathfrak{R}[z]$ is a commutative local ring containing $\mathfrak{R}[y]$ and that $1 + J(\mathfrak{R}[z]) = (1 + J(\mathfrak{O})) \cap \mathfrak{R}[z]$ contains two n^{th} roots, z and $y^{1/n}$, of y. The unicity part of the proposition for $\mathfrak{R}[z]$ now tells us that $z = y^{1/n}$. So $y^{1/n}$ is the unique n^{th} root of y in $1 + J(\mathfrak{O})$, and the proposition is proved.

2. The Clifford extension for blocks

Suppose that H is a finite group, and that K is a normal subgroup of H. Let $\mathfrak{O}$ be the group algebra $\mathfrak{R}H$ of H over the valuation ring $\mathfrak{R}$ of §1. For each coset σ in the factor group $G = H/K$, let $\mathfrak{O}_\sigma$ be the $\mathfrak{R}$-submodule of $\mathfrak{O}$ having the elements of σ as a basis. Then $\mathfrak{O}$, G, and the $\mathfrak{O}_\sigma$, $\sigma \in G$, satisfy:

(2.1a) *$\mathfrak{O}$ is a non-zero $\mathfrak{R}$-order,*
(2.1b) *G is a group,*
(2.1c) *$\mathfrak{O}_\sigma$ is an $\mathfrak{R}$-submodule of $\mathfrak{O}$, for each $\sigma \in G$,*
(2.1d) $\mathfrak{O} = \oplus \sum_{\sigma\in G} \mathfrak{O}_\sigma$ *(as $\mathfrak{R}$-modules),*
(2.1e) $\mathfrak{O}_\sigma\mathfrak{O}_\tau = \mathfrak{O}_{\sigma\tau}$ *(module product), for all σ, $\tau \in G$.*

We shall develop the theory in the general situation (2.1), indicating from time to time the special properties of the case coming from H and K in the above fashion.

First we collect some trivial consequences of (2.1) in the following

PROPOSITION 2.2 (a) *The identity 1 of $\mathfrak{O}$ lies in $\mathfrak{O}_1$.*
(b) *$\mathfrak{O}_1$ is a non-zero suborder of $\mathfrak{O}$.*
(c) *Each $\mathfrak{O}_\sigma$, $\sigma \in G$, is a non-zero $\mathfrak{R}$-lattice.*
(d) *G is a finite group.*

Proofs. (a) By (2.1d) there are unique elements $e_\sigma \in \mathfrak{O}_\sigma$, for $\sigma \in G$, all but a finite number of which are zero, such that $1 = \sum_{\sigma\in G} e_\sigma$. If $y \in \mathfrak{O}_1$, then

$$y = y1 = \sum_{\sigma\in G} ye_\sigma .$$

By (2.1e) the product ye_σ lies in $\mathfrak{O}_1\mathfrak{O}_\sigma = \mathfrak{O}_\sigma$, for all $\sigma \in G$. Since $y \in \mathfrak{O}_1$, equating homogeneous components in the above equation (using (2.1d)) gives $ye_\sigma = 0$, for all $\sigma \neq 1$. Hence $\mathfrak{O}_1 e_\sigma = 0$, for any such σ. If $\tau \in G$, we obtain $\mathfrak{O}_\tau e_\sigma = \mathfrak{O}_\tau \mathfrak{O}_1 e_\sigma = 0$ (from 2.1e). In view of (2.1d), this implies that $\mathfrak{O}e_\sigma = 0$, for all $\sigma \in G - \{1\}$. But then

$$1 = 1\cdot 1 = \sum_{\sigma\in G} 1\cdot e_\sigma = 1\cdot e_1 = e_1 \in \mathfrak{O}_1 .$$

(b) Conditions (2.1a, c, d) imply that $\mathfrak{O}_1$ is an $\mathfrak{R}$-sublattice of $\mathfrak{O}$. By

(2.1e) it is a subring of $\mathfrak{O}$. We have just seen that it contains 1, which is non-zero by (2.1a). Hence $\mathfrak{O}_1$ is a non-zero suborder of $\mathfrak{O}$.

(c) Conditions (2.1a, c, d) imply that each $\mathfrak{O}_\sigma$, $\sigma \in G$, is an $\mathfrak{R}$-lattice. If $\mathfrak{O}_\sigma = 0$, then (2.1e) gives $\mathfrak{O}_1 = \mathfrak{O}_\sigma \mathfrak{O}_{\sigma^{-1}} = 0$, contradicting (b). Therefore each $\mathfrak{O}_\sigma$ is non-zero.

(d) Because each $\mathfrak{O}_\sigma$ is a *non-zero* $\mathfrak{R}$-sublattice of $\mathfrak{O}$, condition (2.1d) implies that the number of elements of G is no larger than the $\mathfrak{R}$-rank of $\mathfrak{O}$, which is finite by (2.1a). This completes the proof of the proposition.

Conditions (2.1) and Proposition 2.2 (a) can be expressed by saying that $\mathfrak{O}, \{\mathfrak{O}_\sigma \mid \sigma \in G\}$ is a *graded Clifford system over* $\mathfrak{R}$ in the sense of [CCT] or [ICE], with the additional hypothesis that $\mathfrak{O}$, as an $\mathfrak{R}$-module, is a non-zero $\mathfrak{R}$-lattice. So we are free to apply the results of these articles to the present case. For example, there is a natural action of the group G on the family $\mathrm{Id}(\mathfrak{O}_1)$ of all two-sided ideals of $\mathfrak{O}_1$, given by

$$(2.3) \qquad \mathfrak{J}^\sigma = \mathfrak{O}_{\sigma^{-1}} \mathfrak{J} \mathfrak{O}_\sigma, \quad \textit{for all} \quad \sigma \in G, \qquad \mathfrak{J} \in \mathrm{Id}(\mathfrak{O}_1)$$

(see [CCT, §2]). We know from [ICE, 1.11] that the Jacobson radical $J(\mathfrak{O}_1)$ is fixed by G under this action. It follows (see [ICE, 1.5]) that

(2.4) $\mathfrak{O}J(\mathfrak{O}_1) = \oplus \sum_{\sigma \in G} \mathfrak{O}_\sigma J(\mathfrak{O}_1) = J(\mathfrak{O}_1)\mathfrak{O}$ *is a graded two-sided ideal of* $\mathfrak{O}$ *with* $\mathfrak{O}_\sigma J(\mathfrak{O}_1) = J(\mathfrak{O}_1)\mathfrak{O}_\sigma$ *as its* σ^{th} *homogeneous component, for any* $\sigma \in G$.

The following will be important.

PROPOSITION 2.5. $\mathfrak{O}J(\mathfrak{O}_1) \subseteq J(\mathfrak{O})$.

Proof. Because $\bar{\mathfrak{O}}_1 = \mathfrak{O}_1/\mathfrak{p}\mathfrak{O}_1$ is a finite-dimensional algebra over $\bar{\mathfrak{F}}$, its radical $J(\bar{\mathfrak{O}}_1)$ satisfies $J(\bar{\mathfrak{O}}_1)^n = 0$, for some $n > 0$. In view of (1.2), this implies that $J(\mathfrak{O}_1)^n \subseteq \mathfrak{p}\mathfrak{O}_1$. Since $\mathfrak{O}J(\mathfrak{O}_1) = J(\mathfrak{O}_1)\mathfrak{O}$, we conclude that

$$[\mathfrak{O}J(\mathfrak{O}_1)]^n = \mathfrak{O}^n J(\mathfrak{O}_1)^n \subseteq \mathfrak{O}\mathfrak{p}\mathfrak{O}_1 = \mathfrak{p}\mathfrak{O} \subseteq J(\mathfrak{O}).$$

The proposition follows immediately from this.

We denote by $\mathfrak{C}$ the centralizer $C(\mathfrak{O}_1 \text{ in } \mathfrak{O})$ of $\mathfrak{O}_1$ in $\mathfrak{O}$, and define $\mathfrak{C}_\sigma$ to be $\mathfrak{C} \cap \mathfrak{O}_\sigma$, for each $\sigma \in G$. We remark that, in the case of the group H and its normal subgroup K:

(2.6) $\mathfrak{C}$ *has an* $\mathfrak{R}$*-basis consisting of the class sums of elements of* H *under conjugation by elements of* K.

In the general case, we have:

PROPOSITION 2.7. (a) $\mathfrak{C}$ *is a suborder of* $\mathfrak{O}$ *containing* $1 = 1_{\mathfrak{O}}$.

(b) $\mathfrak{C}_1 = Z(\mathfrak{O}_1)$ *is a central suborder of* $\mathfrak{C}$ *containing* 1.

(c) $\mathfrak{C}_\sigma$ *is an* $\mathfrak{R}$*-sublattice of* $\mathfrak{C}$, *for each* $\sigma \in G$.

(d) $\mathfrak{C} = \oplus \sum_{\sigma \in G} \mathfrak{C}_\sigma$ (*as* $\mathfrak{R}$*-modules*).
(e) $\mathfrak{C}_\sigma \mathfrak{C}_\tau \subseteq \mathfrak{C}_{\sigma\tau}$, *for all* $\sigma, \tau \in G$.
(f) $\mathfrak{C}_\sigma \mathfrak{C}_1 = \mathfrak{C}_1 \mathfrak{C}_\sigma = \mathfrak{C}_\sigma$, *for all* $\sigma \in G$.

Proof. The definition of $\mathfrak{C}$ and (1.3) give (a). By definition, $\mathfrak{C}_1 = \mathfrak{O}_1 \cap C(\mathfrak{O}_1 \text{ in } \mathfrak{O}) = C(\mathfrak{O}_1 \text{ in } \mathfrak{O}_1)$ is the center $Z(\mathfrak{O}_1)$ of $\mathfrak{O}_1$, which is clearly central in $C(\mathfrak{O}_1 \text{ in } \mathfrak{O})$. So (b) follows from (1.4) and Proposition 2.2 (a, b). In view of (2.1e), each $\mathfrak{O}_\sigma$, $\sigma \in G$, is a two-sided $\mathfrak{O}_1$-submodule of $\mathfrak{O}$. Therefore (d) follows from (2.1d) and the definitions of $\mathfrak{C}$ and the $\mathfrak{C}_\sigma$'s. Clearly each $\mathfrak{C}_\sigma$ is an $\mathfrak{R}$-submodule of $\mathfrak{C}$. Hence (a) and (d) imply (c). The definitions and (2.1e) give (e) while (f) follows from (e) and the fact that $\mathfrak{C}_1$ contains 1. So the proposition is proved.

The above properties of $\mathfrak{C}$, G, and the $\mathfrak{C}_\sigma$ should be compared with those of $\mathfrak{O}$, G, and the $\mathfrak{O}_\sigma$, listed in (2.1) and Proposition 2.2. There is one vital difference: we have inclusion in Proposition 2.7(e) in place of the equality in (2.1e)—and one minor one: the $\mathfrak{C}_\sigma$, as opposed to the $\mathfrak{O}_\sigma$, can be zero. In order to get back to the original properties we shall pass to a suborder of $\mathfrak{C}$ and then to a subgroup of G.

Let B be a block of $\mathfrak{O}_1$ and e be the corresponding primitive central idempotent of that order. Evidently e is a non-zero central idempotent of $\mathfrak{C} = C(\mathfrak{O}_1 \text{ in } \mathfrak{O})$ lying in $\mathfrak{C}_1 = Z(\mathfrak{O}_1)$. This and Proposition 2.7(f) give us the decompositions

$$\mathfrak{C} = e\mathfrak{C} \oplus (1-e)\mathfrak{C} \quad (\textit{as } \mathfrak{R}\textit{-algebras}),$$

$$\mathfrak{C}_\sigma = e\mathfrak{C}_\sigma \oplus (1-e)\mathfrak{C}_\sigma \quad (\textit{as } \mathfrak{R}\textit{-modules}), \qquad \textit{for each} \quad \sigma \in G.$$

Evidently these decompositions and Proposition 2.7 imply:

(2.8a) $e\mathfrak{C}$ *is a suborder of* $\mathfrak{C}$ *with non-zero identity* e,
(2.8b) $e\mathfrak{C}_1$ *is a central suborder of* $e\mathfrak{C}$ *containing* e,
(2.8c) $e\mathfrak{C}_\sigma$ *is an* $\mathfrak{R}$*-sublattice of* $e\mathfrak{C}$, *for each* $\sigma \in G$,
(2.8d) $e\mathfrak{C} = \oplus \sum_{\sigma \in G} e\mathfrak{C}_\sigma$,
(2.8e) $(e\mathfrak{C}_\sigma)(e\mathfrak{C}_\tau) \subseteq e\mathfrak{C}_{\sigma\tau}$ (*module product*), *for all* $\sigma, \tau \in G$,
(2.8f) $(e\mathfrak{C}_\sigma)(e\mathfrak{C}_1) = (e\mathfrak{C}_1)(e\mathfrak{C}_\sigma) = e\mathfrak{C}_\sigma$, *for all* $\sigma \in G$.

Now we define:

(2.9a) $G[B] = \{\sigma \in G \mid (e\mathfrak{C}_\sigma)(e\mathfrak{C}_{\sigma^{-1}}) = e\mathfrak{C}_1\}$,
(2.9b) $\mathfrak{C}[B] = \oplus \sum_{\sigma \in G[B]} e\mathfrak{C}_\sigma$,
(2.9c) $\mathfrak{C}[B]_\sigma = e\mathfrak{C}_\sigma$, *for all* $\sigma \in G[B]$.

At last we have reached a good system.

PROPOSITION 2.10. *The properties* (2.1) *are satisfied with* $\mathfrak{C}[B]$, $G[B]$, *and the* $\mathfrak{C}[B]_\sigma$ *in place of* $\mathfrak{O}$, G, *and the* $\mathfrak{O}_\sigma$, *respectively.*

Proof. By (2.8f) the subset $G[B]$ of G contains 1_G. If $\sigma, \tau \in G[B]$, then

(2.8e, f) and (2.9a) imply that

$$e\mathfrak{C}_1 \supseteq (e\mathfrak{C}_{\sigma\tau})(e\mathfrak{C}_{(\sigma\tau)^{-1}}) \supseteq (e\mathfrak{C}_\sigma)(e\mathfrak{C}_\tau)(e\mathfrak{C}_{(\sigma\tau)^{-1}}) \supseteq (e\mathfrak{C}_\sigma)(e\mathfrak{C}_\tau)(e\mathfrak{C}_{\tau^{-1}})(e\mathfrak{C}_{\sigma^{-1}})$$
$$= (e\mathfrak{C}_\sigma)(e\mathfrak{C}_1)(e\mathfrak{C}_{\sigma^{-1}}) = (e\mathfrak{C}_\sigma)(e\mathfrak{C}_{\sigma^{-1}}) = e\mathfrak{C}_1 .$$

So equality holds and $\sigma\tau$ lies in $G[B]$. Since G is a finite group (by Proposition 2.2(d)), this implies that $G[B]$ is a subgroup of G, which is condition (2.1b) for our system.

Equality in the above chain of inclusions also gives

$$(e\mathfrak{C}_\sigma)(e\mathfrak{C}_\tau)(e\mathfrak{C}_{(\sigma\tau)^{-1}}) = e\mathfrak{C}_1 .$$

Because $(\sigma\tau)^{-1}$ lies in the subgroup $G[B]$, we have $(e\mathfrak{C}_{(\sigma\tau)^{-1}})(e\mathfrak{C}_{\sigma\tau}) = e\mathfrak{C}_1$. This, (2.9c), and (2.8f) imply that

$$\mathfrak{C}[B]_\sigma\mathfrak{C}[B]_\tau = (e\mathfrak{C}_\sigma)(e\mathfrak{C}_\tau)(e\mathfrak{C}_1) = (e\mathfrak{C}_\sigma)(e\mathfrak{C}_\tau)(e\mathfrak{C}_{(\sigma\tau)^{-1}})(e\mathfrak{C}_{\sigma\tau})$$
$$= (e\mathfrak{C}_1)(e\mathfrak{C}_{\sigma\tau}) = \mathfrak{C}[B]_{\sigma\tau} ,$$

for all $\sigma, \tau \in G[B]$, which is (2.1e) for our system.

The definitions (2.9b, c) give (2.1d) for our system. Condition (2.1c) comes directly from (2.8c). Since G is a finite group, its subgroup $G[B]$ is finite. So (2.8c) and (2.9b) imply that $\mathfrak{C}[B]$ is an $\mathfrak{R}$-sublattice of $e\mathfrak{C}$. Since $G[B]$ is closed under multiplication, (2.8e) and (2.9b) are enough to make $\mathfrak{C}[B]$ a subring of $e\mathfrak{C}$. The identity e of $e\mathfrak{C}$ is non-zero and lies in $\mathfrak{C}[B]$ by (2.8a, b). Therefore $\mathfrak{C}[B]$ is a non-zero suborder of $e\mathfrak{C}$, and the proof of the proposition is complete.

By definition e is a primitive idempotent in the center $\mathfrak{C}_1 = Z(\mathfrak{O}_1)$ of $\mathfrak{O}_1$. Since this order is commutative, Proposition 1.12 implies (as in (1.14)) that $\mathfrak{C}[B]_1 = e\mathfrak{C}_1$ satisfies

(2.11) $$\mathfrak{C}[B]_1 / J(\mathfrak{C}[B]_1) \simeq \bar{\mathfrak{F}}.$$

Hence $\mathfrak{C}[B]_1$ is a local ring in the center of $\mathfrak{C}[B]$.

In view of Proposition 2.10, we may apply (2.4) to $\mathfrak{C}[B]$, $G[B]$, and the $\mathfrak{C}[B]_\sigma$. Identifying each

$$\mathfrak{C}[B]_\sigma / \mathfrak{C}[B]_\sigma J(\mathfrak{C}[B]_1)$$

with its image in the factor ring $\mathfrak{C}[B]/\mathfrak{C}[B]J(\mathfrak{C}[B]_1)$, and using the fact that

$$\mathfrak{C}[B]J(\mathfrak{C}[B]_1) \supseteq \mathfrak{C}[B]\mathfrak{p}\mathfrak{C}[B]_1 = \mathfrak{p}\mathfrak{C}[B]$$

(by (1.2)), we see easily that

(2.12) *Conditions* (2.1) *are satisfied with* $\bar{\mathfrak{F}}$, $\mathfrak{C}[B]/\mathfrak{C}[B]J(\mathfrak{C}[B]_1)$, $G[B]$, *and the* $\mathfrak{C}[B]_\sigma / \mathfrak{C}[B]_\sigma J(\mathfrak{C}[B]_1)$ *in place of* $\mathfrak{R}$, $\mathfrak{O}$, G, *and the* $\mathfrak{O}_\sigma$, *respectively.*

Because of (2.11), we can apply [CCT, §14] to the graded Clifford system

$$\mathfrak{C}[B]/\mathfrak{C}[B]J(\mathfrak{C}[B]_1), \qquad \{\mathfrak{C}[B]_\sigma / \mathfrak{C}[B]_\sigma J(\mathfrak{C}[B]_1) \mid \sigma \in G[B]\}$$

over $\bar{\mathfrak{F}}$. It tells us that there is a unique central extension $G[B]^*$ of the multiplicative group $\bar{F}$ of $\bar{\mathfrak{F}}$ by $G[B]$, which is defined, together with its projection $\mathrm{pr} = \mathrm{pr}_{G[B]^*}$ onto $G[B]$ and its injection $\mathrm{in} = \mathrm{in}_{G[B]^*}$ from $\bar{F}$, by:

(2.13a) $\mathrm{pr}^{-1}(\sigma) = [\mathfrak{C}[B]_\sigma/\mathfrak{C}[B]_\sigma J(\mathfrak{C}[B]_1)] - \{0\}$, *for all* $\sigma \in G[B]$,
(2.13b) $G[B]^* = \bigcup_{\sigma \in G[B]} \mathrm{pr}^{-1}(\sigma)$ *is a subgroup of the unit group of*

$$\mathfrak{C}[B]/\mathfrak{C}[B]J(\mathfrak{C}[B]_1),$$

(2.13c) $\mathrm{in}(f) = fe + J(\mathfrak{C}[B]_1) \in \mathrm{pr}^{-1}(1) = [\mathfrak{C}[B]_1/J(\mathfrak{C}[B]_1)] - \{0\}$, *for all* $f \in \bar{F} = \bar{\mathfrak{F}} - \{0\}$.

We call $G[B]^*$ the *Clifford extension* for the block B. Evidently the above equations simply state that the twisted group algebra $\bar{\mathfrak{F}}[G[B]^*]$ of the extension $G[B]^*$ and its one-dimensional subspaces $\bar{\mathfrak{F}}[G[B]^*]_\sigma$, spanned by $\mathrm{pr}^{-1}(\sigma)$ for $\sigma \in G[B]$, are given by:

(2.14a) $\bar{\mathfrak{F}}[G[B]^*] = \mathfrak{C}[B]/\mathfrak{C}[B]J(\mathfrak{C}[B]_1)$,
(2.14b) $\bar{\mathfrak{F}}[G[B]^*]_\sigma = \mathfrak{C}[B]_\sigma/\mathfrak{C}[B]_\sigma J(\mathfrak{C}[B]_1)$, *for all* $\sigma \in G[B]$.

By [CCT, §2] the axioms (2.1) determine a natural action of the group G as ring automorphisms of $\mathfrak{C} = C(\mathfrak{O}_1 \text{ in } \mathfrak{O})$. If $y \in \mathfrak{C}$ and $\sigma \in G$, then $y^\sigma \in \mathfrak{C}$ is the unique element of $\mathfrak{O}$ satisfying

(2.15) $$xy^\sigma = yx, \quad \textit{for all} \quad x \in \mathfrak{O}_\sigma .$$

It is clear from this definition that $y \to y^\sigma$ is an $\mathfrak{R}$-automorphism of the order $\mathfrak{C}$, for any $\sigma \in G$. We know from [CCT, 2.9] that

(2.16) $$(\mathfrak{C}_\tau)^\sigma = \mathfrak{C}_{\tau^\sigma} = \mathfrak{C}_{\sigma^{-1}\tau\sigma} \quad \textit{for all} \quad \sigma, \tau \in G.$$

In particular, the suborder $\mathfrak{C}_1$ is invariant under G. It follows that G permutes the primitive idempotents of $\mathfrak{C}_1$ among themselves. We denote by G_B the subgroup of all elements of G fixing the primitive idempotent e of $\mathfrak{C}_1$.

PROPOSITION 2.17. *If* $\sigma \in G$ *and* $e\mathfrak{C}_\sigma \neq 0$, *then* $\sigma \in G_B$. *Hence* $G[B] \leq G_B$. *Furthermore, the subgroup* $G[B]$ *is normal in* G_B.

Proof. If $\sigma \in G$ and $\sigma \notin G_B$, then e^σ is a primitive idempotent of $\mathfrak{C}_1$ different from e. Since $\mathfrak{C}_1$ is commutative, this implies that $ee^\sigma = e^\sigma e = 0$. Hence $xe^\sigma = xee^\sigma = 0$, for any element $x \in e\mathfrak{C}_\sigma$. In view of (2.15), this implies that $0 = xe^\sigma = ex = x$, i.e., that $e\mathfrak{C}_\sigma = 0$. This proves the first statement of the proposition. The second follows from the first and the definition (2.9a) of $G[B]$ (in view of (2.8a, b)).

If $\sigma \in G_B$ and $\tau \in G[B]$, then (2.9a), (2.16), and the fact that $y \to y^\sigma$ is an automorphism of the order $\mathfrak{C}$ fixing e, imply that

$$e\mathfrak{C}_1 = (e\mathfrak{C}_1)^\sigma = (e\mathfrak{C}_\tau)^\sigma(e\mathfrak{C}_{\tau^{-1}})^\sigma = (e\mathfrak{C}_{\tau^\sigma})(e\mathfrak{C}_{(\tau^\sigma)^{-1}}).$$

Therefore τ^σ also lies in $G[B]$, and the proposition is proved.

We can use the fact that $ee^\sigma = e^\sigma e = 0$, for $\sigma \in G - G_B$, in another way.

It is exactly the hypothesis of [CCT, 4.3]. Using that result, (1.5), and (2.1d), we see that:

(2.18) *Conditions* (2.1) *are satisfied with* $e\mathfrak{O}e$, G_B, *and the* $e\mathfrak{O}_\sigma = \mathfrak{O}_\sigma e$ (*for* $\sigma \in G_B$) *in place of* $\mathfrak{O}$, G, *and the* $\mathfrak{O}_\sigma$, *respectively.*

Proposition 2.17 and (2.16) imply that G_B leaves invariant the suborder

$$\mathfrak{C}[B] = \oplus \sum_{\sigma \in G[B]} e\mathfrak{C}_\sigma,$$

and that $(\mathfrak{C}[B]_\tau)^\sigma = \mathfrak{C}[B]_{\tau^\sigma}$, for all $\tau \in G[B]$, $\sigma \in G_B$. It follows that G_B leaves invariant the Jacobson radical $J(\mathfrak{C}[B]_1)$, and hence the ideal $\mathfrak{C}[B]J(\mathfrak{C}[B]_1)$. By (2.14) there is an induced action of G_B as algebra automorphisms of $\bar{\mathfrak{F}}[G[B]^*]$ such that $(\bar{\mathfrak{F}}[G[B]^*]_\tau)^\sigma = \bar{\mathfrak{F}}[G[B]^*]_{\tau^\sigma}$, for all $\tau \in G[B]$, $\sigma \in G_B$. From (2.13) we conclude that the subgroup $G[B]^*$ of the unit group of $\bar{\mathfrak{F}}[G[B]^*]$ is G_B-invariant, and that the induced conjugation action of G_B as automorphisms of $G[B]^*$ satisfies:

(2.19a) $\operatorname{pr}(\rho^\tau) = \operatorname{pr}(\rho)^\tau = \tau^{-1}\operatorname{pr}(\rho)\tau \in G[B]$, *for all* $\rho \in G[B]^*$, $\tau \in G_B$,
(2.19b) G_B *centralizes* $\operatorname{pr}^{-1}(1) = \operatorname{in}(\bar{F})$.

Evidently this action of G_B on $G[B]^*$ completely determines the action of G_B on $\bar{\mathfrak{F}}[G[B]^*]$.

3. The Clifford correspondence for blocks

Our first goal is to obtain a one-to-one correspondence between the G_B-invariant central idempotents of $\bar{\mathfrak{F}}[G[B]^*]$ and the central idempotents of $e\mathfrak{O}e$. We start by lifting the central idempotents of $\bar{\mathfrak{F}}[G[B]^*]$ back to $\mathfrak{C}[B]$.

Lemma 3.1. *If* d *is a central idempotent of* $\bar{\mathfrak{F}}[G[B]^*]$, *then there is a unique idempotent* d^* *of* $\mathfrak{C}[B]$ *having* d *as its image in* (2.14a). *This idempotent* d^* *is central in* $\mathfrak{C}[B]$. *The map* $d \to d^*$ *sends the central idempotents of* $\bar{\mathfrak{F}}[G[B]^*]$ *one-to-one onto those of* $\mathfrak{C}[B]$. *Furthermore, it preserves the action of* G_B *on these idempotents.*

Proof. (Compare that of Proposition 1.12). Proposition 1.10 gives us at least one idempotent d^* of $\mathfrak{C}[B]$ mapping onto d. We must prove that such a d^* is always central in $\mathfrak{C}[B]$. The suborder $\mathfrak{C}[B]_1 = e\mathfrak{C}_1$ is central in $\mathfrak{C}[B] \subseteq e\mathfrak{C}$ by (2.8b). So the Peirce decomposition of $\mathfrak{C}[B]$ with respect to d^* is a $\mathfrak{C}[B]_1$-decomposition:

$$\begin{aligned}(3.2)\quad \mathfrak{C}[B] = d^*\mathfrak{C}[B]d^* &\oplus d^*\mathfrak{C}[B](e - d^*) \oplus (e - d^*)\mathfrak{C}[B]d^* \\ &\oplus (e - d^*)\mathfrak{C}[B](e - d^*) \quad (\textit{as two-sided } \mathfrak{C}[B]_1\textit{-modules}).\end{aligned}$$

The summand $d^*\mathfrak{C}[B](e - d^*)$ is an $\mathfrak{R}$-lattice, and hence is a finitely-generated right $\mathfrak{C}[B]_1$-module. So it will be zero if its factor module modulo

$$d^*\mathfrak{C}[B](e - d^*)J(\mathfrak{C}[B]_1)$$

is zero. But this factor module is simply the image $d\bar{\mathfrak{F}}[G[B]^*](1 - d)$ of

$d^*\mathfrak{C}[B](e - d^*)$ in $\bar{\mathfrak{F}}[G[B]^*]$, by (2.14a) and (3.2). Since d is a central idempotent of $\bar{\mathfrak{F}}[G[B]^*]$, this image is zero. Therefore $d^*\mathfrak{C}[B](e - d^*) = 0$. Similarly $(e - d^*)\mathfrak{C}[B]d^* = 0$. So d^* is a central idempotent of $\mathfrak{C}[B]$.

We know from Propositions 2.5 and 2.10 that $\mathfrak{C}[B]J(\mathfrak{C}[B]_1) \subseteq J(\mathfrak{C}[B])$. It follows that $\mathfrak{C}[B]J(\mathfrak{C}[B]_1)$ contains no non-zero idempotent of $\mathfrak{C}[B]$. Now the proof of the first three statements of the lemma can be completed as in the proof of Proposition 1.12. The last statement follows directly from the third and the definition of the actions of G_B.

The above lemma gives us a G_B-invariant idempotent d^* of $e\mathfrak{C}$ corresponding to each G_B-invariant central idempotent d of $\bar{\mathfrak{F}}[G[B]^*]$. To show that these are the only such idempotents of $e\mathfrak{C}$ we shall use:

LEMMA 3.3. *The subspace* $\mathfrak{J} = [\oplus \sum_{\sigma \in G_B - G[B]} e\mathfrak{C}_\sigma] \oplus \mathfrak{C}[B]J(\mathfrak{C}[B]_1)$ *is a two-sided ideal of* $e\mathfrak{C} = \oplus \sum_{\sigma \in G_B} e\mathfrak{C}_\sigma$. *Furthermore, this ideal* $\mathfrak{J}$ *is contained in* $J(e\mathfrak{C})$.

Proof. We know by (2.8b) that $J(\mathfrak{C}[B]_1)$ centralizes $e\mathfrak{C}$. It follows that

$$(e\mathfrak{C})J(\mathfrak{C}[B]_1) = J(\mathfrak{C}[B]_1)(e\mathfrak{C})$$

is a two-sided ideal of $e\mathfrak{C}$. Because $\mathfrak{C}[B]_1$ is an order, (1.2) implies the existence of an integer $n > 0$ such that $J(\mathfrak{C}[B]_1)^n \subseteq \mathfrak{p}\mathfrak{C}[B_1]$. Then

$$[(e\mathfrak{C})J(\mathfrak{C}[B]_1)]^n = (e\mathfrak{C})J(\mathfrak{C}[B]_1)^n \subseteq \mathfrak{p}(e\mathfrak{C}),$$

which forces $(e\mathfrak{C})J(\mathfrak{C}[B]_1)$ to be contained in $J(e\mathfrak{C})$.

If $\sigma \in G_B - G[B]$ and $\tau \in G_B$, then

$$(e\mathfrak{C}_\sigma)(e\mathfrak{C}_{\tau^\sigma}) = (e\mathfrak{C}_\sigma)(e\mathfrak{C}_\tau)^\sigma = (e\mathfrak{C}_\tau)(e\mathfrak{C}_\sigma)$$

by (2.15) and (2.16). We know from (2.8d) and Proposition 2.17 that $e\mathfrak{C} = \oplus \sum_{\sigma \in G_B} e\mathfrak{C}_\sigma$. So this implies that $(e\mathfrak{C}_\sigma)(e\mathfrak{C}) = (e\mathfrak{C})(e\mathfrak{C}_\sigma)$ is a two-sided ideal of $e\mathfrak{C}$. Since the group G is finite (by Proposition 2.2(d)), there exists an integer $m > 0$ such that $\sigma^m = 1$. Then $(e\mathfrak{C}_\sigma)^m$ is an ideal of $e\mathfrak{C}_1 = \mathfrak{C}[B]_1$. If $(e\mathfrak{C}_\sigma)^m$ is not contained in $J(\mathfrak{C}[B]_1)$, then it equals the local ring $e\mathfrak{C}_1$. This implies that $(e\mathfrak{C}_\sigma)(e\mathfrak{C}_{\sigma^{-1}}) = e\mathfrak{C}_1$, contradicting the fact that $\sigma \notin G[B]$. Therefore $(e\mathfrak{C}_\sigma)^m \subseteq J(\mathfrak{C}[B]_1)$ and

$$[(e\mathfrak{C})(e\mathfrak{C}_\sigma)]^m = (e\mathfrak{C})(e\mathfrak{C}_\sigma)^m \subseteq (e\mathfrak{C})J(\mathfrak{C}[B]_1) \subseteq J(e\mathfrak{C}),$$

which implies $(e\mathfrak{C})(e\mathfrak{C}_\sigma) \subseteq J(e\mathfrak{C})$.

Evidently $\mathfrak{J}$ is contained in the two-sided ideal

$$(e\mathfrak{C})J(\mathfrak{C}[B]_1) + \sum_{\sigma \in G_B - G[B]} (e\mathfrak{C})(e\mathfrak{C}_\sigma)$$

which, in turn, is contained in $J(e\mathfrak{C})$. Since $e\mathfrak{C} = \oplus \sum_{\sigma \in G_B} e\mathfrak{C}_\sigma$, it is clear that

$$(e\mathfrak{C})J(\mathfrak{C}[B]_1) \subseteq \mathfrak{J}.$$

To show that $(e\mathfrak{C})(e\mathfrak{C}_\sigma) \subseteq \mathfrak{J}$, for any $\sigma \in G_B - G[B]$, it suffices to prove that

$$(e\mathfrak{C}_\tau)(e\mathfrak{C}_\sigma) \subseteq \mathfrak{C}[B]_{\tau\sigma}\, J(\mathfrak{C}[B]_1),$$

for any $\tau \in G_B$ such that $\tau\sigma \in G[B]$. If this is false, then the usual argument (based on Proposition 2.10) shows that

$$\mathfrak{C}[B]_{(\tau\sigma)^{-1}}(e\mathfrak{C}_\tau)(e\mathfrak{C}_\sigma) \nsubseteq J(\mathfrak{C}[B]_1),$$

which implies $\mathfrak{C}[B]_{(\tau\sigma)^{-1}}(e\mathfrak{C}_\tau)(e\mathfrak{C}_\sigma) = e\mathfrak{C}_1$, since $e\mathfrak{C}_1$ is a local ring. But this forces $(e\mathfrak{C}_{\sigma^{-1}})(e\mathfrak{C}_\sigma) = e\mathfrak{C}_1$, and hence $\sigma^{-1} \in G[B]$, contradicting the choice of $\sigma \notin G[B]$. We conclude that

$$(e\mathfrak{C})(e\mathfrak{C}_\sigma) \subseteq \mathfrak{J} \quad \textit{for all} \quad \sigma \in G_B - G[B],$$

and hence that $\mathfrak{J} = (e\mathfrak{C})J(\mathfrak{C}[B]_1) + \sum_{\sigma \in G_B - G[B]} (e\mathfrak{C})(e\mathfrak{C}_\sigma)$. In view of the above results, this proves the lemma.

We shall also need to compute the center $Z(e\mathfrak{O}e)$ of $e\mathfrak{O}e$. Evidently it is contained in $C(e\mathfrak{O}_1 \text{ in } e\mathfrak{O}e)$, which is equal to $eC(\mathfrak{O}_1 \text{ in } \mathfrak{O}) = e\mathfrak{C}$. Since $e\mathfrak{O}e = \oplus \sum_{\sigma \in G_B} e\mathfrak{O}_\sigma$, it follows immediately from (2.15) that

$$Z(e\mathfrak{O}e) = C(G_B \text{ in } e\mathfrak{C}), \tag{3.4}$$

where, of course, $C(G_B \text{ in } e\mathfrak{C})$ denotes the centralizer of G_B in $e\mathfrak{C}$, i.e., the suborder of all $y \in e\mathfrak{C}$ such that $y^\sigma = y$ for all $\sigma \in G_B$. We conclude that the central idempotents of $e\mathfrak{O}e$ are simply the G_B-invariant (and hence central) idempotents of $e\mathfrak{C}$.

Now we can compute the central idempotents of $e\mathfrak{O}e$.

THEOREM 3.5. *The central idempotents of $e\mathfrak{O}e$ are precisely the G_B-invariant (and hence central) idempotents of $\mathfrak{C}[B]$. By Lemma 3.1, they correspond one-to-one to their images in $\bar{\mathfrak{F}}[G[B]^*]$, which are the G_B-invariant central idempotents of that algebra. In particular, the primitive central idempotents of $e\mathfrak{O}e$ are precisely those central idempotents of $\mathfrak{C}[B]$ whose images in $\bar{\mathfrak{F}}[G[B]^*]$ have the form $d_1 + \cdots + d_n$, where $d_1, \cdots, d_n$ is a G_B-conjugacy class of primitive central idempotents of $\bar{\mathfrak{F}}[G[B]^*]$. Thus there is a one-to-one correspondence between the blocks of $e\mathfrak{O}e$ and the G_B-conjugacy classes of blocks of $\bar{\mathfrak{F}}[G[B]^*]$.*

Proof. In view of (2.14a) and Lemma 3.3, the algebra $\bar{\mathfrak{F}}[G[B]^*]$ is naturally isomorphic to $e\mathfrak{C}/\mathfrak{J}$. Clearly Proposition 2.17 implies that the epimorphism of $e\mathfrak{C}$ onto $\bar{\mathfrak{F}}[G[B]^*]$ preserves the actions of G_B on these two rings. So the image d of a G_B-invariant idempotent d_1 of $e\mathfrak{C}$ is a G_B-invariant central idempotent of $\bar{\mathfrak{F}}[G[B]^*]$. By Lemma 3.1 there is a central idempotent d^* of $\mathfrak{C}[B]$ which is equally G_B-invariant and has the same image d. Now d_1 and d^* are two central idempotents of $e\mathfrak{C}$ which are congruent modulo $J(e\mathfrak{C})$ (by Lemma 3.3). It follows that $d_1 = d^* \in \mathfrak{C}[B]$. This proves the first statement of the proposition. The others follow from this, Lemma 3.1 and (3.4).

The passage from blocks of $e\mathfrak{O}e$ to those of $\mathfrak{O}$ lying over B is a well-known result of Reynolds for group rings (see [8]). The action of G on $\mathfrak{C}$ permutes the primitive central idempotents of $\mathfrak{C}_1$ among themselves. Let s be the sum of the idempotents in the G-orbit of e. Then s is a central idempotent of $\mathfrak{O}$ lying in $\mathfrak{O}_1$ (by (2.15)). Hence $\mathfrak{O}$ is the direct sum (as rings) of its subrings $s\mathfrak{O}$ and $(1 - s)\mathfrak{O}$. Those blocks of $\mathfrak{O}$ which come from blocks of $s\mathfrak{O}$, i.e., whose primitive central idempotents in $\mathfrak{O}$ are those of $s\mathfrak{O}$, are said to *lie over* B. Obviously every block of $\mathfrak{O}$ lies over the members of a unique G-conjugacy class of blocks of $\mathfrak{O}_1$.

PROPOSITION 3.6. *The map $y \to ye = ey$ is an isomorphism of the order $Z(s\mathfrak{O})$ onto $Z(e\mathfrak{O}e)$. The inverse map sends $z \in Z(e\mathfrak{O}e) \subseteq e\mathfrak{C}$ into $\sum z^{\tau}$, summed over representatives τ of the left cosets $(G_B)\tau$ of G_B in G. These maps define a one-to-one correspondence between the primitive central idempotents of $s\mathfrak{O}$ and those of $e\mathfrak{O}e$, and hence between the blocks of $\mathfrak{O}$ lying over B and the blocks of $e\mathfrak{O}e$.*

Proof. Since the central idempotent s of $\mathfrak{O}$ lies in $\mathfrak{O}_1$, it is trivial to verify that $s\mathfrak{O}$, G, and the $s\mathfrak{O}_{\sigma}$ also satisfy (2.1). The definition of s implies that e satisfies [CCT, 4.2] and [CCT, 4.5] with respect to the Clifford system $s\mathfrak{O}$, $\{s\mathfrak{O}_{\sigma} \mid \sigma \in G\}$. So [CCT, 4.4] and [CCT, 3.6] give the first statement of the proposition (see also Proposition 4.9 below).

If $z \in Z(e\mathfrak{O}e)$, then (3.4) implies that $y = \sum z^{\tau} \in C(G \text{ in } \mathfrak{C}) = Z(\mathfrak{O})$. Evidently $sz = ez = z$ forces $sz^{\tau} = z^{\tau}$, for all τ, and hence $sy = y$. Therefore $y \in Z(s\mathfrak{O})$. Because $ez^{\tau} = ee^{\tau}z^{\tau} = 0$, for any $\tau \in G - G_B$, we have $z = ey$, which proves the second statement of the proposition. The rest of the proposition follows directly from these two statements.

Putting together the preceding two results, we obtain:

THEOREM 3.7. *The map sending c into the image d in $\bar{\mathfrak{F}}[G[B]^*]$ of $ec \in \mathfrak{C}[B]$ defines a one-to-one correspondence between primitive central idempotents c of $s\mathfrak{O}$ and sums $d = d_1 + \cdots + d_n$ of G_B-conjugacy classes $d_1, \cdots, d_n$ of primitive central idempotents of $\bar{\mathfrak{F}}[G[B]^*]$. The inverse map sends such a sum $d \in \bar{\mathfrak{F}}[G[B]^*]$ into $c = \sum (d^*)^{\tau}$, where τ runs over representatives of the left cosets $(G_B)\tau$ of G_B in G and d^* is the unique central idempotent of $\mathfrak{C}[B]$ having d as its image. Thus these maps define a one-to-one correspondence between blocks of $\mathfrak{O}$ lying over B and G_B-conjugacy classes of blocks of $\bar{\mathfrak{F}}[G[B]^*]$.*

4. Green's theory

Green [7] has given a definition of defect groups in a very general setting. It is only necessary to have a finite group E acting as automorphisms of a ring $\mathfrak{D}$ with identity (actually, Green puts conditions on the ring $\mathfrak{D}$, forcing it to be an algebra over a field or a $\mathfrak{p}$-adic ring. These conditions are unnecessary if one uses maximal two-sided ideals in place of primitive idempotents, as we do below). We outline his theory here.

For simplicity, we denote by $\mathfrak{D}(E)$ the centralizer

$$C(E \text{ in } \mathfrak{D}) = \{d \in \mathfrak{D} \mid d^{\sigma} = d, \text{ for all } \sigma \in E\}.$$

Evidently $\mathfrak{D}(E)$ is a subring of $\mathfrak{D}$ containing the identity $1 = 1_{\mathfrak{D}}$. If D is a subgroup of E, let E/D be the family of left cosets $D\sigma$ of D in E. Then d^{τ}, for any $d \in \mathfrak{D}(D)$ and $\tau \in E/D$, can be defined as the common value of the d^{σ}, for $\sigma \in \tau$. Furthermore the *trace* $\mathrm{tr}_{D\to E}$ can be defined, as usual, by

$$\text{(4.1)} \qquad \mathrm{tr}_{D\to E}(d) = \textstyle\sum_{\tau \in E/D} d^{\tau}, \textit{ for all } d \in \mathfrak{D}(D).$$

Evidently $\mathrm{tr}_{D\to E}$ is a homomorphism of the additive group of $\mathfrak{D}(D)$ into $\mathfrak{D}(E)$ which preserves both left and right multiplication by elements of $\mathfrak{D}(E)$. So its image $\mathfrak{D}(E \mid D) = \mathrm{tr}_{D\to E}(\mathfrak{D}(D))$ is a two-sided ideal of the ring $\mathfrak{D}(E)$.

The ideals $\mathfrak{D}(E \mid D)$ have the following obvious properties:

$$\text{(4.2a)} \qquad \mathfrak{D}(E \mid D^{\sigma}) = \mathfrak{D}(E \mid D), \textit{ for all } \sigma \in E, D \le E,$$

$$\text{(4.2b)} \qquad \mathfrak{D}(E \mid D) \subseteq \mathfrak{D}(E \mid C), \textit{ for all } D \le C \le E,$$

$$\text{(4.2c)} \qquad \mathfrak{D}(E \mid E) = \mathfrak{D}(E).$$

Since $\mathrm{tr}_{D\to E}(d) = [E:D]d$, for any $d \in \mathfrak{D}(E)$, we have

$$\text{(4.3)} \qquad [E:D]\mathfrak{D}(E) \subseteq \mathfrak{D}(E \mid D), \textit{ for all } D \le E.$$

Because each map $d \to d^{\sigma}$, $\sigma \in E$, is a ring automorphism of $\mathfrak{D}$, one can compute (see (4.18) in [7]) that

$$\text{(4.4)} \qquad \mathfrak{D}(E \mid C)\mathfrak{D}(E \mid D) \subseteq \textstyle\sum_{\sigma \in E} \mathfrak{D}(E \mid C \cap D^{\sigma}), \textit{ for all } C, D \le E.$$

Let $\mathfrak{M}$ be a maximal two-sided ideal of $\mathfrak{D}(E)$. Since each $\mathfrak{D}(E \mid D)$, for $D \le E$, is a two-sided ideal of $\mathfrak{D}(E)$, it satisfies either $\mathfrak{D}(E \mid D) \subseteq \mathfrak{M}$ or $\mathfrak{D}(E \mid D) + \mathfrak{M} = \mathfrak{D}(E)$. A *defect group* of $\mathfrak{M}$ is a subgroup D of E which is minimal under inclusion among those satisfying the latter condition. Using (4.2)–(4.4) one easily verifies (see Theorems 4i and 4k in [7]) that:

(4.5) *The defect groups of $\mathfrak{M}$ form a single E-conjugacy class. If $\mathfrak{D}(E)/\mathfrak{M}$ has prime characteristic p, then any defect group of $\mathfrak{M}$ is a p-subgroup of E. If $\mathfrak{D}(E)/\mathfrak{M}$ has characteristic zero, then $\{1\}$ is the only defect group of $\mathfrak{M}$.*

If N is a normal subgroup of E, then the factor group E/N acts naturally as ring automorphisms of $\mathfrak{D}(N)$, with $y^{\tau} = y^{\sigma}$, for any coset $\tau \in E/N$ and any $\sigma \in \tau$. Evidently $\mathfrak{D}(E) = \mathfrak{D}(N)(E/N)$. So we can use this action of E/N on $\mathfrak{D}(N)$ to define defect groups in E/N of the maximal two-sided ideal $\mathfrak{M}$ of $\mathfrak{D}(E)$.

PROPOSITION 4.6. *If, in the above situation, D is a defect group of $\mathfrak{M}$ in E, then DN/N is a defect group of $\mathfrak{M}$ in E/N.*

Proof. Let C be any subgroup of E containing N. Then (4.1) implies that

$$\operatorname{tr}_{C/N\to E/N}(y) = \operatorname{tr}_{C\to E}(y),$$

for all $y \in \mathfrak{D}(N)(C/N) = \mathfrak{D}(C)$. Hence $\mathfrak{D}(N)(E/N \mid C/N) = \mathfrak{D}(E \mid C)$. It follows that the defect groups of $\mathfrak{M}$ in E/N have the form C^*/N where C^* is a minimal member of the family of all subgroups C of E containing N such that $\mathfrak{D}(E \mid C) \not\subseteq \mathfrak{M}$. In view of (4.2b) and (4.5), the last condition just says that C contains an E-conjugate D^σ of D. So C^* is simply a minimal member of the family of all subgroups of E containing $D^\sigma N$, for some $\sigma \in E$, i.e., $C^* = D^\sigma N = (DN)^\sigma$, for some $\sigma \in E$. In view of (4.5), this completes the proof of the proposition.

Suppose that d is an idempotent of $\mathfrak{D}$ satisfying:

(4.7a) *If $\sigma \in E$ and $d^\sigma \neq d$, then $d^\sigma d = 0$,*
(4.7b) *d centralizes $\mathfrak{D}(E)$.*

Let E_d be the subgroup of all elements of E fixing d. Condition (4.7a) implies that any two distinct E-conjugates of d are perpendicular idempotents of $\mathfrak{D}$. Hence the sum $c = \operatorname{tr}_{E_d\to E}(d)$ of the distinct E-conjugates of d is an idempotent in $\mathfrak{D}(E)$. In view of (4.7b), the idempotent c is central in $\mathfrak{D}(E)$. So we have the decomposition

$$\mathfrak{D}(E) = c\mathfrak{D}(E) \oplus (1 - c)\mathfrak{D}(E) \quad (\textit{as rings}).$$

The maximal two-sided ideal $\mathfrak{M}$ of $\mathfrak{D}(E)$ must contain either $c\mathfrak{D}(E)$ or $(1 - c)\mathfrak{D}(E)$. In the latter case we say that $\mathfrak{M}$ *lies over d*. Then it has the form

$$\mathfrak{M} = c\mathfrak{M} \oplus (1 - c)\mathfrak{D}(E), \tag{4.8}$$

where $c\mathfrak{M}$ is a maximal two-sided ideal of $c\mathfrak{D}(E)$.

Because E_d fixes d, it leaves invariant the subring $d\mathfrak{D}d$ of $\mathfrak{D}$. We use the action of E_d on $d\mathfrak{D}d$ to define the defect groups in E_d of any maximal two-sided ideal of $d\mathfrak{D}d(E_d)$.

PROPOSITION 4.9. *In the above situation, the map $y \to dy = yd$ is an isomorphism of the ring $c\mathfrak{D}(E)$ onto $d\mathfrak{D}d(E_d)$. The inverse map sends any $z \in d\mathfrak{D}d(E_d)$ into*

$$\operatorname{tr}_{E_d\to E}(z) \in c\mathfrak{D}(E).$$

If $\mathfrak{M}$ is a maximal two-sided ideal of $\mathfrak{D}(E)$ lying over d, then any defect group in E_d of the corresponding maximal two-sided ideal $d\mathfrak{M} = dc\mathfrak{M}$ of $d\mathfrak{D}d(E_d)$ is also a defect group of $\mathfrak{M}$ in E.

Proof. Since d is an idempotent in $\mathfrak{D}$, condition (4.7b) implies that

$$\varphi : y \to dy = yd$$

is a ring homomorphism of $c\mathfrak{D}(E)$ into $d\mathfrak{D}d$. Because E_d fixes both d and $y \in c\mathfrak{D}(E)$, the image $\varphi(c\mathfrak{D}(E))$ is contained in $d\mathfrak{D}d(E_d)$. Using the E-

invariance of y, we obtain

$$\mathrm{tr}_{E_d \to E}(dy) = \sum_{\sigma \in E/E_d} (dy)^\sigma = \sum_{\sigma \in E/E_d} d^\sigma y = cy = y.$$

Hence φ is a ring monomorphism with $\mathrm{tr}_{E_d \to E}$ as a left inverse.

If $z \in d\mathfrak{D}d(E_d)$, then $z = dzd$. In view of (4.7a), this implies that $d^\sigma z = zd^\sigma = 0$, for any E-conjugate $d^\sigma \neq d$ of d. We conclude that $d^\rho z^\tau = z^\tau d^\rho = 0$, for any two distinct elements $\rho, \tau \in E/E_d$. It follows that $\mathrm{tr}_{E_d \to E}(z) \in \mathfrak{D}(E)$ satisfies

$$\begin{aligned} c\, \mathrm{tr}_{E_d \to E}(z) &= \left(\sum_{\rho \in E/E_d} d^\rho\right)\left(\sum_{\tau \in E/E_d} z^\tau\right) = \sum_{\tau \in E/E_d} d^\tau z^\tau = \mathrm{tr}_{E_d \to E}(dz). \\ &= \mathrm{tr}_{E_d \to E}(z). \end{aligned}$$

Therefore $\mathrm{tr}_{E_d \to E}$ sends $d\mathfrak{D}d(E_d)$ into $c\mathfrak{D}(E)$. Furthermore,

$$d\, \mathrm{tr}_{E_d \to E}(z) = d \sum_{\tau \in E/E_d} z^\tau = dz = z.$$

So $\mathrm{tr}_{E_d \to E}$ is also a right inverse to φ, which proves the first two statements of the proposition.

Since $c\mathfrak{M}$ is a maximal two-sided ideal of $c\mathfrak{D}(E)$ and $dc = d$ (by (4.7a)), the above result implies that $dc\mathfrak{M} = d\mathfrak{M}$ is a maximal two-sided ideal of $d\mathfrak{D}d(E_d)$. Let D be any defect group in E_d of $d\mathfrak{M}$. Then there exists an element $w \in d\mathfrak{D}d(D)$ such that

$$\mathrm{tr}_{D \to E_d}(w) \in d\mathfrak{D}d(E_d) - d\mathfrak{M}.$$

The above results tell us that $\mathrm{tr}_{D \to E}(w) = \mathrm{tr}_{E_d \to E}(\mathrm{tr}_{D \to E_d}(w))$ lies in $c\mathfrak{D}(E) - c\mathfrak{M}$, and hence in $\mathfrak{D}(E) - \mathfrak{M}$. Therefore $\mathfrak{D}(E \mid D) \not\subseteq \mathfrak{M}$ and D contains a defect group C of $\mathfrak{M}$ in E.

There exists an element $x \in \mathfrak{D}(C)$ such that $\mathrm{tr}_{C \to E}(x) \notin \mathfrak{M}$. In view of (4.8), the projection $c\, \mathrm{tr}_{C \to E}(x)$ of $\mathrm{tr}_{C \to E}(x) \in \mathfrak{D}(E)$ into $c\mathfrak{D}(E)$ cannot lie in $c\mathfrak{M}$. By the above results, this implies that $dc\, \mathrm{tr}_{C \to E}(x) = d\, \mathrm{tr}_{C \to E}(x) \notin d\mathfrak{M}$. But

$$\begin{aligned} d\, \mathrm{tr}_{C \to E}(x) &= d\, \mathrm{tr}_{C \to E}(x) d = \sum_{\sigma \in E/C} dx^\sigma d = \sum_\tau \left\{ \sum_{\rho \in E_d/(E_d \cap C^\tau)} (dx^\tau d)^\rho \right\} \\ &= \sum_\tau \mathrm{tr}_{E_d \cap C^\tau \to E_d}(dx^\tau d), \end{aligned}$$

where τ runs over a family of representatives for the double cosets $C\tau E_d$ in E. Evidently each $dx^\tau d$ lies in $d\mathfrak{D}d(E_d \cap C^\tau)$. So the above equation implies that some $\mathrm{tr}_{E_d \cap C^\tau \to E_d}(dx^\tau d)$ does not lie in $d\mathfrak{M}$, and therefore that $E_d \cap C^\tau$ contains a defect group of $d\mathfrak{M}$ in E_d. In view of (4.5), such a defect group is conjugate to D. Hence

$$|C| = |C^\tau| \geq |E_d \cap C^\tau| \geq |D|.$$

Since $C \leq D$, this implies $C = D$, which completes the proof of the proposition.

5. An operator group for $\mathfrak{C}$

We shall first apply Green's theory to the case in which $\mathfrak{D}$ is the order $\mathfrak{C}$ of §2. To do things with the proper generality we assume (in addition to the hypotheses of §1 and §2) that:

(5.1a) *The residue class field* $\bar{\mathfrak{F}}$ *has prime characteristic* p.

(5.1b) *A finite group* E *acts as* $\mathfrak{R}$*-automorphisms of the order* $\mathfrak{C}$, *and as automorphisms of the group* G.

(5.1c) $(\mathfrak{C}_\sigma)^\tau = \mathfrak{C}_{\sigma^\tau}$, *for all* $\sigma \in G$, $\tau \in E$.

Evidently $\mathfrak{C}(E)$ is a pure $\mathfrak{R}$-submodule and hence a suborder of $\mathfrak{C}$. In view of (4.5), conditions (5.1a) and (1.2) imply:

(5.2) *Any defect group of any* $\mathfrak{M} \in \operatorname{Max}(\mathfrak{C}(E))$ *is a* p*-subgroup of* E.

It is clear from (5.1c) that E permutes among themselves the primitive idempotents of the central suborder $\mathfrak{C}_1$ of $\mathfrak{C}$. Evidently the primitive idempotent e of that suborder satisfies (4.7) for E. Denoting by E_B the subgroup of all $\sigma \in E$ fixing e, we obtain from Proposition 4.9:

(5.3) *The map* $y \to ey = ye$ *is an isomorphism of the suborder* $\operatorname{tr}_{E_B\to E}(e)\,\mathfrak{C}(E)$ *onto* $e\mathfrak{C}(E_B)$ *whose inverse sends* $z \in e\mathfrak{C}(E_B)$ *into* $\operatorname{tr}_{E_B\to E}(z)$. *If* $\mathfrak{M} \in \operatorname{Max}(\mathfrak{C}(E))$ *lies over* e, *then the defect groups in* E_B *of the corresponding ideal* $e\mathfrak{M} \in \operatorname{Max}(e\mathfrak{C}(E_B))$ *are also defect groups of* $\mathfrak{M}$ *in* E.

From (5.1c) we obtain

$$(e\mathfrak{C}_\sigma)^\tau = e\mathfrak{C}_{\sigma^\tau}, \text{ for all } \sigma \in G,\ \tau \in E_B. \tag{5.4}$$

In view of (2.9) and (5.1b), this implies:

(5.5a) E_B *leaves invariant the subgroup* $G[B]$ *of* G *and the suborder* $\mathfrak{C}[B]$ *of* $e\mathfrak{C}$,

(5.5b) $(\mathfrak{C}[B]_\sigma)^\tau = \mathfrak{C}[B]_{\sigma^\tau}$, *for all* $\sigma \in G[B]$, $\tau \in E_B$.

We pass easily from $e\mathfrak{C}$ to $\mathfrak{C}[B]$.

Proposition 5.6. *The order* $e\mathfrak{C}(E_B)$ *satisfies*

$$e\mathfrak{C}(E_B) = \mathfrak{C}[B](E_B) + J(e\mathfrak{C}(E_B)). \tag{5.7}$$

So there is a one-to-one correspondence between ideals $\mathfrak{M} \in \operatorname{Max}(e\mathfrak{C}(E_B))$ *and ideals* $\mathfrak{N} \in \operatorname{Max}(\mathfrak{C}[B](E_B))$, *in which* $\mathfrak{M}$ *corresponds to* $\mathfrak{M} \cap \mathfrak{C}[B](E_B)$ *and* $\mathfrak{N}$ *to* $\mathfrak{N} + J(e\mathfrak{C}(E_B))$. *If two such* $\mathfrak{M}$ *and* $\mathfrak{N}$ *correspond, then their defect groups, defined by the actions of* E_B *on* $e\mathfrak{C}$ *and* $\mathfrak{C}[B]$ *respectively, coincide.*

Proof. Let $\mathfrak{L} = \oplus \sum_{\sigma\in G-G[B]} e\mathfrak{C}_\sigma$. In view of (5.4), (5.5a), (2.8d), and (2.9b), the order $e\mathfrak{C}$ is the direct sum of its two E_B-invariant $\mathfrak{R}$-sublattices $\mathfrak{C}[B]$ and $\mathfrak{L}$. Defining $\mathfrak{L}(E_B)$, as usual, to be the centralizer of E_B in $\mathfrak{L}$, we conclude that

$$e\mathfrak{C}(E_B) = \mathfrak{L}(E_B) \oplus \mathfrak{C}[B](E_B).$$

Proposition 2.17 and Lemma 3.3 imply that $\mathfrak{L} \subseteq J(e\mathfrak{C})$. Applying Proposition 1.9, we obtain $\mathfrak{L}(E_B) \subseteq J(e\mathfrak{C}) \cap e\mathfrak{C}(E_B) \subseteq J(e\mathfrak{C}(E_B))$. Therefore (5.7) holds.

The second statement of the proposition follows directly from (5.7) and Proposition 1.9. For the third statement, notice that $\mathfrak{M} = \mathfrak{L}(E_B) \oplus \mathfrak{N}$ while

$$e\mathfrak{C}(E_B \mid D) = \mathfrak{L}(E_B \mid D) \oplus \mathfrak{C}[B](E_B \mid D),$$

for any subgroup D of E_B (where, of course, $\mathfrak{L}(E_B \mid D) = \mathrm{tr}_{D\to E_B}(\mathfrak{L}(D))$). Since $\mathfrak{L}(E_B \mid D) \subseteq \mathfrak{L}(E_B)$, we conclude that $e\mathfrak{C}(E_B \mid D) \not\subseteq \mathfrak{M}$ if and only if $\mathfrak{C}[B](E_B \mid D) \not\subseteq \mathfrak{N}$, which implies the rest of the proposition.

Equation (5.5b) implies that $\mathfrak{C}[B]_1$ is E_B-invariant. Hence so are its radical $J(\mathfrak{C}[B]_1)$ and the ideal $\mathfrak{C}[B]J(\mathfrak{C}[B]_1)$ which it generates. From this, (2.14), and (5.5), we conclude that:

(5.8a) *The action of E_B on $\mathfrak{C}[B]$ induces an action of E_B as $\bar{\mathfrak{F}}$-automorphisms of the algebra $\bar{\mathfrak{F}}[G[B]^*] = \mathfrak{C}[B]/\mathfrak{C}[B]J(\mathfrak{C}[B]_1)$,*

(5.8b) $(\bar{\mathfrak{F}}[G[B]^*]_\sigma)^\tau = \bar{\mathfrak{F}}[G[B]^*]_{\sigma^\tau}$, *for all $\sigma \in G[B]$, $\tau \in E_B$.*

The image $\mathfrak{C}[B](E_B)^J$ of $\mathfrak{C}[B](E_B)$ in $\bar{\mathfrak{F}}[G[B]^*]$ can very well be a proper subalgebra of $\bar{\mathfrak{F}}[G[B]^*](E_B)$ (see Example 13.11 below), a circumstance which considerably complicates the analysis. Nevertheless, we can obtain a useful characterization of defect groups of ideals $\mathfrak{M} \in \mathrm{Max}(\mathfrak{C}[B](E_B))$ in terms of defect groups of primitive idempotents in $\mathfrak{C}[B](E_B)^J$.

Let y be a non-zero element of $\bar{\mathfrak{F}}[G[B]^*]$. Then $y = \sum_{\sigma\in G[B]} y_\sigma$, where the elements $y_\sigma \in \bar{\mathfrak{F}}[G[B]^*]_\sigma$, for $\sigma \in G[B]$, are unique and not all zero. We denote by $\Delta(y)$ the family of all p-Sylow subgroups of all centralizers $C(\sigma \text{ in } E_B)$ in E_B of elements $\sigma \in G[B]$ for which $y_\sigma \neq 0$. Then $\Delta(y)$ is a non-empty family of p-subgroups of E_B whose maximal elements (under inclusion) will be called the *defect groups* of y. Evidently these defect groups do not necessarily form a single E_B-conjugacy class. In fact, they need not even be closed under E_B-conjugation.

By convention $\Delta(0)$ will consist only of the trivial subgroup $\{1\}$ of E_B, which is therefore the only defect group of $0 \in \bar{\mathfrak{F}}[G[B]^*]$.

LEMMA 5.9. *If $D \leq E_B$, then the image $\mathfrak{C}[B](E_B \mid D)^J$ in $\bar{\mathfrak{F}}[G[B]^*]$ of $\mathfrak{C}[B](E_B \mid D)$ is the set of all elements $y \in \mathfrak{C}[B](E_B)^J$ such that each subgroup $P \in \Delta(y)$ is contained in some E_B-conjugate of D.*

Proof. Suppose that $y \in \mathfrak{C}[B](E_B \mid D)^J$. Then y certainly lies in $\mathfrak{C}[B](E_B)^J$. If $y = 0$, then by convention the only group $\{1\}$ in $\Delta(y)$ is contained in D. So we can suppose that $0 \neq y = \sum_{\sigma\in G[B]} y_\sigma$, where $y_\sigma \in \bar{\mathfrak{F}}[G[B]^*]_\sigma$, for all $\sigma \in G[B]$.

Because y lies in $\mathfrak{C}[B](E_B \mid D)^J$, there is some element $x \in \mathfrak{C}[B](D)$ such that y is the image of $\mathrm{tr}_{D\to E_B}(x)$. Hence $y = \mathrm{tr}_{D\to E_B}(z)$ where $z \in \bar{\mathfrak{F}}[G[B]^*](D)$ is the image of x. Write $z = \sum_{\sigma\in G[B]} z_\sigma$, where $z_\sigma \in \bar{\mathfrak{F}}[G[B]^*]_\sigma$, for all $\sigma \in G[B]$. The D-invariance of z and (5.8b) imply that $(z_\sigma)^\tau = z_{\sigma^\tau}$, for all $\sigma \in G[B]$,

$\tau \in D$. It follows that $C(\sigma \text{ in } D)$ centralizes z_σ, for all $\sigma \in G[B]$, and that

$$z = \sum_{\sigma \in S} \operatorname{tr}_{C(\sigma \text{ in } D) \to D}(z_\sigma),$$

where S is a family of representatives for the D-orbits in $G[B]$. Hence

$$y = \operatorname{tr}_{D \to E_B}(z) = \sum_{\sigma \in S} \operatorname{tr}_{C(\sigma \text{ in } D) \to E_B}(z_\sigma) = \sum_{\sigma \in S} \operatorname{tr}_{C(\sigma \text{ in } E_B) \to E_B}(w_\sigma),$$

where $w_\sigma = \operatorname{tr}_{C(\sigma \text{ in } D) \to C(\sigma \text{ in } E_B)}(z_\sigma)$, for all $\sigma \in S$.

Suppose that $w_\sigma \neq 0$, for some $\sigma \in S$. By (5.8b), w_σ lies in the one-dimensional subspace $\bar{\mathfrak{F}}[G[B]^*]_\sigma$. Hence $\bar{\mathfrak{F}}[G[B]^*]_\sigma$ equals $\bar{\mathfrak{F}} w_\sigma$. So it is centralized by $C(\sigma \text{ in } E_B)$. In particular, $C(\sigma \text{ in } E_B)$ centralizes $z_\sigma \in \bar{\mathfrak{F}}[G[B]^*]_\sigma$. Hence

$$w_\sigma = \operatorname{tr}_{C(\sigma \text{ in } D) \to C(\sigma \text{ in } E_B)}(z_\sigma) = [C(\sigma \text{ in } E_B) : C(\sigma \text{ in } D)] z_\sigma .$$

Because w_σ is non-zero, this and (5.1a) imply that p does not divide the index

$$[C(\sigma \text{ in } E_B) : C(\sigma \text{ in } D)],$$

and thus that $C(\sigma \text{ in } D)$ contains a p-Sylow subgroup P of $C(\sigma \text{ in } E_B)$. Clearly (5.8b) and the above expression for y imply that any $\rho \in G[B]$ satisfying $y_\rho \neq 0$ is E_B-conjugate to some $\sigma \in S$ for which $w_\sigma \neq 0$. So the above argument shows that the members of $\Delta(y)$ are all E_B-conjugate to p-Sylow subgroups P of $C(\sigma \text{ in } D)$, for such σ, and hence are all contained in E_B-conjugates of D.

Now let y be any element of $\mathfrak{C}[B](E_B)^J$ such that each subgroup $P \in \Delta(y)$ is contained in some E_B-conjugate of D. Since 0 certainly lies in $\mathfrak{C}[B](E_B \mid D)^J$, we can assume that $0 \neq y = \sum_{\sigma \in G[B]} y_\sigma$, where $y_\sigma \in \bar{\mathfrak{F}}[G[B]^*]_\sigma$, for all $\sigma \in G[B]$. Choose any element $c \in \mathfrak{C}[B](E_B)$ having y as image, and write $c = \sum_{\sigma \in G[B]} c_\sigma$, where $c_\sigma \in \mathfrak{C}[B]_\sigma$, for all $\sigma \in G[B]$. The E_B-invariance of c and (5.5b) give $(c_\sigma)^\tau = c_{\sigma^\tau}$, for all $\sigma \in G[B]$, $\tau \in E_B$. Hence $C(\sigma \text{ in } E_B)$ centralizes c_σ, for any $\sigma \in G[B]$, and

$$c = \sum_{\sigma \in T} \operatorname{tr}_{C(\sigma \text{ in } E_B) \to E_B}(c_\sigma),$$

where T is a family of representatives for the E_B-orbits in $G[B]$.

If the image y_σ of c_σ is non-zero, then any p-Sylow subgroup P of $C(\sigma \text{ in } E_B)$ lies in $\Delta(y)$ and hence is contained in an E_B-conjugate of D. Evidently we can choose the family T so that D contains a p-Sylow subgroup P of $C(\sigma \text{ in } E_B)$ whenever $\sigma \in T$ and $y_\sigma \neq 0$. Then the index $[C(\sigma \text{ in } E_B) : C(\sigma \text{ in } D)]$ is not divisible by p. By (5.1a) its image is a unit in $\mathfrak{R}$. So we can form the element

$$b_\sigma = [C(\sigma \text{ in } E_B) : C(\sigma \text{ in } D)]^{-1} c_\sigma \in \mathfrak{C}[B]_\sigma(C(\sigma \text{ in } E_B)).$$

Then

$$\operatorname{tr}_{C(\sigma \text{ in } E_B) \to E_B}(c_\sigma) = \operatorname{tr}_{C(\sigma \text{ in } D) \to E_B}(b_\sigma) = \operatorname{tr}_{D \to E_B}(a_\sigma),$$

where $a_\sigma = \operatorname{tr}_{C(\sigma \text{ in } D) \to D}(b_\sigma) \in \mathfrak{C}[B](D)$. It follows that y is the image of

$$\sum_{\sigma \in T,\ y_\sigma \neq 0} \operatorname{tr}_{C(\sigma \text{ in } E_B) \to E_B}(c_\sigma) = \sum_{\sigma \in T,\ y_\sigma \neq 0} \operatorname{tr}_{D \to E_B}(a_\sigma) \in \mathfrak{C}[B](E_B \mid D).$$

Therefore $y \in \mathfrak{C}[B](E_B \mid D)^J$, and the lemma is proved.

The above lemma leads to the following method for computing defect groups.

PROPOSITION 5.10. *Any ideal* $\mathfrak{M} \in \mathrm{Max}\,(\mathfrak{C}[B](E_B))$ *is the inverse image of its image* $\mathfrak{M}^J$ *in* $\mathfrak{C}[B](E_B)^J$. *So* $\mathfrak{M} \to \mathfrak{M}^J$ *sends* $\mathrm{Max}\,(\mathfrak{C}[B](E_B))$ *one to one onto* $\mathrm{Max}\,(\mathfrak{C}[B](E_B)^J)$. *If* $\mathfrak{M} \in \mathrm{Max}\,(\mathfrak{C}[B](E_B))$ *and* d *is any primitive idempotent of the algebra* $\mathfrak{C}[B](E_B)^J$ *corresponding to* $\mathfrak{M}^J$ *(i.e., satisfying* $d \notin \mathfrak{M}^J$*), then the defect groups in* E_B *of the non-zero element* d *of* $\bar{\mathfrak{F}}[G[B]^*]$ *coincide with the defect groups in* E_B *of* $\mathfrak{M}$. *In particular, these defect groups of* d *form a single* E_B*-conjugacy class.*

Proof. Propositions 2.5 and 2.10 tell us that $\mathfrak{C}[B]J(\mathfrak{C}[B]_1) \subseteq J(\mathfrak{C}[B])$. In view of Proposition 1.9, this implies that the kernel

$$\mathfrak{C}[B](E_B) \cap \mathfrak{C}[B]J(\mathfrak{C}[B]_1)$$

of the epimorphism

$$\mathfrak{C}[B](E_B) \to \mathfrak{C}[B](E_B)^J$$

is contained in $J(\mathfrak{C}[B](E_B))$. The first two statements of the proposition follow directly from this.

For the third and fourth statements, let D be any defect group of $\mathfrak{M}$ in E_B. Then the primitive idempotent d lies in

$$\mathfrak{C}[B](E_B)^J = \mathfrak{M}^J + \mathfrak{C}[B](E_B \mid D)^J.$$

Since both $\mathfrak{M}^J$ and $\mathfrak{C}[B](E_B \mid D)^J$ are two-sided ideals of the finite-dimensional $\bar{\mathfrak{F}}$-algebra $\mathfrak{C}[B](E_B)^J$, we conclude (see Lemma 3.3a of [6]), that d lies in at least one of them. But $d \notin \mathfrak{M}^J$. Hence $d \in \mathfrak{C}[B](E_B \mid D)^J$. By Lemma 5.9, every group $P \in \Delta(d)$ is contained in some E_B-conjugate of D. So we can complete the proof by showing that every E_B-conjugate of D lies in $\Delta(d)$.

Write $d = \sum_{\sigma \in G[B]} d_\sigma$, where $d_\sigma \in \bar{\mathfrak{F}}[G[B]^*]_\sigma$, for all $\sigma \in G[B]$. We know that d is the image of an element $y \in \mathfrak{C}[B](E_B)$, which we can write as $y = \sum_{\sigma \in G[B]} y_\sigma$, where $y_\sigma \in \mathfrak{C}[B]_\sigma$, for all $\sigma \in G[B]$. In view of (5.5b), the E_B-invariance of y implies that of $y_S = \sum_{\sigma \in S} y_\sigma$, for every E_B-orbit S of $G[B]$. Hence the image $d_S = \sum_{\sigma \in S} d_\sigma$ of y_S also lies in $\mathfrak{C}[B](E_B)^J$. Because the sum d of the various d_S does not lie in $\mathfrak{M}^J$, we can fix an E_B-orbit S of $G[B]$ such that $d_S \notin \mathfrak{M}^J$. Evidently $d_S \neq 0$. The E_B-invariance of d_S and (5.8b) imply that $d_\sigma \neq 0$ for all $\sigma \in S$. So $\Delta(d_S)$ consists of the E_B-conjugates of a p-Sylow subgroup P of $C(\sigma \text{ in } E_B)$, for some $\sigma \in S$. Applying Lemma 5.9, we see that $d_S \in \mathfrak{C}[B](E_B \mid P)^J - \mathfrak{M}^J$. Hence P contains an E_B-conjugate D^τ of the defect group D of $\mathfrak{M}$. On the other hand $P \in \Delta(d_S) \subseteq \Delta(d)$ is contained in an E_B-conjugate of D. We conclude that $P = D^\tau$, and hence that $\Delta(d)$ contains the set $\Delta(d_S)$ of all E_B-conjugates of D. As remarked above, this completes the proof of the proposition.

6. Defect groups in G

We shall use Proposition 5.10 in conjunction with Lemma 3.1 to compute defect groups of blocks. To do so, we must assume, in addition to the hypotheses of §§1, 2 and 5, that:

(6.1a) *G is a normal subgroup of E.*

(6.1b) *The action of E on G is by conjugation in E: $\sigma^{\tau} = \tau^{-1}\sigma\tau$, for all $\sigma \in G$, $\tau \in E$.*

(6.1c) *The restriction to G of the action of E on $\mathfrak{C}$ is the action* (2.15) *of G on $\mathfrak{C}$.*

Condition (6.1c) implies that $\mathfrak{C}(G)$ is simply the center $Z(\mathfrak{O})$ of $\mathfrak{O}$. Hence it and its suborder $\mathfrak{C}(E)$ are both commutative. Because the order $\mathfrak{C}(E)$ is commutative, there is, in view of Proposition 1.12, a natural one-to-one correspondence between primitive idempotents d of $\mathfrak{C}(E)$ and maximal ideals $\mathfrak{M} \in \operatorname{Max}(\mathfrak{C}(E))$ in which

(6.2) *$\mathfrak{M}$ corresponds to d if and only if* $\mathfrak{M} = (1 - d)\,\mathfrak{C}(E) \oplus J(d\,\mathfrak{C}(E))$.

By (6.1a) and (5.1b) the group E permutes among themselves the primitive idempotents of $\mathfrak{C}(G) = Z(\mathfrak{O})$, which correspond to the blocks of $\mathfrak{O}$. Evidently this defines an action of E on the blocks of $\mathfrak{O}$. The primitive idempotents of the suborder $\mathfrak{C}(E)$ of the commutative order $\mathfrak{C}(G)$ are clearly the sums of the elements of the E-orbits of primitive idempotents of $\mathfrak{C}(G)$. In view of the one-to-one correspondence (6.2), we obtain a natural one-to-one correspondence between E-orbits S of blocks of $\mathfrak{O}$ and ideals $\mathfrak{M} \in \operatorname{Max}(\mathfrak{C}(E))$. We define the *defect groups* in E of such an E-orbit S to be those of the corresponding ideal $\mathfrak{M}$. If S has only one element $\tilde{B}$, we also call these the defect groups of $\tilde{B}$ in E. In general, we denote by $E_{\tilde{B}}$ the subgroup of all elements of E fixing a block $\tilde{B} \in S$. Then $E_{\tilde{B}} = E_d$, where d is the primitive central idempotent of $\mathfrak{O}$ (and hence central idempotent of $\mathfrak{C}$) corresponding to $\tilde{B}$. Evidently (4.7) holds with $\mathfrak{C}$ in place of $\mathfrak{D}$. So Proposition 4.9 and the correspondence (6.2) give:

(6.3) *If $\tilde{B}$ is a block of $\mathfrak{O}$, then any defect group in $E_{\tilde{B}}$ of $\tilde{B}$ is also a defect group in E of the E-orbit of $\tilde{B}$.*

It follows easily from (6.1c), (2.14b), and (2.15) that the induced action of $G[B]$ on $\bar{\mathfrak{F}}[G[B]^*]$ is the ordinary conjugation action

(6.4) $y^{\sigma} = y^{\rho} = \rho^{-1}y\rho$, *for all* $y \in \bar{\mathfrak{F}}[G[B]^*]$, $\sigma \in G[B]$, $\rho \in \operatorname{pr}^{-1}(\sigma)$.

This implies that $\bar{\mathfrak{F}}[G[B]^*](G[B])$ is the center $Z(\bar{\mathfrak{F}}[G[B]^*])$ of $\bar{\mathfrak{F}}[G[B]^*]$. Since $G[B]$ is a normal subgroup of E_B, by (5.5a) and (6.1a, b), we can repeat the above analysis, obtaining natural one-to-one correspondences between E_B-orbits of blocks of $\bar{\mathfrak{F}}[G[B]^*]$, primitive idempotents of $\bar{\mathfrak{F}}[G[B]^*](E_B)$, and ideals in $\operatorname{Max}(\bar{\mathfrak{F}}[G[B]^*](E_B))$. As above, we define the defect groups in E_B of an E_B-orbit T of blocks of $\bar{\mathfrak{F}}[G[B]^*]$ to be those of the corresponding ideal $\mathfrak{N} \in \operatorname{Max}(\bar{\mathfrak{F}}[G[B]^*](E_B))$.

The subgroup G_B is normal in E_B by (6.1a, c). So the E_B-orbits of blocks of $\bar{\mathfrak{F}}[G[B]^*]$ are simply the unions of the E_B-orbits of G_B-conjugacy classes of blocks of $\bar{\mathfrak{F}}[G[B]^*]$. Theorem 3.7 gives a one-to-one correspondence between these G_B-conjugacy classes and the blocks of $\mathfrak{O}$ lying over B. This correspondence is clearly E_B-invariant, and hence defines a one-to-one correspondence between E_B-orbits of blocks of $\bar{\mathfrak{F}}[G[B]^*]$ and E_B-orbits of blocks of $\mathfrak{O}$ lying over B.

We shall say that an E-orbit S of blocks of $\mathfrak{O}$ *lies over* B if some $\tilde{B} \in S$ lies over B. Since $(\tilde{B})^\sigma$ lies over B^σ, for all $\sigma \in E$, it is clear that the members of S lying over B form an E_B-orbit R, and that $S \leftrightarrow R$ is a one-to-one correspondence between E-orbits S lying over B of blocks of $\mathfrak{O}$ and E_B-orbits R of blocks of $\mathfrak{O}$ lying over B. Combined with the results of the preceding paragraph, this gives us a natural one-to-one correspondence between E_B-orbits of blocks of $\bar{\mathfrak{F}}[G[B]^*]$ and E-orbits lying over B of blocks of $\mathfrak{O}$.

THEOREM 6.5. *Let T be an E_B-orbit of blocks of $\bar{\mathfrak{F}}[G[B]^*]$ and S be the corresponding E-orbit lying over B of blocks of $\mathfrak{O}$. Then the defect groups of T in E_B are precisely the defect groups in E_B of the corresponding primitive central idempotent d of $\bar{\mathfrak{F}}[G[B]^*](E_B)$ considered as an element of $\bar{\mathfrak{F}}[G[B]^*]$. Furthermore, they are among the defect groups of S in E.*

Proof. Notice that conditions (2.1) are satisfied with $\bar{\mathfrak{F}}$, $\bar{\mathfrak{F}}[G[B]^*]$, $G[B]$, and the $\bar{\mathfrak{F}}[G[B]^*]_\sigma$ in place of $\mathfrak{R}$, $\mathfrak{O}$, G, and the $\mathfrak{O}_\sigma$, respectively. In this case 1 is the only primitive idempotent of $\bar{\mathfrak{F}}[G[B]^*]_1 \simeq \bar{\mathfrak{F}}$. It follows that the suborders corresponding to $\mathfrak{C}$, $e\mathfrak{C}$, and $\mathfrak{C}[B]$ all coincide with $\bar{\mathfrak{F}}[G[B]^*]$. Furthermore $J(\bar{\mathfrak{F}}[G[B]^*]_1) = \{0\}$, which implies that the algebra corresponding to $\bar{\mathfrak{F}}[G[B]^*]$ is

$$\bar{\mathfrak{F}}[G[B]^*]/\{0\} = \bar{\mathfrak{F}}[G[B]^*].$$

In view of (5.8), conditions (5.1) are also satisfied in this case, with E_B in place of E. Now Proposition 5.10 gives the first statement of the theorem.

The one-to-one correspondence of Lemma 3.1 between central idempotents of $\bar{\mathfrak{F}}[G[B]^*]$ and those of $\mathfrak{C}[B]$ obviously preserves the actions of E_B. It follows that any idempotent

$$d \in \bar{\mathfrak{F}}[G[B]^*](E_B) \subseteq Z(\bar{\mathfrak{F}}[G[B]^*])$$

is the image of an idempotent $d^* \in \mathfrak{C}[B](E_B)$. We conclude that $\bar{\mathfrak{F}}[G[B]^*](E_B)$ and its subalgebra $\mathfrak{C}[B](E_B)^J$ have the same idempotents. In particular, any primitive idempotent d of $\bar{\mathfrak{F}}[G[B]^*](E_B)$ is also a primitive idempotent of $\mathfrak{C}[B](E_B)^J$. The ideal $\mathfrak{N} \in \operatorname{Max}(\mathfrak{C}[B](E_B))$ corresponding to such a d in Proposition 5.10 obviously corresponds to d^* via (6.2). Furthermore, the ideal

$$\mathfrak{M} = \mathfrak{N} + J(e\mathfrak{C}(E_B)) \in \operatorname{Max}(e\mathfrak{C}(E_B))$$

corresponding to $\mathfrak{N}$ in Proposition 5.6 corresponds to the primitive idempotent d^* of $e\mathfrak{C}(E_B)$ via (6.2). Finally, the ideal

$$\mathfrak{L} = \operatorname{tr}_{E_B \to E}(\mathfrak{M}) \oplus (1 - \operatorname{tr}_{E_B \to E}(e))\mathfrak{C}(E) \in \operatorname{Max}(\mathfrak{C}(E))$$

corresponding to $\mathfrak{M}$ in (5.3) corresponds to the primitive idempotent $\operatorname{tr}_{E_B \to E}(d^*)$ of $\mathfrak{C}(E)$ via (6.2). From Theorem 3.7 we see that $\operatorname{tr}_{E_B \to E}(d^*)$, and hence $\mathfrak{L}$, correspond to S. Now (5.3) and Propositions 5.6 and 5.10 tell us that the defect groups in E_B of d considered as an element of $\bar{\mathfrak{F}}[G[B]^*]$ are among the defect groups of S in E. This completes the proof of the theorem.

COROLLARY 6.6. *Any defect group in G_B of a G_B-conjugacy class of blocks of $\bar{\mathfrak{F}}[G[B]^*]$ is also a defect group in G of the corresponding block of $\mathfrak{O}$.*

Proof. This is the special case $E = G$ of the theorem.

As a consequence of the above theorem, we have:

PROPOSITION 6.7. *Assume that E fixes a block $\tilde{B}$ of $\mathfrak{O}$. If D is a defect group in E of $\tilde{B}$, then $D \cap G$ is a defect group of $\tilde{B}$ in G, while $DG/G \simeq D/(D \cap G)$ is a p-Sylow subgroup of E/G.*

Proof. We can choose B so that $\tilde{B}$ lies over B. Then $S = \{\tilde{B}\}$ is an E-orbit lying over B. Let T be the corresponding E_B-orbit of blocks of $\bar{\mathfrak{F}}[G[B]^*]$. Since E fixes $\tilde{B}$, it leaves invariant the family of all blocks of $\mathfrak{O}_1$ lying under $\tilde{B}$, which is just the G-orbit of B. Hence $E = E_B G$. The last statement of Theorem 6.5 implies that D is E-conjugate to a defect group D_0 in E_B of T. Hence D is G-conjugate to an E_B-conjugate D_0^σ of D_0. Since we can replace D by a G-conjugate without changing the results of this proposition, and since D_0^σ is also a defect group in E_B of T, we can assume that $D = D_0$.

Let d be the primitive idempotent of $\bar{\mathfrak{F}}[G[B]^*](G_B)$ corresponding to $\tilde{B}$ in Theorem 3.7. Then d is also the primitive idempotent of $\bar{\mathfrak{F}}[G[B]^*](E_B)$ corresponding to T. Theorem 6.5 now tells us that D is a defect group in E_B of d, i.e., a maximal member of the family $\Delta(d)$ of all p-Sylow subgroups P of all centralizers $C(\sigma \text{ in } E_B)$ of all elements $\sigma \in G[B]$ such that the "σ^{th} component" d_σ of d is non-zero. For any such σ and P, the intersection $P \cap G_B = P \cap G$ is a p-Sylow subgroup of the normal subgroup

$$C(\sigma \text{ in } G_B) = C(\sigma \text{ in } E_B) \cap G$$

of $C(\sigma \text{ in } E_B)$. So the corresponding family $\Delta'(d)$, defined with G in place of E, is $\{P \cap G \mid P \in \Delta(d)\}$. The above analysis shows that any maximal element D_1 of $\Delta'(d)$ is a defect group in G of $\tilde{B}$. Since $D \cap G \in \Delta'(d)$, we can choose D_1 to contain $D \cap G$. But $D_1 = P \cap G$, for some $P \in \Delta(d)$, and P is contained in some maximal element of $\Delta(d)$, which must be an E_B-conjugate D^τ of D. Hence $D \cap G \leq D_1 \leq D^\tau \cap G = (D \cap G)^\tau$. Comparing orders, we see that $D_1 = D \cap G$. So $D \cap G$ is a defect group of $\tilde{B}$ in G.

Let c be the primitive idempotent of $Z(\mathfrak{O}) = \mathfrak{C}(G)$ corresponding to $\tilde{B}$. Then c is also the primitive idempotent of $\mathfrak{C}(E)$ corresponding to $S = \{\tilde{B}\}$. Applying Proposition 4.6, we see that DG/G is a defect group of c defined with respect to the induced action of E/G on $\mathfrak{C}(G)$. Because c is an E-invariant central idempotent of $\mathfrak{C}(G)$, this order is the ring direct sum of its E-invariant suborders $c\mathfrak{C}(G)$ and $(1 - c)\mathfrak{C}(G)$. It follows that DG/G is a defect group in E/G for the unique maximal ideal $J(c\mathfrak{C}(E))$ of $c\mathfrak{C}(E) = c\mathfrak{C}(G)(E/G)$.

The order $c\mathfrak{C}(G)$, being local, satisfies $c\mathfrak{C}(G) = \mathfrak{R}c + J(c\mathfrak{C}(G))$. It follows that

$$c\mathfrak{C}(C) = \mathfrak{R}c + [c\mathfrak{C}(C) \cap J(c\mathfrak{C}(G))],$$

for any group C such that $G \leq C \leq E$. If $y \in c\mathfrak{C}(C) \cap J(c\mathfrak{C}(G))$, then $y^{\rho} \in J(c\mathfrak{C}(G))$, for all $\rho \in E$. Hence

$$\mathrm{tr}_{C\to E}(y) \in c\mathfrak{C}(E) \cap J(c\mathfrak{C}(G)) \subseteq J(c\mathfrak{C}(E)),$$

by Proposition 1.9. Therefore

$$c\mathfrak{C}(E \mid C) = \mathfrak{R}\ \mathrm{tr}_{C\to E}(c) + [c\mathfrak{C}(E \mid C) \cap J(c\mathfrak{C}(E))].$$

It follows that $c\mathfrak{C}(E \mid C) \nsubseteq J(c\mathfrak{C}(E))$ if and only if $\mathrm{tr}_{C\to E}(c) = [E:C]c \notin J(c\mathfrak{C}(E))$. In view of (1.2) and (5.1a), this occurs if and only if p does not divide $[E:C] = [E/G:C/G]$. We conclude that the defect groups in E/G of $J(c\mathfrak{C}(E))$ are precisely the p-Sylow subgroups of that group. Since DG/G is among these defect groups, this completes the proof of the proposition.

7. Defect groups in H

If we return to the situation $\mathfrak{O} = \mathfrak{R}H$ of the beginning of §2, we see that the above results are inadequate. Using them, we can only compute the defect groups in $G = H/K$ of blocks of $\mathfrak{R}H$, while what we really want are their defect groups in H. So we must go a bit deeper into the structure. Instead of just studying $\mathfrak{R}H$, we shall consider the more general situation in which:

(7.1a) *$\mathfrak{D}$, H, and the $\mathfrak{D}_\sigma$ satisfy* (2.1) *in place of $\mathfrak{O}$, G, and the $\mathfrak{O}_\sigma$, respectively.*

(7.1b) *$\mathfrak{D}_1$ is a local ring contained in the center $Z(\mathfrak{D})$ of $\mathfrak{D}$.*

(7.1c) *The residue class field $\bar{\mathfrak{F}}$ has prime characteristic p.*

Of course, we obtain the special case $\mathfrak{R}H$ by taking $\mathfrak{D} = \mathfrak{R}H$ and $\mathfrak{D}_\sigma = \mathfrak{R}\sigma$, for all $\sigma \in H$.

Evidently (7.1b) implies that $\mathfrak{D} = C(\mathfrak{D}_1 \text{ in } \mathfrak{D})$. Therefore H acts naturally as $\mathfrak{R}$-automorphisms of $\mathfrak{D}$ via (2.15). So we can introduce groups K and E satisfying:

(7.2a) *H is a normal subgroup of the finite group E.*

(7.2b) *The action* (2.15) *of H on $\mathfrak{D} = C(\mathfrak{D}_1$ in $\mathfrak{D})$ extends to an action of E as $\mathfrak{R}$-automorphisms of the order $\mathfrak{D}$.*

(7.2c) $(\mathfrak{D}_\sigma)^\tau = \mathfrak{D}_{\sigma^\tau} = \mathfrak{D}_{\tau^{-1}\sigma\tau}$, *for all $\sigma \in H$, $\tau \in E$.*

(7.2d) *K is an E-invariant normal subgroup of H.*

Notice that conditions (5.1) and (6.1) are now satisfied with E, $\mathfrak{D}$, H, and the $\mathfrak{D}_\sigma$ in place of E, $\mathfrak{C}$, G, and the $\mathfrak{C}_\sigma$, respectively. So we can define as usual the defect groups in E of E-conjugacy classes of blocks of $\mathfrak{D}$.

As at the beginning of §2, we set $G = H/K$, $\mathfrak{O} = \mathfrak{D}$, and $\mathfrak{O}_\tau = \oplus \sum_{\sigma\in\tau} \mathfrak{D}_\sigma$, for each coset $\tau \in G = H/K$. Evidently $\mathfrak{O}$, G, and the $\mathfrak{O}_\sigma$ satisfy (2.1).

From (2.15) for $\mathfrak{D}$ it is clear that $\mathfrak{C} = C(\mathfrak{O}_1 \text{ in } \mathfrak{O}) = C(K \text{ in } \mathfrak{D}) = \mathfrak{D}(K)$. Since K is a normal subgroup of E (by (7.2d)), there is an induced action of E/K on $\mathfrak{C} = \mathfrak{D}(K)$. Of course, $G = H/K$ is a normal subgroup of E/K. Using (2.15) for both $\mathfrak{D}$ and $\mathfrak{O}$, one verifies easily that (5.1) and (6.1) are satisfied with E/K in place of E. So we can talk about the defect groups in E/K of E/K-conjugacy classes of blocks of $\mathfrak{O}$ (which are the same as the E-conjugacy classes of blocks of $\mathfrak{D}$).

Finally, conditions (5.1) and (6.1) hold with E, $\mathfrak{O}_1$, K, and the $\mathfrak{D}_\sigma$ (for $\sigma \in K$) in place of E, $\mathfrak{C}$, G, and the $\mathfrak{C}_\sigma$, respectively. So we can consider defect groups in E of E-conjugacy classes of blocks of $\mathfrak{O}_1$. In particular, we can consider the defect groups in E_B of the block B of $\mathfrak{O}_1$.

THEOREM 7.3. *Let S be an E/K-orbit lying over B of blocks of $\mathfrak{O}$, and T be the corresponding E_B/K-orbit of blocks of $\bar{\mathfrak{F}}[G[B]^*]$. Choose a defect group $\bar{D}$ of T in E_B/K and a p-Sylow subgroup P of E_B/K containing $\bar{D}$. Then P is the image of a defect group C of B in E_B. For any such C, the inverse image $\tilde{D}$ of $\bar{D}$ in C is a defect group in E of S considered as an E-orbit of blocks of $\mathfrak{D}$.*

Proof. Let C be any defect group of B in E_B. We can apply Proposition 6.7 with E_B, B, $\mathfrak{O}_1$, and K in place of E, $\tilde{B}$, $\mathfrak{O}$, and G, respectively. It tells us that $C \cap K$ is a defect group of B in K while CK/K is a p-Sylow subgroup of E_B/K. By E_B-conjugation, we can choose C so that $CK/K = P$, which is the first conclusion of the theorem.

Next we show that:

(7.4) *C is a defect group in CK of B.*

Since C is a defect group of B in E_B and e is a primitive central idempotent of $\mathfrak{O}_1$, there exists an element $y \in e\mathfrak{O}_1(C)$ such that $e = \mathrm{tr}_{C \to E_B}(y)$. In view of (4.1), we can write this as

$$e = \textstyle\sum_\sigma \mathrm{tr}_{C^\sigma \cap CK \to CK}(y^\sigma),$$

where σ runs over a family of representatives for the double cosets $C\sigma CK$ in E_B. The idempotent e is primitive in $\mathfrak{O}_1(CK)$ which is contained in the commutative order $\mathfrak{O}_1(K) = Z(\mathfrak{O}_1)$. Hence $e\mathfrak{O}_1(CK)$ is a local ring. Since each trace $\mathrm{tr}_{C^\sigma \cap CK \to CK}(y^\sigma)$ lies in $e\mathfrak{O}_1(CK)$, we conclude from the above equation that

$$\mathrm{tr}_{C^\sigma \cap CK \to CK}(y^\sigma) \notin J(e\mathfrak{O}_1(CK)),$$

for some $\sigma \in E_B$. So the ideal $e\mathfrak{O}_1(CK \mid C^\sigma \cap CK)$ is equal to the local ring $e\mathfrak{O}_1(CK)$. Therefore $e = \mathrm{tr}_{C^\sigma \cap CK \to CK}(z)$, for some element $z \in e\mathfrak{O}_1(C^\sigma \cap CK)$.

We know that $CK/K = P$ is a p-Sylow subgroup of E_B/K. Hence the index $[E_B{:}CK]$ is not divisible by p. In view of (7.1c), the image of this index is a unit of $\mathfrak{R}$. Therefore

$$e = \mathrm{tr}_{CK \to E_B}([E_B{:}CK]^{-1}e) = \mathrm{tr}_{C^\sigma \cap CK \to E_B}([E_B{:}CK]^{-1}z).$$

It follows that $C^{\sigma} \cap CK$ contains a defect group of B in E_B. Since C^{σ} is such a defect group, this implies that $C^{\sigma} = C^{\sigma} \cap CK \leq CK$. Because K is a normal subgroup of E_B we have $| C^{\sigma} \cap K | = | (C \cap K)^{\sigma} | = | C \cap K |$ and $| C^{\sigma} | = | C |$. We conclude that $| C^{\sigma}K | = | CK |$, and hence that $CK = C^{\sigma}K = (CK)^{\sigma}$. But then $z^{\sigma^{-1}} \in e\mathfrak{O}_1(C)$ and

$$e = e^{\sigma^{-1}} = [\mathrm{tr}_{C^{\sigma} \to (CK)^{\sigma}}(z)]^{\sigma^{-1}} = \mathrm{tr}_{C \to CK}(z^{\sigma^{-1}}).$$

Therefore C contains a defect group C_1 of B in CK. But $e = \mathrm{tr}_{C_1 \to CK}(w)$, for some $w \in e\mathfrak{O}_1(C_1)$, implies $e = \mathrm{tr}_{C_1 \to E_B}([E_B:CK]^{-1}w)$. Hence C_1 contains a defect group for B in E_B. This forces $C_1 = C$, and proves (7.4).

The E-orbit S of blocks of $\mathfrak{D}$ corresponds, as usual, to a primitive idempotent d of $\mathrm{tr}_{E_B \to E}(e)\mathfrak{D}(E)$. Since e centralizes $\mathfrak{D}(E) \subseteq \mathfrak{C}$ and satisfies (4.7a), we can apply Proposition 4.9. This tells us that any defect group D of the corresponding primitive idempotent ed of $e\mathfrak{D}e(E_B)$, defined by the action of E_B on $e\mathfrak{D}e = e\mathfrak{O}e$, is also a defect group of S in E. So the theorem will be proved if we can show that D can be chosen equal to $\tilde{D}$.

Proposition 4.6 tells us that DK/K is a defect group in E_B/K for the primitive central idempotent ed of $e\mathfrak{D}e(E_B) = e\mathfrak{D}e(K)(E_B/K)$. Since e is a central idempotent of $\mathfrak{C} = \mathfrak{D}(K)$, we have $e\mathfrak{D}e(K) = e[\mathfrak{D}(K)] = e\mathfrak{C}$. So DK/K is a defect group of the primitive idempotent ed of $e\mathfrak{C}(E_B/K)$ defined by the action of E_B/K on $e\mathfrak{C}$.

We know from (2.18) that (2.1) holds for $e\mathfrak{O}e = e\mathfrak{D}e$, G_B, and the $e\mathfrak{O}_{\sigma} = \mathfrak{O}_{\sigma}e$, for $\sigma \in G_B$. Furthermore, $e\mathfrak{C} = C(e\mathfrak{O}_1 \text{ in } e\mathfrak{O}e)$. We see easily that conditions (5.1) and (6.1) are now satisfied with $e\mathfrak{O}e$, G_B, $e\mathfrak{C}$, E_B/K, the $e\mathfrak{O}_{\sigma}$, and the $e\mathfrak{C}_{\sigma}$ in place of $\mathfrak{O}$, G, $\mathfrak{C}$, E, the $\mathfrak{O}_{\sigma}$, and the $\mathfrak{C}_{\sigma}$, respectively. If B' is the unique block of $e\mathfrak{O}_1$ corresponding to the primitive central idempotent e, then it is clear from the definitions that $e\mathfrak{C}[B'] = \mathfrak{C}[B]$ and that $G_B[B']^* = G[B]^*$ as extensions of $\bar{F}$ by $G_B[B'] = G[B]$ operated on by E_B/K. If S' is the E_B/K-orbit of blocks of $e\mathfrak{O}e$ corresponding to ed, then Theorem 3.7 says that the corresponding E_B/K-orbit of blocks of $\bar{\mathfrak{F}}[G[B']^*] = \bar{\mathfrak{F}}[G[B]^*]$ is defined by the image of $ed \in e\mathfrak{C}[B'] = \mathfrak{C}[B]$. But this E_B/K-orbit is T by Theorem 3.7. So Theorem 6.5 says that the defect group DK/K in E_B/K of S' is E_B/K-conjugate to the defect group $\bar{D}$ of T in E_B/K. Since D is only defined to within E_B-conjugacy, we may choose it so that

$$DK/K = \bar{D}. \tag{7.5}$$

Now D is a subgroup of the inverse image CK of $P \geq \bar{D}$ in E_B. Since D is a defect group for ed in E_B, there exists an element $x \in e\mathfrak{D}e(D)$ such that $ed = \mathrm{tr}_{D \to E_B}(x)$. Because $ex = x$ and e is E_B-invariant, we have

$$\mathrm{tr}_{D \to CK}(x) = \mathrm{tr}_{D \to CK}(ex) = e\ \mathrm{tr}_{D \to CK}(x).$$

By (7.4), the idempotent e lies in $e\mathfrak{D}e(CK \mid C)$. Therefore (4.4) gives

$$\mathrm{tr}_{D \to CK}(x) \in e\mathfrak{D}e(CK \mid C)e\mathfrak{D}e(CK \mid D) \subseteq \textstyle\sum_{\sigma \in CK} e\mathfrak{D}e(CK \mid C \cap D^{\sigma}).$$

Hence

$$ed = \mathrm{tr}_{CK\to E_B}(\mathrm{tr}_{D\to CK}(x)) \in \sum_{\sigma\in CK} \mathrm{tr}_{CK\to E_B}(e\mathfrak{D}e(CK \mid C \cap D^\sigma)) = \sum_{\sigma\in CK} e\mathfrak{D}e(E_B \mid C \cap D^\sigma).$$

We conclude that there is some element $\sigma \in CK$ such that $C \cap D^\sigma$ contains a defect group in E_B of ed. Since $C \cap D^\sigma$ is contained in D^σ, which is such a defect group, we have $D^\sigma = C \cap D^\sigma \leq C$. Writing $\sigma = \tau\rho$, where $\tau \in K$, $\rho \in C$, we see that $D^\tau \leq C^{\rho^{-1}} = C$ satisfies $D^\tau K = DK$. In view of (7.5), this implies that D^τ is contained in the inverse image $\tilde{D}$ of $\bar{D}$ in C.

Replacing D by D^τ, we have now chosen D to be contained in $\tilde{D}$ as well as to satisfy (7.5). Since $DK/K = \tilde{D}K/K = \bar{D}$, the subgroup D will be equal to $\tilde{D}$ if and only if $D \cap K = \tilde{D} \cap K$. By construction $\tilde{D} \cap K$ is $C \cap K$, which is a defect group of B in K (see the first paragraph of this proof). So the theorem will be proved once we show that $D \cap K$ contains a defect group of B in K.

As above, $ed = \mathrm{tr}_{D\to E_B}(x)$, for some $x \in e\mathfrak{D}e(D)$. The element $u = \mathrm{tr}_{D\to DK}(x)$ lies in $e\mathfrak{D}e(K) = e\mathfrak{C}$. The ideal $\mathfrak{J}$ of Lemma 3.3 is E_B-invariant by (5.5a). If u lies in $\mathfrak{J}$, then so do all of its E_B-conjugates. Hence

$$ed = \mathrm{tr}_{DK\to E_B}(u) \in \mathfrak{J},$$

which is impossible since ed is a non-zero idempotent and $\mathfrak{J} \subseteq J(e\mathfrak{C})$ by Lemma 3.3. Therefore $u \notin \mathfrak{J}$. Writing $u = \sum_{\sigma\in G_B} u_\sigma$, where $u_\sigma \in e\mathfrak{C}_\sigma$ for all $\sigma \in G_B$, we conclude that there is an element $\rho \in G[B]$ such that $u_\rho \notin \mathfrak{C}[B]_\rho J(\mathfrak{C}[B]_1)$. By (2.14b) the image $\bar{u}_\rho$ of u_ρ is a non-zero element of $\bar{\mathfrak{F}}[G[B]^*]_\rho$, i.e., $\bar{u}_\rho \in G[B]^*$. Hence no power of $\bar{u}_\rho$ is zero. In particular, if $n > 0$ is an integer such that $\rho^n = 1$, then $\bar{u}_\rho^n$ is a non-zero element of $\bar{\mathfrak{F}}[G[B]^*]_1$. Therefore $u_\rho^n \in e\mathfrak{C}_1 - J(e\mathfrak{C}_1)$.

By (2.18) we have a unique decomposition of $x \in e\mathfrak{D}e = e\mathfrak{O}e$ in the form $x = \sum_{\sigma\in G_B} x_\sigma$, where $x_\sigma \in e\mathfrak{O}_\sigma$, for each $\sigma \in G_B$. Since each $e\mathfrak{O}_\sigma$ is K-invariant, and x is D-invariant, each x_σ is $D \cap K$-invariant. Furthermore

$$u = \mathrm{tr}_{D\to DK}(x) = \mathrm{tr}_{D\cap K\to K}(x) = \sum_{\sigma\in G_B} \mathrm{tr}_{D\cap K\to K}(x_\sigma).$$

It follows that $u_\sigma = \mathrm{tr}_{D\cap K\to K}(x_\sigma)$, for all $\sigma \in G_B$. In particular,

$$u_\rho = \mathrm{tr}_{D\cap K\to K}(x_\rho).$$

Since $u_\rho^{n-1} \in e\mathfrak{C}$ is K-invariant, this implies that $u_\rho^n = \mathrm{tr}_{D\cap K\to K}(x_\rho u_\rho^{n-1})$. But

$$x_\rho u_\rho^{n-1} \in e\mathfrak{O}_\rho\, e\mathfrak{C}_{\rho^{n-1}} \subseteq e\mathfrak{O}_{\rho^n} = e\mathfrak{O}_1 .$$

Therefore

$$u_\rho^n \in e\mathfrak{O}_1(K \mid D \cap K) - J(e\mathfrak{O}_1(K)).$$

We conclude that $D \cap K$ contains a defect group in K of B, which, as noted above, completes the proof of the theorem.

COROLLARY 7.6. *The intersection $\tilde{D} \cap K$ is a defect group of B in K.*

Proof. In the first paragraph of the above proof it was shown that $C \cap K$ is a defect group of B in K. Since $\tilde{D} \cap K = C \cap K$ by definition, this implies the corollary.

Since we have operated at several levels of generality in the last three sections, it is perhaps wise to specify what happens when Theorem 7.3 is applied to a finite group H and normal subgroup K, as at the beginning of §2. We assume, of course, that the characteristic of the residue class field $\bar{\mathfrak{F}}$ is a prime p. We fix a block B of $\mathfrak{R}K$, and set $G = H/K$. Then §2 gives us the normal subgroup $G[B]$ of $G_B = H_B/K$ and the Clifford extension $G[B]^*$ of $\bar{F}$ by $G[B]$, together with the conjugation action of G_B on $G[B]^*$. Theorem 3.7 gives a one-to-one correspondence between blocks $\tilde{B}$ of $\mathfrak{R}H$ lying over B and G_B-conjugacy classes T of blocks of $\bar{\mathfrak{F}}[G[B]^*]$. Fix such a block $\tilde{B}$ and a block $\bar{B}$ in the corresponding class T. Green's theory gives us defect groups $\bar{D}$ of $\bar{B}$ in $G_{B,\bar{B}} = G_B \cap G_{\bar{B}}$ and C of B in H_B.

THEOREM 7.7. *In the above situation, CK/K is a p-Sylow subgroup of $H_B/K = G_B$. We can choose C so that this p-Sylow subgroup contains the p-group $\bar{D}$. Then the inverse image $\tilde{D}$ of $\bar{D}$ in C is a defect group of $\tilde{B}$ in H. Furthermore, $\tilde{D} \cap K = C \cap K$ is a defect group of B in K.*

Proof. The first statement is a consequence of Proposition 6.7 with H_B, B, $\mathfrak{R}K$, K, and the $\mathfrak{R}\sigma$, $\sigma \in K$, in place of E, $\tilde{B}$, $\mathfrak{O}$, G, and the $\mathfrak{O}_\sigma$, $\sigma \in G$, respectively. The second and third statements are the result of Theorem 7.3 with $E = H$, $\mathfrak{D} = \mathfrak{R}H$, $\mathfrak{D}_\sigma = \mathfrak{R}\sigma$, for $\sigma \in H$, and $S = \{\tilde{B}\}$ together with (6.3) with G_B in place of E, $\bar{\mathfrak{F}}[G[B]^*]$ in place of $\mathfrak{O}$, and $\bar{B}$ in place of $\tilde{B}$. The last statement follows from Corollary 7.6.

8. Brauer's First Main Theorem

Curiously enough, Brauer's First Main Theorem (Theorem (10B) of [1]) is valid even in the general setting (5.1). To show this, we shall use the notation and assumptions of §5.

Fix a p-subgroup D of E_B. The centralizer $C(D \text{ in } G[B])$ is then the subgroup of all $\sigma \in G[B]$ satisfying $\sigma^\tau = \sigma$ for all $\tau \in D$. Since $C(D \text{ in } G[B])$ is a subgroup of $G[B]$, Proposition 2.10 implies that (2.1) holds with

$$\mathfrak{C}[B]_{(D)} = \oplus \sum\nolimits_{\sigma \in C(D \text{ in } G[B])} \mathfrak{C}[B]_\sigma ,$$

$C(D \text{ in } G[B])$, and the $\mathfrak{C}[B]_\sigma$ in place of $\mathfrak{O}$, G, and the $\mathfrak{O}_\sigma$, respectively. Notice that

$$\begin{aligned}\mathfrak{C}[B]_{(D)}\, J(\mathfrak{C}[B]_1) &= \oplus \sum\nolimits_{\sigma \in C(D \text{ in } G[B])} \mathfrak{C}[B]_\sigma\, J(\mathfrak{C}[B]_1)\\ &= \mathfrak{C}[B]_{(D)} \cap \mathfrak{C}[B] J(\mathfrak{C}[B]_1).\end{aligned}$$

So we can naturally identify the $\bar{\mathfrak{F}}$-algebra $\mathfrak{C}[B]_{(D)}/\mathfrak{C}[B]_{(D)}\, J(\mathfrak{C}[B]_1)$ with the image

$$\bar{\mathfrak{F}}[G[B]^*]_{(D)} = \oplus \sum\nolimits_{\sigma \in C(D \text{ in } G[B])} \bar{\mathfrak{F}}[G[B]^*]_\sigma$$

of $\mathfrak{C}[B]_{(D)}$ in $\bar{\mathfrak{F}}[G[B]^*]$. Equation (5.5b) implies that $\mathfrak{C}[B]_{(D)}$ is invariant under the normalizer $N(D) = N(D \text{ in } E_B)$ of D in E_B. We conclude easily that the results (5.8)–(5.10) are also valid with $\mathfrak{C}[B]_{(D)}$, $\bar{\mathfrak{F}}[G[B]^*]_{(D)}$, $N(D)$, and $C(D \text{ in } G[B])$ in place of $\mathfrak{C}[B]$, $\bar{\mathfrak{F}}[G[B]^*]$, E_B, and $G[B]$, respectively.

PROPOSITION 8.1. *The restriction S to $\mathfrak{C}[B](E_B)^J$ of the natural projection of*

$$\bar{\mathfrak{F}}[G[B]^*] = \oplus \sum_{\sigma \in G[B]} \bar{\mathfrak{F}}[G[B]^*]_\sigma$$

onto

$$\bar{\mathfrak{F}}[G[B]^*]_{(D)} = \oplus \sum_{\sigma \in C(D \text{ in } G[B])} \bar{\mathfrak{F}}[G[B]^*]_\sigma$$

is an identity-preserving $\bar{\mathfrak{F}}$-homomorphism of this subalgebra into the image

$$\mathfrak{C}[B]_{(D)}(N(D))^J$$

in $\bar{\mathfrak{F}}[G[B]^*]$ of $\mathfrak{C}[B]_{(D)}(N(D))$.

Proof. Suppose that $z = \sum_{\sigma \in G[B]} z_\sigma \in \mathfrak{C}[B](E_B)$, where $z_\sigma \in \mathfrak{C}[B]_\sigma$ for all $\sigma \in G[B]$. Equation (5.5b) and the E_B-invariance of z imply $(z_\sigma)^\tau = z_{\sigma^\tau}$, for all $\sigma \in G[B]$, $\tau \in E_B$. It follows that $N(D)$ centralizes

$$x = \sum_{\sigma \in C(D \text{ in } G[B])} z_\sigma ,$$

i.e., that x lies in $\mathfrak{C}[B]_{(D)}(N(D))$. By definition S sends the image z^J of z in $\bar{\mathfrak{F}}[G[B]^*]$ into the image x^J of x. We conclude that S is an $\bar{\mathfrak{F}}$-linear map of $\mathfrak{C}[B](E_B)^J$ into $\mathfrak{C}[B]_{(D)}(N(D))^J$. Evidently $1 \in \bar{\mathfrak{F}}[G[B]^*]_1$ implies that $S(1) = 1$. So it only remains to be shown that $S(yw) = S(y)S(w)$, for all $y, w \in \mathfrak{C}[B](E_B)^J$.

This is just a repetition of the original proof of Brauer (see page 426 of [1]). Write $y = \sum_{\sigma \in G[B]} y_\sigma$ and $w = \sum_{\sigma \in G[B]} w_\sigma$, where $y_\sigma, w_\sigma \in \bar{\mathfrak{F}}[G[B]^*]_\sigma$, for all $\sigma \in G[B]$. Fix $\pi \in C(D \text{ in } G[B])$. Set

$$T = \{\rho \times \sigma \in G[B] \times G[B] \mid \rho\sigma = \pi\},$$

$$T' = \{\rho \times \sigma \in C(D \text{ in } G[B]) \times C(D \text{ in } G[B]) \mid \rho\sigma = \pi\}.$$

The group D fixes π, and hence acts naturally on T, with $(\rho \times \sigma)^\tau = \rho^\tau \times \sigma^\tau$, for all $\rho \times \sigma \in T$, $\tau \in D$. Denote the D-orbits of T by $T_1, \cdots, T_n$, and choose $\rho_i \times \sigma_i \in T_i$, for $i = 1, \cdots, n$. Evidently T' is precisely the subset of all $\rho \times \sigma \in T$ centralized by D, i.e., the set of all $\rho_i \times \sigma_i$, $i = 1, \cdots, n$, for which $|T_i| = 1$.

For any $i = 1, \cdots, n$ and any $\tau \in D$, the E_B-invariance of y and of w implies that

$$(y_{\rho_i})^\tau = y_{\rho_i^\tau} \quad \text{and} \quad (w_{\sigma_i})^\tau = w_{\sigma_i^\tau} .$$

Hence

$$y_{\rho_i^\tau} w_{\sigma_i^\tau} = (y_{\rho_i})^\tau (w_{\sigma_i})^\tau = (y_{\rho_i} w_{\sigma_i})^\tau .$$

The finite p-group D acts as linear transformations of the one-dimensional subspace $\bar{\mathfrak{F}}[G[B]^*]_\pi$ over the field $\bar{\mathfrak{F}}$ of characteristic p. Hence D centralizes

$\mathfrak{F}[G[B]^*]_\pi$. In particular, $\tau \in D$ fixes $y_{\rho_i} w_{\sigma_i}$, which lies in $\mathfrak{F}[G[B]^*]_\pi$ since $\rho_i \sigma_i = \pi$. We conclude that

$$\sum_{\rho \times \sigma \in T_i} y_\rho w_\sigma = | T_i | y_{\rho_i} w_{\sigma_i} .$$

Since D is a p-group, $| T_i |$ is a power of p. In view of (5.1a), the above expression is zero unless $| T_i | = 1$. Hence

$$\sum_{\rho \times \sigma \in T} y_\rho w_\sigma = \sum_{i=1}^{n} (\sum_{\rho \times \sigma \in T_i} y_\rho w_\sigma) = \sum_{i=1, | T_i | = 1}^{n} y_{\rho_i} w_{\sigma_i} = \sum_{\rho \times \sigma \in T'} y_\rho w_\sigma .$$

But the first sum is the π^{th} component of yw while the last is the π^{th} component of $S(y)S(w)$ (in the decomposition $\mathfrak{F}[G[B]^*] = \oplus \sum_{\sigma \in G[B]} \mathfrak{F}[G[B]^*]_\sigma$). Their equality for any $\pi \in C(D \text{ in } G[B])$ is precisely the desired equation $S(yw) = S(y)S(w)$. So the proposition is proved.

The above homomorphism S is called the *Brauer homomorphism* defined by D.

We next define a subset $G[B](E_B \| D)$ of $G[B]$ and a subspace $\mathfrak{C}[B](E_B \| D)^J$ of $\mathfrak{C}[B](E_B)^J$ by:

(8.2a) $G[B](E_B \| D) = \{\sigma \in G[B] \mid D$ *is* E_B*-conjugate to a p-Sylow subgroup of* $C(\sigma \text{ in } E_B)\}$.

(8.2b) $\mathfrak{C}[B](E_B \| D)^J = \mathfrak{C}[B](E_B)^J \cap (\oplus \sum_{\sigma \in G[B](E_B \| D)} \mathfrak{F}[G[B]^*]_\sigma)$.

Evidently the non-zero elements of $\mathfrak{C}[B](E_B \| D)^J$ are precisely those non-zero elements y of $\mathfrak{C}[B](E_B)^J$ for which $\Delta(y)$ consists of the E_B-conjugates of D.

PROPOSITION 8.3. *Choose subgroups* $D_1 = D, D_2, \cdots, D_n$ *of* D *so that any subgroup of* D *is* E_B*-conjugate to exactly one of* $D_1, \cdots, D_n$. *Then*

(8.4) $\mathfrak{C}[B](E_B \mid D)^J = \oplus \sum_{i=1}^{n} \mathfrak{C}[B](E_B \| D_i)^J$ (*as* $\mathfrak{F}$*-vector spaces*).

Proof. It follows from Lemma 5.9 that

$$\mathfrak{C}[B](E_B \mid D)^J = \mathfrak{C}[B](E_B)^J \cap (\oplus \sum_{\sigma \in T} \mathfrak{F}[G[B]^*]_\sigma),$$

where $T = \bigcup_{i=1}^{n} G[B](E_B \| D_i)$. Hence each $\mathfrak{C}[B](E_B \| D_i)^J$ is contained in $\mathfrak{C}[B](E_B \mid D)^J$. Since the subsets $G[B](E_B \| D_i)$ are pairwise disjoint, the sum of the subspaces $\mathfrak{C}[B](E_B \| D_i)^J$ is direct. Therefore the left side of (8.4) contains the right side.

Now let y be any element of $\mathfrak{C}[B](E_B \mid D)^J$. We can write $y = \sum_{\sigma \in T} y_\sigma$, where $y_\sigma \in \mathfrak{F}[G[B]^*]_\sigma$, for each $\sigma \in T$. Furthermore y is the image of an element $z \in \mathfrak{C}[B](E_B)$, which we can write as $z = \sum_{\sigma \in G[B]} z_\sigma$, where $z_\sigma \in \mathfrak{C}[B]_\sigma$ for all $\sigma \in G[B]$. Because z is E_B-invariant, we have $(z_\sigma)^\tau = z_{\sigma^\tau}$, for all $\sigma \in G[B]$, $\tau \in E_B$. All the subsets $G[B](E_B \| D_i)$, $i = 1, \cdots, n$, are clearly E_B-invariant. It follows that $z_i = \sum_{\sigma \in G[B](E_B \| D_i)} z_\sigma$ lies in $\mathfrak{C}[B](E_B)$, for each $i = 1, \cdots, n$. By (8.2b) its image $y_i = \sum_{\sigma \in G[B](E_B \| D_i)} y_\sigma$ lies in $\mathfrak{C}[B](E_B \| D_i)^J$, for $i = 1, \cdots, n$. Hence $y = y_1 + \cdots + y_n$ lies in the right side of (8.4). This completes the proof of the proposition.

Since D is also a p-subgroup of $N(D)$ we can similarly define

$$C(D \text{ in } G[B])(N(D) \parallel D) \quad \text{and} \quad \mathfrak{C}[B]_{(D)}(N(D) \parallel D)^J.$$

The critical property of the map S is then given by:

LEMMA 8.5. *The Brauer homomorphism S defined by D sends $\mathfrak{C}[B](E_B \parallel D)^J$ one-to-one onto $\mathfrak{C}[B]_{(D)}(N(D) \parallel D)^J$.*

Proof. If $\sigma \in G[B](E_B \parallel D) \cap C(D \text{ in } G[B])$, then $C = C(\sigma \text{ in } E_B)$ contains D and has an E_B-conjugate D^τ as a p-Sylow subgroup. It follows that D is a p-Sylow subgroup of C. Hence D is a p-Sylow subgroup of $C \cap N(D) = C(\sigma \text{ in } N(D))$. Therefore

$$\sigma \in C(D \text{ in } G[B])(N(D) \parallel D).$$

In view of (8.2b) and Proposition 8.1 we conclude that S maps

$$\mathfrak{C}[B](E_B \parallel D)^J \quad \text{into} \quad \mathfrak{C}[B]_{(D)}(N(D) \parallel D)^J.$$

Let

$$y = \sum\nolimits_{\sigma \in G[B](E_B \parallel D)} y_\sigma$$

be a non-zero element of $\mathfrak{C}[B](E_B \parallel D)^J$, where $y_\sigma \in \bar{\mathfrak{F}}[G[B]^*]_\sigma$ for each σ. The E_B-invariance of y gives $y_{\sigma^\tau} = (y_\sigma)^\tau$, for all $\sigma \in G[B](E_B \parallel D)$, $\tau \in E_B$. It follows from this and (8.2a) that some $\sigma \in G[B](E_B \parallel D)$ satisfies simultaneously $y_\sigma \neq 0$ and $D \leq C(\sigma \text{ in } E_B)$, i.e., $\sigma \in C(D \text{ in } G[B])$. We conclude from the definition of S that $S(y) \neq 0$. So S is a one-to-one map of $\mathfrak{C}[B](E_B \parallel D)^J$ into $\mathfrak{C}[B]_{(D)}(N(D) \parallel D)^J$.

Now let

$$w = \sum\nolimits_{\sigma \in C(D \text{ in } G[B])(N(D) \parallel D)} w_\sigma$$

be any element of $\mathfrak{C}[B]_{(D)}(N(D) \parallel D)^J$, where $w_\sigma \in \bar{\mathfrak{F}}[G[B]^*]_\sigma$, for each σ. To complete the proof of the lemma, we must construct an element

$$y \in \mathfrak{C}[B](E_B \parallel D)^J$$

such that $S(y) = w$.

If σ is any element of $C(D \text{ in } G[B])(N(D) \parallel D)$, then D is clearly the unique p-Sylow subgroup of $C(\sigma \text{ in } N(D))$. Hence D is contained in a p-Sylow subgroup P of $C = C(\sigma \text{ in } E_B)$. If D is properly contained in P, then it is properly contained in its normalizer $N(D \text{ in } P)$ in P. But then $N(D \text{ in } P)$ is a p-subgroup of $C \cap N(D) = C(\sigma \text{ in } N(D))$ strictly containing the p-Sylow subgroup D of that group. This is impossible. Hence $D = P$ is a p-Sylow subgroup of C and $\sigma \in G[B](E_B \parallel D)$.

When $w_\sigma = 0$, the subgroup C centralizes w_σ. We use an argument of Reynolds [9] to show that C also centralizes w_σ when $w_\sigma \neq 0$. In that case the one-dimensional subspace $\bar{\mathfrak{F}}[G[B]^*]_\sigma$ of $\bar{\mathfrak{F}}[G[B]^*]$ is equal to $\bar{\mathfrak{F}}w_\sigma$. From (5.8b) it is clear that $C = C(\sigma \text{ in } E_B)$ leaves this subspace invariant. Hence there is a function λ from C to the multiplicative group $\bar{F}$ of $\bar{\mathfrak{F}}$ such that $(w_\sigma)^\tau = \lambda(\tau)w_\sigma$, for all $\tau \in C$. Because the action of C on $\bar{\mathfrak{F}}[G[B]^*]_\sigma$ is $\bar{\mathfrak{F}}$-

linear, the map λ is a homomorphism of the finite group C into $\bar{F}$. In particular, its image $\lambda(C)$ is a finite subgroup of $\bar{F}$, and hence is cyclic of order not divisible by the characteristic p of $\bar{\mathfrak{F}}$. We conclude that the kernel K of λ is a normal subgroup of C containing every p-Sylow subgroup of C. In particular, the p-Sylow subgroup D of C is a p-Sylow subgroup of K. Now the Frattini argument tells us that $C = N(D \text{ in } C)K$. But the $N(D)$-invariance of w implies that w_σ is centralized by $N(D \text{ in } C) = C \cap N(D)$. Therefore $N(D \text{ in } C) \leq K$ and $C = K$ centralizes w_σ.

We know that w is the image of an element $z \in \mathfrak{C}[B]_{(D)}(N(D))$, which we can write as $z = \sum_{\rho \in C(D \text{ in } G[B])} z_\rho$, where $z_\rho \in \mathfrak{C}[B]_\rho$ for each ρ. Clearly w_σ is the image of z_σ. The $N(D)$-invariance of z implies that $C \cap N(D) = C(\sigma \text{ in } N(D))$ centralizes z_σ. Hence $\operatorname{tr}_{C \cap N(D) \to C}(z_\sigma)$ is defined. This is a sum of various C-conjugates of z_σ by (4.1). Since $C = C(\sigma \text{ in } E_B)$ leaves invariant $\mathfrak{C}[B]_\sigma$, which contains z_σ, the above trace is an element of that submodule.

The intersection $C \cap N(D)$ contains the p-Sylow subgroup D of C. Hence the index $[C:C \cap N(D)]$ is not divisible by p. In view of (5.1a), we can now define a C-invariant element $x_\sigma \in \mathfrak{C}[B]_\sigma$ by

$$x_\sigma = [C:C \cap N(D)]^{-1} \operatorname{tr}_{C \cap N(D) \to C}(z_\sigma).$$

Since C centralizes the image w_σ of z_σ, the image of x_σ in $\bar{\mathfrak{F}}[G[B]^*]_\sigma$ is

$$[C:C \cap N(D)]^{-1} \operatorname{tr}_{C \cap N(D) \to C}(w_\sigma) = [C:C \cap N(D)]^{-1}[C:C \cap N(D)]w_\sigma = w_\sigma.$$

We must prove:

(8.6) *If* $\sigma^\tau \in C(D \text{ in } G[B])(N(D) \parallel D)$, *for some* $\tau \in E_B$, *then* $x_{\sigma^\tau} = (x_\sigma)^\tau$.

Since both D and D^τ are p-Sylow subgroups of $C^\tau = C(\sigma^\tau \text{ in } E_B)$, there exists an element $\rho \in C^\tau$ such that $D^{\tau\rho} = D$. Because ρ centralizes x_{σ^τ}, it suffices to prove (8.6) with $\tau\rho$ in place of τ, i.e., we can assume that τ lies in $N(D)$.

The $N(D)$-invariance of z now gives $z_{\sigma^\tau} = (z_\sigma)^\tau$. Therefore

$$\begin{aligned} x_{\sigma^\tau} &= [C^\tau:C^\tau \cap N(D)]^{-1} \operatorname{tr}_{C^\tau \cap N(D) \to C^\tau}((z_\sigma)^\tau) \\ &= [C^\tau:(C \cap N(D))^\tau]^{-1} \operatorname{tr}_{(C \cap N(D))^\tau \to C^\tau}((z_\sigma)^\tau) \\ &= ([C:C \cap N(D)]^{-1} \operatorname{tr}_{C \cap N(D) \to C}(z_\sigma))^\tau = (x_\sigma)^\tau. \end{aligned}$$

This proves (8.6).

Now we define x_π, for any $\pi \in G[B](E_B \parallel D)$, to be $(x_{\pi^\tau})^{\tau^{-1}}$, for any $\tau \in E_B$ such that $\pi^\tau \in C(D \text{ in } G[B])(N(D) \parallel D)$. We know from the first two paragraphs of this proof that such a τ exists, and from (8.6) that x_π is a well-defined element of $\mathfrak{C}[B]_\pi$. Clearly $(x_\pi)^\tau = x_{\pi^\tau}$, for any $\pi \in G[B](E_B \parallel D)$, $\tau \in E_B$. It follows that $x = \sum_{\pi \in G[B](E_B \parallel D)} x_\pi$ lies in $\mathfrak{C}[B](E_B)$ and that its image y lies in $\mathfrak{C}[B](E_B \parallel D)^J$. Because w_σ is the image of x_σ, for any

$$\sigma \in G[B](E_B \parallel D) \cap C(D \text{ in } G[B]) = C(D \text{ in } G[B])(N(D) \parallel D),$$

this element y satisfies $S(y) = w$. So the lemma is proved.

We denote by Max $(\mathfrak{C}[B](E_B) \mid D)$ the family of all maximal ideals $\mathfrak{M} \in \operatorname{Max}(\mathfrak{C}[B](E_B))$ having D as a defect group with respect to the action of E_B on $\mathfrak{C}[B]$. We define Max $(\mathfrak{C}[B]_{(D)}(N(D)) \mid D)$ similarly, using the action of $N(D)$ on $\mathfrak{C}[B]_{(D)}$.

THEOREM 8.7 (Brauer's First Main Theorem). *For any p-subgroup D of E_B, the Brauer homomorphism*

$$S : \mathfrak{C}[B](E_B)^J \to \mathfrak{C}[B]_{(D)}(N(D))^J$$

induces a one-to-one correspondence between ideals $\mathfrak{M} \in \operatorname{Max}(\mathfrak{C}[B](E_B) \mid D)$ and ideals $\mathfrak{N} \in \operatorname{Max}(\mathfrak{C}[B]_{(D)}(N(D)) \mid D)$. Two such $\mathfrak{M}$ and $\mathfrak{N}$ correspond if and only if the image $\mathfrak{M}^J$ of $\mathfrak{M}$ in $\mathfrak{C}[B](E_B)^J$ is the inverse image $S^{-1}(\mathfrak{N}^J)$ of the image $\mathfrak{N}^J$ of $\mathfrak{N}$ in $\mathfrak{C}[B]_{(D)}(N(D))^J$. In that case S induces an $\bar{\mathfrak{F}}$-isomorphism of the algebra

$$\mathfrak{C}[B](E_B)/\mathfrak{M} \simeq \mathfrak{C}[B](E_B)^J/\mathfrak{M}^J$$

onto

$$\mathfrak{C}[B]_{(D)}(N(D))/\mathfrak{N} \simeq \mathfrak{C}[B]_{(D)}(N(D))^J/\mathfrak{N}^J,$$

and $\mathfrak{N}$ is the only maximal two-sided ideal $\mathfrak{N}_0$ of $\mathfrak{C}[B]_{(D)}(N(D))$ such that $S^{-1}(\mathfrak{N}_0^J) \subseteq \mathfrak{M}^J$.

Proof. We denote by $\mathfrak{K}$ the sum of the ideals $\mathfrak{C}[B](E_B \mid C)^J$, where C runs over all subgroups satisfying $C < D$ (if $D = \{1\}$, then $\mathfrak{K} = \{0\}$). Then Proposition 8.3 implies immediately that

$$(8.8) \qquad \mathfrak{C}[B](E_B \mid D)^J = \mathfrak{C}[B](E_B \parallel D)^J \oplus \mathfrak{K} \quad (\text{as } \bar{\mathfrak{F}}\text{-spaces}).$$

Evidently D is contained in a p-Sylow subgroup of $C(\sigma \text{ in } N(D))$, for all $\sigma \in C(D \text{ in } G[B])$. It follows that $C(D \text{ in } G[B])(N(D) \parallel C)$ is empty, and that $\mathfrak{C}[B]_{(D)}(N(D) \parallel C)^J = \{0\}$, for all $C \ngeq D$. So (8.4) gives:

$$(8.9a) \quad \mathfrak{C}[B]_{(D)}(N(D) \mid C)^J = \{0\}, \textit{for all } C \ngeq D,$$
$$(8.9b) \quad \mathfrak{C}[B]_{(D)}(N(D) \mid D)^J = \mathfrak{C}[B]_{(D)}(N(D) \parallel D)^J.$$

The same argument shows that no $\sigma \in C(D \text{ in } G[B])$ can be contained in $G[B](E_B \parallel C)$, for any $C < D$. This and Proposition 8.3 tell us that $\mathfrak{K}$ lies in the kernel of S. So (8.8), (8.9b), and Lemma 8.5 give us an exact sequence:

$$(8.10) \quad 0 \longrightarrow \mathfrak{K} \xrightarrow{\subseteq} \mathfrak{C}[B](E_B \mid D)^J \xrightarrow{S} \mathfrak{C}[B]_{(D)}(N(D) \mid D)^J \longrightarrow 0.$$

Let $\mathfrak{N}_1, \cdots, \mathfrak{N}_n$ be the distinct members of

$$\operatorname{Max}(\mathfrak{C}[B]_{(D)}(N(D)) \mid D).$$

In view of (8.9a) and Proposition 5.10, their images $\mathfrak{N}_1^J, \cdots, \mathfrak{N}_n^J$ are the distinct maximal two-sided ideals of $\mathfrak{C}[B]_{(D)}(N(D))^J$ not containing the two-sided ideal $\mathfrak{C}[B]_{(D)}(N(D) \mid D)^J$. Since $\mathfrak{J} = J(\mathfrak{C}[B]_{(D)}(N(D))^J)$ is the intersection of all the maximal two-sided ideals of this algebra, we conclude that inclusion induces an exact sequence:

$$(8.11)\quad \begin{aligned} 0 \longrightarrow \mathfrak{C}[B]_{(D)}(N(D) \mid D)^J \cap \mathfrak{J} &\xrightarrow{\subseteq} \mathfrak{C}[B]_{(D)}(N(D) \mid D)^J \\ &\longrightarrow \mathfrak{C}[B]_{(D)}(N(D))^J/\mathfrak{N}_1^J \cap \cdots \cap \mathfrak{N}_n^J \longrightarrow 0. \end{aligned}$$

In particular, the intersections

$$\mathfrak{N}_1^J \cap \mathfrak{C}[B]_{(D)}(N(D) \mid D)^J, \cdots, \mathfrak{N}_n^J \cap \mathfrak{C}[B]_{(D)}(N(D) \mid D)^J$$

are all distinct. It follows from this and the exactness of (8.10) that the inverse images $S^{-1}(\mathfrak{N}_1^J), \cdots, S^{-1}(\mathfrak{N}_n^J)$ are distinct two-sided ideals of $\mathfrak{C}[B](E_B)^J$, and hence that their respective inverse images $\mathfrak{M}_1, \cdots, \mathfrak{M}_n$ are distinct two-sided ideals of $\mathfrak{C}[B](E_B)$.

Since $\mathfrak{N}_i^J + \mathfrak{C}[B]_{(D)}(N(D) \mid D)^J = \mathfrak{C}[B]_{(D)}(N(D))^J$, for any $i = 1, \cdots, n$, the exactness of (8.10) implies that

$$\mathfrak{N}_i^J + S(\mathfrak{C}[B](E_B)^J) = \mathfrak{C}[B]_{(D)}(N(D))^J.$$

We conclude that S induces an algebra isomorphism of

$$\mathfrak{C}[B](E_B)/\mathfrak{M}_i \simeq \mathfrak{C}[B](E_B)^J/S^{-1}(\mathfrak{N}_i^J)$$

onto

$$\mathfrak{C}[B]_{(D)}(N(D))/\mathfrak{N}_i \simeq \mathfrak{C}[B]_{(D)}(N(D))^J/\mathfrak{N}_i^J.$$

Since $\mathfrak{N}_i$ is a maximal two-sided ideal of $\mathfrak{C}[B]_{(D)}(N(D))$, this implies that $\mathfrak{M}_i$ is a maximal two-sided ideal of $\mathfrak{C}[B](E_B)$. Because

$$\mathfrak{N}_i^J \not\supseteq \mathfrak{C}[B]_{(D)}(N(D) \mid D)^J,$$

the exactness of (8.10) implies that

$$\mathfrak{M}_i^J = S^{-1}(\mathfrak{N}_i^J) \not\supseteq \mathfrak{C}[B](E_B \mid D)^J.$$

On the other hand,

$$\mathfrak{M}_i^J \supseteq \mathfrak{K} \supseteq \mathfrak{C}[B](E_B \mid C)^J,$$

for all $C < D$. Therefore D is a defect group of $\mathfrak{M}_i$. So $\mathfrak{N}_i \to \mathfrak{M}_i$ is a one-to-one map of Max $(\mathfrak{C}[B]_{(D)}(N(D)) \mid D)$ into Max $(\mathfrak{C}[B](E_B) \mid D)$.

If $\mathfrak{N}_0$ is any member of Max $(\mathfrak{C}[B]_{(D)}(N(D)))$ different from $\mathfrak{N}_1, \cdots, \mathfrak{N}_n$, then (8.9) and Proposition 5.10 imply that $\mathfrak{N}_0^J$ contains $\mathfrak{C}[B]_{(D)}(N(D) \mid D)^J$. The exactness of (8.10) tells us that $S^{-1}(\mathfrak{N}_0^J)$ contains $\mathfrak{C}[B](E_B \mid D)^J$, which is contained in none of $\mathfrak{M}_i^J, \cdots, \mathfrak{M}_n^J$, since each $\mathfrak{M}_i$ has D as a defect group. Hence $S^{-1}(\mathfrak{N}_0^J) \not\subseteq \mathfrak{M}_i^J$, for $i = 1, \cdots, n$, which completes the proof of the last statement of the theorem.

Now let $\mathfrak{M}$ be any member of Max $(\mathfrak{C}[B](E_B) \mid D)$. Then $\mathfrak{M}^J$ contains $\mathfrak{K}$, while

$$\begin{aligned} \mathfrak{C}[B](E_B \mid D)^J/(\mathfrak{C}[B](E_B \mid D)^J \cap \mathfrak{M}^J) &\simeq (\mathfrak{C}[B](E_B \mid D)^J + \mathfrak{M}^J)/\mathfrak{M}^J \\ &\simeq \mathfrak{C}[B](E_B)^J/\mathfrak{M}^J, \end{aligned}$$

as algebras. It follows from the exactness of (8.10) that

$$S(\mathfrak{C}[B](E_B \mid D)^J \cap \mathfrak{M}^J)$$

is a two-sided ideal of the subalgebra (without identity!) $\mathfrak{C}[B]_{(D)}(N(D) \mid D)^J$ such that the corresponding factor algebra is isomorphic to the simple algebra (with identity) $\mathfrak{C}[B](E_B)^J/\mathfrak{M}^J$. Evidently such an ideal must contain the nilpotent ideal $\mathfrak{C}[B]_{(D)}(N(D) \mid D)^J \cap \mathfrak{J}$. Because (8.11) is exact, there must be an $i = 1, \cdots, n$ such that

$$S(\mathfrak{C}[B](E_B \mid D)^J \cap \mathfrak{M}^J) = \mathfrak{N}_i^J \cap \mathfrak{C}[B]_{(D)}(N(D) \mid D)^J.$$

Now $\mathfrak{M}_i$ is another ideal in Max $(\mathfrak{C}[B](E_B) \mid D)$ such that $\mathfrak{M}_i^J = S^{-1}(\mathfrak{N}_i^J)$ and $\mathfrak{M}^J$ have the same intersection with the two-sided ideal $\mathfrak{C}[B](E_B \mid D)^J$. If $\mathfrak{M}_i \neq \mathfrak{M}$, then multiplying the equations

$$\mathfrak{M}_i^J + \mathfrak{M}^J = \mathfrak{C}[B](E_B)^J \quad \text{and} \quad \mathfrak{C}[B](E_B \mid D)^J + \mathfrak{M}^J = \mathfrak{C}[B](E_B)^J$$

gives

$$\mathfrak{C}[B](E_B)^J \subseteq \mathfrak{M}_i^J\, \mathfrak{C}[B](E_B \mid D)^J + \mathfrak{M}^J \subseteq (\mathfrak{M}_i^J \cap \mathfrak{C}[B](E_B \mid D)^J) + \mathfrak{M}^J = \mathfrak{M}^J,$$

which is impossible. Hence $\mathfrak{M}_i = \mathfrak{M}$. Therefore $\mathfrak{N}_i \to \mathfrak{M}_i$ is onto, and the theorem is proved.

9. The Brauer analysis

We continue to use the notation and assumptions of §8. The remaining parts of the Brauer analysis (in §11 of [1]) of the blocks of a finite group G_0 having a fixed defect group D_0 take us from such blocks of the normalizer $N_{G_0}(D_0)$ to certain $N_{G_0}(D_0)$-conjugacy classes of irreducible characters of the centralizer $C_{G_0}(D_0)$. In our generalization of his analysis, the role of the group algebra of $C_{G_0}(D_0)$ will be played by the image $\mathfrak{C}[B]_{(D)}(D)^J$ in $\bar{\mathfrak{F}}[G[B]^*]$ of the suborder $\mathfrak{C}[B]_{(D)}(D)$ of $\mathfrak{C}[B]_{(D)}$. To describe this image, we define

$$(9.1)\quad G[B](D) = \{\sigma \in C(D \text{ in } G[B]) \mid \mathfrak{C}[B]_\sigma(D) \nsubseteq \mathfrak{C}[B]_\sigma J(\mathfrak{C}[B]_1)\},$$

where, as usual, $\mathfrak{C}[B]_\sigma(D)$ is the $\mathfrak{R}$-sublattice of all elements of $\mathfrak{C}[B]_\sigma$ fixed by D.

PROPOSITION 9.2. *The subset $G[B](D)$ is an $N(D)$-invariant normal subgroup of $C(D$ in $G[B])$. The factor group $C(D$ in $G[B])/G[B](D)$ is a p-group. Furthermore*

$$(9.3)\qquad \mathfrak{C}[B]_{(D)}(D)^J = \oplus \sum_{\sigma \in G[B](D)} \bar{\mathfrak{F}}[G[B]^*]_\sigma .$$

Hence $\mathfrak{C}[B]_{(D)}(D)^J$ is a twisted group algebra of $G[B](D)$.

Proof. Since $\bar{\mathfrak{F}}[G[B]^*]$ is a twisted group algebra, each $\bar{\mathfrak{F}}[G[B]^*]_\sigma$, for $\sigma \in C(D$ in $G[B])$, is one-dimensional over $\bar{\mathfrak{F}}$. In view of (2.14b) and (9.1), this implies that such a σ lies in $G[B](D)$ if and only if $\bar{\mathfrak{F}}[G[B]^*]_\sigma$ is the image

$\mathfrak{C}[B]_\sigma(D)^J$ in $\bar{\mathfrak{F}}[G[B]^*]$ of $\mathfrak{C}[B]_\sigma(D)$. If $\sigma, \tau \in G[B](D)$, then

$$\mathfrak{C}[B]_\sigma(D)^J = \bar{\mathfrak{F}}[G[B]^*]_\sigma \quad \text{and} \quad \mathfrak{C}[B]_\tau(D)^J = \bar{\mathfrak{F}}[G[B]^*]_\tau$$

imply that the image of $\mathfrak{C}[B]_\sigma(D)\,\mathfrak{C}[B]_\tau(D) \subseteq \mathfrak{C}[B]_{\sigma\tau}(D)$ is

$$\bar{\mathfrak{F}}[G[B]^*]_\sigma\,\bar{\mathfrak{F}}[G[B]^*]_\tau = \bar{\mathfrak{F}}[G[B]^*]_{\sigma\tau}\,.$$

Therefore $\sigma\tau$ also lies in $G[B](D)$. Evidently

$$e \in \mathfrak{C}[B]_1(D) \not\subseteq J(\mathfrak{C}[B]_1).$$

So $1 \in G[B](D)$. Since $G[B]$ is a finite group, its subset $G[B](D)$ is therefore a subgroup.

The subgroup $C(D \text{ in } G[B])$ is clearly $N(D)$-invariant, as is the suborder $\mathfrak{C}[B](D)$. If $\sigma \in G[B](D)$ and $\pi \in N(D)$, we conclude that $\sigma^\pi \in C(D \text{ in } G[B])$ and that the image of $\mathfrak{C}[B]_{\sigma^\pi}(D) = (\mathfrak{C}[B]_\sigma(D))^\pi$ is

$$\bar{\mathfrak{F}}[G[B]^*]_{\sigma^\pi} = (\bar{\mathfrak{F}}[G[B]^*]_\sigma)^\pi.$$

Hence $\sigma^\pi \in G[B](D)$ and $G[B](D)$ is $N(D)$-invariant.

Suppose that $\sigma \in C(D \text{ in } G[B])$ and $y_\sigma \in \mathfrak{C}[B]_\sigma - \mathfrak{C}[B]_\sigma J(\mathfrak{C}[B]_1)$. Since $\mathfrak{C}[B]_1 = e\mathfrak{C}_1$ is a local ring, we know from [ICE, 1.16] that y_σ is a unit of $\mathfrak{C}[B]$ whose inverse y_σ^{-1} lies in $\mathfrak{C}[B]_{\sigma^{-1}} - \mathfrak{C}[B]_{\sigma^{-1}} J(\mathfrak{C}[B]_1)$. Furthermore, $y_\sigma\,\mathfrak{C}[B]_{\sigma^{-1}} = \mathfrak{C}[B]_1$, by [ICE, 1.14]. Multiplying this last equation on the right by $\mathfrak{C}[B]_\sigma$, we obtain

$$\mathfrak{C}[B]_\sigma = \mathfrak{C}[B]_1\,\mathfrak{C}[B]_\sigma = y_\sigma\,\mathfrak{C}[B]_{\sigma^{-1}}\,\mathfrak{C}[B]_\sigma = y_\sigma\,\mathfrak{C}[B]_1\,.$$

If $\rho \in D$, then $\sigma^\rho = \sigma$. So the above equation and (5.1c) give us an element $z_1 \in \mathfrak{C}[B]_1$ such that $(y_\sigma)^\rho = y_\sigma z_1$. Clearly z_1 is also a unit of $\mathfrak{C}[B]$ and $(y_\sigma^{-1})^\rho = z_1^{-1} y_\sigma^{-1}$.

Now let τ be any element of $G[B](D)$. Choose

$$x_\tau \in \mathfrak{C}[B]_\tau(D) - \mathfrak{C}[B]_\tau J(\mathfrak{C}[B]_1).$$

Then $y_\sigma^{-1} x_\tau y_\sigma \in \mathfrak{C}[B]_{\tau^\sigma}$. Since $z_1 \in \mathfrak{C}[B]_1$ is central in $\mathfrak{C}[B]$ (by (2.8b) and (2.9)), we have

$$(y_\sigma^{-1} x_\tau y_\sigma)^\rho = (y_\sigma^{-1})^\rho (x_\tau)^\rho (y_\sigma)^\rho = z_1^{-1} y_\sigma^{-1} x_\tau y_\sigma z_1 = y_\sigma^{-1} x_\tau y_\sigma\,.$$

Hence $y_\sigma^{-1} x_\tau y_\sigma \in \mathfrak{C}[B]_{\tau^\sigma}(D)$. Evidently the image of $y_\sigma^{-1} x_\tau y_\sigma$ in $\bar{\mathfrak{F}}[G[B]^*]_{\tau^\sigma}$ is conjugate to that of x_τ, and hence is non-zero. Therefore $\tau^\sigma \in G[B](D)$ and $G[B](D)$ is a normal subgroup of $C(D \text{ in } G[B])$. This completes the proof of the first statement of the proposition.

For the second statement, it suffices to show that any element $\sigma \in C(D \text{ in } G[B])$ having order n not divisible by p is an element of $G[B](D)$. Let y_σ be, as above, any element of $\mathfrak{C}[B]_\sigma$ having a non-zero image $\bar{y}_\sigma$ in $\bar{\mathfrak{F}}[G[B]^*]_\sigma$. Then $(\bar{y}_\sigma)^n$ is a non-zero element of

$$\bar{\mathfrak{F}}[G[B]^*]_{\sigma^n} = \bar{\mathfrak{F}}[G[B]^*]_1 \simeq \bar{\mathfrak{F}}.$$

Because $\bar{\mathfrak{F}}$ is algebraically closed, it contains an element $\bar{f}$ such that

$$(\bar{f}\bar{y}_\sigma)^n = \bar{f}^n(\bar{y}_\sigma)^n = 1.$$

Replacing y_σ by fy_σ, where f is any element of $\mathfrak{R}$ having $\bar{f}$ as image in $\bar{\mathfrak{F}} = \mathfrak{R}/\mathfrak{p}$, we see that y_σ can be chosen so that $(y_\sigma)^n \equiv 1 \pmod{J(\mathfrak{C}[B]_1)}$. Now Proposition 1.16 gives us an element $z_1 \in \mathfrak{C}[B]_1$ such that $(z_1)^n = (y_\sigma)^n$. Since $\mathfrak{C}[B]_1$ is central in $\mathfrak{C}[B]$, we can replace y_σ by $z_1^{-1}y_\sigma$ to obtain $(y_\sigma)^n = 1$.

We know that every element of $\mathfrak{C}[B]_\sigma$ has a unique expression in the form $y_\sigma w_1$, where $w_1 \in \mathfrak{C}[B]_1$. Furthermore, $(y_\sigma w_1)^n = y_\sigma^n w_1^n = w_1^n$, since $\mathfrak{C}[B]_1$ is central in $\mathfrak{C}[B]$. Therefore $(y_\sigma w_1)^n = 1$ if and only if w_1 is an n^{th}-root of 1 in $\mathfrak{C}[B]_1$. Because $\bar{\mathfrak{F}}$ has characteristic p which does not divide n, the valuation ring $\mathfrak{R}$ contains precisely n n^{th} roots of 1, of the form $\zeta, \zeta^2, \cdots, \zeta^n = 1$, whose images in $\bar{\mathfrak{F}}$ are the distinct n^{th} roots of 1 in that field. In view of (2.11), the image of w_1 in $\mathfrak{C}[B]_1/J(\mathfrak{C}[B]_1)$ must coincide with that of ζ^i, for a unique $i = 1, \cdots, n$. Hence

$$\zeta^{-i}w_1 \equiv 1 \pmod{J(\mathfrak{C}[B]_1)}$$

is another n^{th}-root of 1 in $\mathfrak{C}[B]_1$. Now Proposition 1.16 tells us that $\zeta^{-i}w_1 = 1$, i.e., that $w_1 = \zeta^i$. We conclude that $\zeta y_\sigma, \zeta^2 y_\sigma, \cdots, \zeta^n y_\sigma = y_\sigma$ are the only n^{th} roots of 1 in $\mathfrak{C}[B]_\sigma$. Evidently these elements must be permuted among themselves by any $\rho \in D$. Therefore $(y_\sigma)^\rho = \zeta^i y_\sigma$, for some unique $i = 1, \cdots, n$. Since ρ centralizes $\zeta \in \mathfrak{R}$, we have $(y_\sigma)^{\rho^n} = \zeta^{in}y_\sigma = y_\sigma$. But n is relatively prime to the order of $\rho \in D$, since D is a p-group. Therefore ρ centralizes y_σ, and

$$y_\sigma \in \mathfrak{C}[B]_\sigma(D) - \mathfrak{C}[B]_\sigma/J(\mathfrak{C}[B]_1).$$

So $\sigma \in G[B](D)$, which completes the proof of the second statement of the proposition.

If $y = \sum_{\sigma \in C(D \text{ in } G[B])} y_\sigma \in \mathfrak{C}[B]_{(D)}$, where $y_\sigma \in \mathfrak{C}[B]_\sigma$ for all σ, then clearly $y \in \mathfrak{C}[B]_{(D)}(D)$ if and only if $(y_\sigma)^\rho = y_{\sigma^\rho} = y_\sigma$ for all $\rho \in D$, $\sigma \in C(D \text{ in } G[B])$. We conclude that

$$\mathfrak{C}[B]_{(D)}(D) = \oplus \sum_{\sigma \in C(D \text{ in } G[B])} \mathfrak{C}[B]_\sigma(D).$$

Equation (9.3) follows directly from this, (9.1), and the one-dimensionality of the $\bar{\mathfrak{F}}$-spaces $\bar{\mathfrak{F}}[G[B]^*]_\sigma$. Since $G[B](D)$ is a subgroup of $G[B]$ and $\bar{\mathfrak{F}}[G[B]^*]$ is a twisted group algebra of $G[B]$, this implies that $\mathfrak{C}[B]_{(D)}(D)^J$ is a twisted group algebra of $G[B](D)$. So the proposition is proved.

We shall need an additional hypothesis to carry out the rest of the Brauer analysis. Of course, everything would work under the conditions (6.1). But we can get away with the weaker assumption that

$$\mathfrak{C}[B]_{(D)}(N(D))^J \subseteq Z(\mathfrak{C}[B]_{(D)}(D)^J), \tag{9.4}$$

where, as usual, the right term denotes the center of the algebra $\mathfrak{C}[B]_{(D)}(D)^J$.

We should remark that (9.4) certainly holds when (6.1) is satisfied, since

$C(D \text{ in } G[B])$ is then a normal subgroup of $N(D) = N(D \text{ in } E_B)$ (by (5.5a) and (6.1)), which implies that

$$\mathfrak{C}[B]_{(D)}(N(D)) \subseteq \mathfrak{C}[B]_{(D)}(C(D \text{ in } G[B])) = Z(\mathfrak{C}[B]_{(D)}),$$

by the definition (2.15) of the action of $G[B]$ on $\mathfrak{C}[B]$.

It follows from Proposition 9.2 that the subalgebra $\mathfrak{C}[B]_{(D)}(D)^J$ of $\bar{\mathfrak{F}}[G[B]^*]$ is $N(D)$-invariant. So $N(D)$ permutes among themselves the ideals $\mathfrak{N} \in \operatorname{Max}(\mathfrak{C}[B]_{(D)}(D)^J)$. For any such $\mathfrak{N}$, its stabilizer $N(D)_{\mathfrak{N}}$ in $N(D)$ acts naturally as algebra automorphisms of the simple factor algebra $\mathfrak{C}[B]_{(D)}(D)^J/\mathfrak{N}$. Obviously D, which centralizes $\mathfrak{C}[B]_{(D)}(D)^J$, is a subgroup of $N(D)_{\mathfrak{N}}$. So the $\bar{\mathfrak{F}}$-subspace

$$[\mathfrak{C}[B]_{(D)}(D)^J/\mathfrak{N}](N(D)_{\mathfrak{N}} \mid D)$$

of $\mathfrak{C}[B]_{(D)}(D)^J/\mathfrak{N}$ is defined.

The Brauer analysis can now be completed by:

THEOREM 9.5. *When* (9.4) *holds there is a one-to-one correspondence between the ideals* $\mathfrak{M} \in \operatorname{Max}(\mathfrak{C}[B]_{(D)}(N(D)) \mid D)$ *and the* $N(D)$*-orbits of ideals* $\mathfrak{N} \in \operatorname{Max}(\mathfrak{C}[B]_{(D)}(D)^J)$ *satisfying*

$$[\mathfrak{C}[B]_{(D)}(D)^J/\mathfrak{N}](N(D)_{\mathfrak{N}} \mid D) \neq \{0\}. \tag{9.6}$$

Such an $\mathfrak{M}$ *corresponds to the orbit of such an* $\mathfrak{N}$ *if and only if*

$$\mathfrak{M}^J = \mathfrak{N} \cap \mathfrak{C}[B]_{(D)}(N(D))^J.$$

In that case the $N(D)$*-conjugates of* $\mathfrak{N}$ *are the only ideals*

$$\mathfrak{N}_0 \in \operatorname{Max}(\mathfrak{C}[B]_{(D)}(D)^J)$$

satisfying

$$\mathfrak{N}_0 \cap \mathfrak{C}[B]_{(D)}(N(D))^J \subseteq \mathfrak{M}^J.$$

Proof. For any $\mathfrak{N} \in \operatorname{Max}(\mathfrak{C}[B]_{(D)}(D)^J)$, we denote by $\mathfrak{A}[\mathfrak{N}]$ the factor algebra $\mathfrak{C}[B]_{(D)}(D)^J/\mathfrak{N}$. This is a finite-dimensional simple algebra over the algebraically closed field $\bar{\mathfrak{F}}$. Hence its center $Z(\mathfrak{A}[\mathfrak{N}])$ is just $\bar{\mathfrak{F}}\cdot 1$. Since $\mathfrak{C}[B]_{(D)}(N(D))^J$ contains the identity of $\mathfrak{C}[B]_{(D)}(D)^J$, condition (9.4) implies that its image in $\mathfrak{A}[\mathfrak{N}]$ is precisely $\bar{\mathfrak{F}}\cdot 1$. So

$$\mathfrak{C}[B]_{(D)}(N(D))^J/(\mathfrak{N} \cap \mathfrak{C}[B]_{(D)}(N(D))^J) \simeq \bar{\mathfrak{F}} \quad (\textit{as } \bar{\mathfrak{F}}\textit{-algebras}),$$

and

$$\mathfrak{N} \cap \mathfrak{C}[B]_{(D)}(N(D))^J \in \operatorname{Max}(\mathfrak{C}[B]_{(D)}(N(D))^J).$$

Hence the inverse image $\mathfrak{M} = \mathfrak{M}(\mathfrak{N})$ of $\mathfrak{N} \cap \mathfrak{C}[B]_{(D)}(N(D))^J$ is a maximal two-sided ideal of $\mathfrak{C}[B]_{(D)}(N(D))$ satisfying $\mathfrak{M}^J = \mathfrak{N} \cap \mathfrak{C}[B]_{(D)}(N(D))^J$.

Suppose that $\mathfrak{N}$ satisfies (9.6). Since D centralizes

$$\mathfrak{A}[\mathfrak{N}] = \mathfrak{C}[B]_{(D)}(D)^J/\mathfrak{N},$$

there exists an $x \in \mathfrak{A}[\mathfrak{N}]$ such that $w = \operatorname{tr}_{D\to N(D)}(x) \neq 0$. Because the algebra

$\mathfrak{S}[B]_{(D)}(D)^J$ is finite dimensional, it contains an element y having x as image in $\mathfrak{A}[\mathfrak{N}]$ such that $y \in \mathfrak{N}_0$ for all $\mathfrak{N}_0 \neq \mathfrak{N}$ in Max $(\mathfrak{S}[B]_{(D)}(D)^J)$. This implies that $y^\sigma \in \mathfrak{N} = (\mathfrak{N}^{\sigma^{-1}})^\sigma$, for all $\sigma \in N(D) - N(D)_{\mathfrak{N}}$. So the image in $\mathfrak{A}[\mathfrak{N}]$ of

$$\mathrm{tr}_{D \to N(D)}(y) = \sum\nolimits_{\sigma \in N(D)/D} y^\sigma$$

is just $\sum_{\sigma \in N(D)_{\mathfrak{N}}/D} x^\sigma = w \neq 0$. We conclude that $\mathrm{tr}_{D \to N(D)}(y) \notin \mathfrak{M}^J$. If z is any element of $\mathfrak{S}[B]_{(D)}(D)$ having y as image, then this implies that $\mathrm{tr}_{D \to N(D)}(z) \in \mathfrak{S}[B]_{(D)}(N(D) \mid D) - \mathfrak{M}$. Therefore D contains a defect group C of $\mathfrak{M}$.

If $C < D$, then $\mathfrak{S}[B]_{(D)}(N(D) \mid C)^J = \{0\}$ by (8.9a). Since $\mathfrak{M}$ is the inverse image of $\mathfrak{M}^J$, it therefore contains $\mathfrak{S}[B]_{(D)}(N(D) \mid C)$. This is impossible for the defect group C of $\mathfrak{M}$. Hence $C = D$ and

$$\mathfrak{M} \in \mathrm{Max}\ (\mathfrak{S}[B]_{(D)}(N(D)) \mid D).$$

Since $N(D)$ centralizes $\mathfrak{M}^J$, it is clear that $\mathfrak{M}(\mathfrak{N}^\sigma) = \mathfrak{M}(\mathfrak{N})^\sigma = \mathfrak{M}$, for all $\sigma \in N(D)$. Suppose that $\mathfrak{N}_0 \in \mathrm{Max}\ (\mathfrak{S}[B]_{(D)}(D)^J)$ is not $N(D)$-conjugate to $\mathfrak{N}$. Then, by construction, $y^\sigma \in \mathfrak{N}_0$ for all $\sigma \in N(D)$. Hence $\mathrm{tr}_{D \to N(D)}(y) \in \mathfrak{N}_0$. We conclude that $\mathrm{tr}_{D \to N(D)}(z)$ lies in the inverse image $\mathfrak{M}(\mathfrak{N}_0)$ of $\mathfrak{N}_0 \cap \mathfrak{S}[B]_{(D)}(N(D))^J$. Since this element does not lie in $\mathfrak{M}$, we must have $\mathfrak{M}(\mathfrak{N}_0) \nsubseteq \mathfrak{M}$. This proves the last statement of the theorem, and shows that we have a one-to-one correspondence between all the $N(D)$-orbits of ideals

$$\mathfrak{N} \in \mathrm{Max}\ (\mathfrak{S}[B]_{(D)}(D)^J)$$

satisfying (9.6) and some of the ideals

$$\mathfrak{M} \in \mathrm{Max}\ (\mathfrak{S}[B]_{(D)}(N(D)) \mid D).$$

It remains to be shown that we obtain every such $\mathfrak{M}$ in this way. First notice that $\mathfrak{M}$ certainly has the form $\mathfrak{M}(\mathfrak{N})$ for some ideal

$$\mathfrak{N} \in \mathrm{Max}\ (\mathfrak{S}[B]_{(D)}(D)^J).$$

Indeed, if this is false, then $\mathfrak{M}^J$, which lies in Max $(\mathfrak{S}[B]_{(D)}(N(D))^J)$ by Proposition 5.10 (for $N(D)$ and $C(D \text{ in } G[B])$ in place of E_B and $G[B]$), contains none of the maximal two-sided ideals

$$\mathfrak{N} \cap \mathfrak{S}[B]_{(D)}(N(D))^J,$$

for $\mathfrak{N} \in \mathrm{Max}\ (\mathfrak{S}[B]_{(D)}(D)^J)$. Hence $\mathfrak{M}^J$ does not contain their product. But this product is contained in their intersection, which is

$$\mathfrak{S}[B]_{(D)}(N(D))^J \cap J(\mathfrak{S}[B]_{(D)}(D)^J) \subseteq J(\mathfrak{S}[B]_{(D)}(N(D))^J) \subseteq \mathfrak{M}^J,$$

by Proposition 1.9. The contradiction proves the above statement.

Now we fix an ideal $\mathfrak{N} \in \mathrm{Max}\ (\mathfrak{S}[B]_{(D)}(D)^J)$ such that $\mathfrak{M} = \mathfrak{M}(\mathfrak{N})$. We must show that $\mathfrak{N}$ satisfies (9.6). If $\mathfrak{N}_1 = \mathfrak{N}, \mathfrak{N}_2, \cdots, \mathfrak{N}_n$ form the $N(D)$-

conjugacy class of $\mathfrak{N}$, then the natural map of $\mathfrak{C}[B]_{(D)}(D)^J$ into

$$\mathfrak{A}[\mathfrak{N}_1] \oplus \cdots \oplus \mathfrak{A}[\mathfrak{N}_n]$$

(as algebras) is an epimorphism with kernel $\mathfrak{N}_1 \cap \cdots \cap \mathfrak{N}_n$. Hence $N(D)$ acts naturally on $\mathfrak{A}[\mathfrak{N}_1] \oplus \cdots \oplus \mathfrak{A}[\mathfrak{N}_n]$ so that this map is $N(D)$-invariant. Because $\mathfrak{M}$ has D as a defect group, there is an element $z \in \mathfrak{C}[B]_{(D)}(D)$ such that $\mathrm{tr}_{D\to N(D)}(z) \notin \mathfrak{M}$. It follows that the image x of z in $\mathfrak{A}[\mathfrak{N}_1] \oplus \cdots \oplus \mathfrak{A}[\mathfrak{N}_n]$ satisfies $\mathrm{tr}_{D\to N(D)}(x) \neq 0$. Write $x = x_1 \oplus \cdots \oplus x_n$, where $x_i \in \mathfrak{A}[\mathfrak{N}_i]$ for $i = 1, \cdots, n$. Then

$$0 \neq \mathrm{tr}_{D\to N(D)}(x) = \mathrm{tr}_{D\to N(D)}(x_1) + \cdots + \mathrm{tr}_{D\to N(D)}(x_n).$$

So there exists an $i = 1, \cdots, n$ such that $\mathrm{tr}_{D\to N(D)}(x_i) \neq 0$. It follows that $\mathrm{tr}_{D\to N(D)_{\mathfrak{N}_i}}(x_i)$ is a non-zero element of $\mathfrak{A}[\mathfrak{N}_i]$. If $\sigma \in N(D)$ satisfies $\mathfrak{N}_i^\sigma = \mathfrak{N}$, then $x_0 = x_i^\sigma$ will be an element of $\mathfrak{A}[\mathfrak{N}] = \mathfrak{A}[\mathfrak{N}_1]$ such that

$$\mathrm{tr}_{D\to N(D)_{\mathfrak{N}}}(x_0) = \mathrm{tr}_{D^\sigma\to(N(D)_{\mathfrak{N}_i})^\sigma}(x_i^\sigma) = [\mathrm{tr}_{D\to N(D)_{\mathfrak{N}_i}}(x_i)]^\sigma \neq 0.$$

Therefore $\mathfrak{N}$ satisfies (9.6). This completes the proof of the theorem.

Condition (9.6) does not look very much like Brauer's conditions in Theorem (11B) of [1]. However, we shall show that it is equivalent to them under the hypothesis (6.1).

PROPOSITION 9.7. *Suppose that* (6.1) *holds. Then an ideal*

$$\mathfrak{N} \in \mathrm{Max}\,(\mathfrak{C}[B]_{(D)}(D)^J)$$

satisfies (9.6) *if and only if it satisfies the two conditions*:

(9.8a) $$[N(D)_{\mathfrak{N}}:D\cdot G[B](D)] \not\equiv 0 \pmod p,$$

(9.8b) $$[\mathfrak{C}[B]_{(D)}(D)^J/\mathfrak{N}](G[B](D) \mid D \cap G[B](D)) \neq \{0\}.$$

Proof. It follows from (6.1c) that the action on $\mathfrak{C}[B]_{(D)}(D)^J$ of any element $\sigma \in G[B](D)$ is that of conjugation by any non-zero element of $\bar{\mathfrak{F}}[G[B]^*]_\sigma$. In view of (9.3), this implies that

$$[\mathfrak{C}[B]_{(D)}(D)^J](G[B](D))$$

is the center $Z(\mathfrak{C}[B]_{(D)}(D)^J)$. As in the proof of Theorem 9.5, the image of this subalgebra in $\mathfrak{A}[\mathfrak{N}] = \mathfrak{C}[B]_{(D)}(D)^J/\mathfrak{N}$ is the center $\bar{\mathfrak{F}}\cdot 1$ of that algebra, which is centralized by $N(D)_{\mathfrak{N}}$. By Proposition 9.2 and (6.1b), $G[B](D)$ is a normal subgroup of $N(D)$. From the above description of its action on $\mathfrak{C}[B]_{(D)}(D)^J$ and (9.3), it is clear that $G[B](D) \leq N(D)_{\mathfrak{N}}$. Hence $D\cdot G[B](D)$ is a subgroup of $N(D)_{\mathfrak{N}}$. If $x \in \mathfrak{A}[\mathfrak{N}]$, then

$$\mathrm{tr}_{D\to D\cdot G[B](D)}(x) = \mathrm{tr}_{D\cap G[B](D)\to G[B](D)}(x)$$

lies in the image $\bar{\mathfrak{F}}\cdot 1$ of $[\mathfrak{C}[B]_{(D)}(D)^J](G[B](D))$ in $\mathfrak{A}[\mathfrak{N}]$. So we have

$$\begin{aligned}\mathrm{tr}_{D\to N(D)_{\mathfrak{N}}}(x) &= \mathrm{tr}_{D\cdot G[B](D)\to N(D)_{\mathfrak{N}}}(\mathrm{tr}_{D\to D\cdot G[B](D)}(x)) \\ &= [N(D)_{\mathfrak{N}}:D\cdot G[B](D)]\cdot \mathrm{tr}_{D\cap G[B](D)\to G[B](D)}(x).\end{aligned}$$

It follows that

$$\mathfrak{A}[\mathfrak{N}](N(D)_{\mathfrak{N}} \mid D) = [N(D)_{\mathfrak{N}} : D \cdot G[B](D)]\mathfrak{A}[\mathfrak{N}](G[B](D) \mid D \cap G[B](D)).$$

Because $\bar{\mathfrak{F}}$ has characteristic p, the equivalence of (9.6) and (9.8), which is the proposition, is a direct consequence of this equation.

Condition (9.8a) is obviously comparable to Brauer's index condition on the inertial subgroup in (11B) of [1], to which it reduces in his special case. To bring (9.8b) into the form of Brauer's statement we need only apply Theorem 9.5 once more.

PROPOSITION 9.9. *The ideals* $\mathfrak{N} \in \mathrm{Max}\,(\mathfrak{C}[B]_{(D)}(D)^J)$ *satisfying* (9.8b) *correspond one-to-one to the blocks* $\bar{B}$ *of the twisted group algebra* $\mathfrak{C}[B]_{(D)}(D)^J$ *having defect group* $D \cap G[B](D)$ *in* $G[B](D)$. *Two such* $\mathfrak{N}$ *and* $\bar{B}$ *correspond if and only if* $\mathfrak{N} \in \bar{B}$. *In that case* $\mathfrak{N}$ *is the only ideal in* $\mathrm{Max}\,(\mathfrak{C}[B]_{(D)}(D)^J)$ *lying in* $\bar{B}$.

Proof. In view of Proposition 9.2, the axioms (2.1), (5.1), and (6.1) are all satisfied if $\mathfrak{N}$, $\mathfrak{O}$, G, the $\mathfrak{O}_\sigma$, $\sigma \in G$, and E are replaced by $\bar{\mathfrak{F}}$, $\mathfrak{C}[B]_{(D)}(D)^J$, $G[B](D)$, the $\bar{\mathfrak{F}}[G[B]^*]_\sigma$, $\sigma \in G[B](D)$, and $G[B](D)$, respectively. In this case the orders and algebras corresponding to $\mathfrak{C}$, $e\mathfrak{C}$, $\mathfrak{C}[B]$, and $\bar{\mathfrak{F}}[G[B]^*]$ all coincide with $\mathfrak{C}[B]_{(D)}(D)^J$, while the groups corresponding to E_B, G_B, and $G[B]$ all coincide with $G[B](D)$.

We use $D \cap G[B](D)$ in place of D. By Proposition 9.2 the group corresponding to $C(D \text{ in } G[B])$ is now $G[B](D)$. Hence the order corresponding to $\mathfrak{C}[B]_{(D)}$ is $\mathfrak{C}[B]_{(D)}(D)^J$. Since $D \cap G[B](D) \leq D$ centralizes $\mathfrak{C}[B]_{(D)}(D)^J$, the order and algebra corresponding to $\mathfrak{C}[B]_{(D)}(D)$ and $\mathfrak{C}[B]_{(D)}(D)^J$ also coincide with $\mathfrak{C}[B]_{(D)}(D)^J$.

The group corresponding to $N(D) = N(D \text{ in } E_B)$ is now $G[B](D)$, which operates on the twisted group algebra $\mathfrak{C}[B]_{(D)}(D)^J$ of $G[B](D)$ in the usual manner by conjugation. It follows that the order $[\mathfrak{C}[B]_{(D)}(D)^J](G[B](D))$ corresponding to $\mathfrak{C}[B]_{(D)}(N(D))$ is now the center $Z(\mathfrak{C}[B]_{(D)}(D)^J)$. Hence its maximal ideals $\mathfrak{M}$ correspond one-to-one to the blocks $\bar{B}$ of $\mathfrak{C}[B]_{(D)}(D)^J$. By definition the defect groups of $\mathfrak{M}$ are those of the corresponding block $\bar{B}$. Since $G[B](D)$ (which corresponds to $N(D)$) acts as inner automorphisms of $\mathfrak{C}[B]_{(D)}(D)^J$, it leaves invariant all ideals $\mathfrak{N} \in \mathrm{Max}\,(\mathfrak{C}[B]_{(D)}(D)^J)$. Now Theorem 9.5, applied to the new situation, gives the present proposition.

10. First half of the analysis of $G[B]^*$

We can use the Brauer analysis to compute the Clifford extension of a block in terms of those of its corresponding characters. We shall do this under the assumption that (7.1) holds. So we can apply the above theories to $\mathfrak{D}$, H, and the $\mathfrak{D}_\sigma$, $\sigma \in H$, in place of $\mathfrak{O}$, G, and the $\mathfrak{O}_\sigma$, $\sigma \in G$. Condition (7.1b) tells us that $\mathfrak{D}_1$ is a local ring which is central in $\mathfrak{D}$. It follows that the suborders corresponding to $\mathfrak{C}$, $e\mathfrak{C}$, and $\mathfrak{C}[B]$ all coincide with $\mathfrak{D}$, while the subgroups corresponding to G_B and $G[B]$ both coincide with H. We denote by

H^* the central extension of $\bar{F}$ by H corresponding to $G[B]^*$. In view of (2.14), its twisted group algebra $\bar{\mathfrak{F}}[H^*]$ is given by:

(10.1a) $\bar{\mathfrak{F}}[H^*] = \mathfrak{D}/\mathfrak{D}J(\mathfrak{D}_1)$,

(10.1b) $\bar{\mathfrak{F}}[H^*]_\sigma = \mathfrak{D}_\sigma/\mathfrak{D}_\sigma J(\mathfrak{D}_1)$, *for all* $\sigma \in H$.

We shall use the superscript I to denote the images of objects in $\bar{\mathfrak{F}}[H^*]$, reserving the superscript J, as usual, for their images in $\bar{\mathfrak{F}}[G[B]^*]$ (as defined below).

We fix a normal subgroup K of H. In view of (2.16) and (7.1c), conditions (5.1) are satisfied with K, $\mathfrak{D}$, H, and the $\mathfrak{D}_\sigma$ in place of E, $\mathfrak{C}$, G, and the $\mathfrak{C}_\sigma$, respectively, using the operation (2.15) of $K \leq H$ on $\mathfrak{D}$ and the conjugation action of K on H. The subgroup corresponding to E_B is now K. For any p-subgroup D of K, we define, as in §§8, 9:

(10.2a) $\mathfrak{D}_{(D)} = \oplus \sum_{\sigma \in C(D \text{ in } H)} \mathfrak{D}_\sigma$,

(10.2b) $\bar{\mathfrak{F}}[H^*]_{(D)} = \mathfrak{D}^I_{(D)} = \oplus \sum_{\sigma \in C(D \text{ in } H)} \bar{\mathfrak{F}}[H^*]_\sigma$,

(10.2c) $N(D) = N(D \text{ in } K)$,

(10.2d) $H(D) = \{\sigma \in C(D \text{ in } H) \mid \mathfrak{D}_\sigma(D) \not\subseteq \mathfrak{D}_\sigma J(\mathfrak{D}_1)\}$.

We denote by S the Brauer homomorphism of $\mathfrak{D}(K)^I$ into $\mathfrak{D}_{(D)}(N(D))^I$ defined (in Proposition 8.1) by the subgroup D.

Let G be the factor group H/K. As in Theorem 7.3, we define:

(10.3a) $\mathfrak{O} = \mathfrak{D}$,

(10.3b) $\mathfrak{O}_\tau = \oplus \sum_{\sigma \in \tau} \mathfrak{D}_\sigma$, *for all cosets* $\tau \in G = H/K$.

Then $\mathfrak{O}$, G, and the $\mathfrak{D}_\tau$ also satisfy (2.1). It is clear from (2.15) that:

(10.4a) $\mathfrak{C} = C(\mathfrak{O}_1 \text{ in } \mathfrak{O}) = C(K \text{ in } \mathfrak{D}) = \mathfrak{D}(K) = \mathfrak{O}(K)$,

(10.4b) $\mathfrak{C}_\tau = C(\mathfrak{O}_1 \text{ in } \mathfrak{O}_\tau) = C(K \text{ in } \mathfrak{O}_\tau) = \mathfrak{O}_\tau(K)$, *for all* $\tau \in G$.

As usual, we fix a block B of $\mathfrak{O}_1$, and define e, $e\mathfrak{C}$, $G[B]$, $\mathfrak{C}[B]$, and $G[B]^*$ as in §2. Since K acts as automorphisms of the order $\mathfrak{O}_1 = \oplus \sum_{\sigma \in K} \mathfrak{D}_\sigma$, we can choose the above p-group D to satisfy:

(10.5) *D is a defect group in K of B.*

The hypotheses of Brauer's First Main Theorem 8.7 are now satisfied with K, $\mathfrak{O}_1$, K, and the $\mathfrak{D}_\sigma$, $\sigma \in K$, in place of E_B, $\mathfrak{C}[B]$, $G[B]$, and the $\mathfrak{C}[B]_\sigma$, $\sigma \in G[B]$, respectively. In view of Proposition 8.1, the Brauer homomorphism used in that theorem is simply the restriction to $\mathfrak{O}_1(K)^I = \mathfrak{C}_1^I$ of the above homomorphism S. Since $\mathfrak{C}_1$ is a commutative order, its primitive idempotent e corresponds to the unique maximal ideal

(10.6) $$\mathfrak{M} = J(e\mathfrak{C}_1) \oplus (1 - e)\mathfrak{C}_1,$$

in which it doesn't lie. Condition (10.5) says that $\mathfrak{M}$ lies in Max $(\mathfrak{O}_1(K) \mid D)$. So Theorem 8.7 gives us a unique ideal $\mathfrak{N} \in$ Max $(\mathfrak{O}_{1,(D)}(N(D)) \mid D)$ such that

(10.7) $$\mathfrak{M}^I = S^{-1}(\mathfrak{N}^I) \cap \mathfrak{O}_1(K)^I,$$

where, of course, $\mathfrak{O}_{1,(D)} = \mathfrak{O}_1 \cap \mathfrak{D}_{(D)} = \oplus \sum_{\sigma \in C(D \text{ in } K)} \mathfrak{D}_\sigma$.

More generally, we define:

(10.8a) $\mathfrak{O}_{(D)} = \mathfrak{D}_{(D)} = \oplus \sum_{\sigma \in C(D \text{ in } H)} \mathfrak{D}_\sigma$,

(10.8b) $\mathfrak{O}_{\tau,(D)} = \mathfrak{O}_\tau \cap \mathfrak{O}_{(D)} = \oplus \sum_{\sigma \in C(D \text{ in } \tau)} \mathfrak{D}_\sigma$, *for all cosets* $\tau \in G = H/K$.

Evidently (2.16) and the $N(D)$-invariance of $C(D \text{ in } H)$ imply that $\mathfrak{O}_{(D)}$ is an $N(D)$-invariant suborder of $\mathfrak{O}$. Hence $\mathfrak{O}_{(D)}(N(D))$ is a well-defined suborder of $\mathfrak{O}_{(D)}$. Since $N(D) = N(D \text{ in } K)$ is a subgroup of the normal subgroup K of H, it leaves invariant each coset τ of K in H. Therefore $\mathfrak{O}_\tau$ and $\mathfrak{O}_{\tau,(D)}$ are $N(D)$-invariant $\mathfrak{R}$-sublattices of $\mathfrak{O}$, and $\mathfrak{O}_{\tau,(D)}(N(D)) = C(N(D) \text{ in } \mathfrak{O}_{\tau,(D)})$ is a well-defined $\mathfrak{R}$-sublattice of $\mathfrak{O}_{\tau,(D)}$.

PROPOSITION 10.9. (a) *The suborder* $\mathfrak{O}_{(D)}(N(D))$ *of* $\mathfrak{O}$ *is invariant under* $N(D \text{ in } H)$ *and centralized by* $N(D) = N(D \text{ in } K)$,

(b) $\mathfrak{O}_{1,(D)}(N(D))$ *is a central suborder of* $\mathfrak{O}_{(D)}(N(D))$ *containing the identity of* $\mathfrak{O}$,

(c) $\mathfrak{O}_{(D)}(N(D)) = \oplus \sum_{\tau \in G} \mathfrak{O}_{\tau,(D)}(N(D))$ *(as* $\mathfrak{R}$*-modules)*,

(d) $\mathfrak{O}_{\rho,(D)}(N(D))\mathfrak{O}_{\tau,(D)}(N(D)) \subseteq \mathfrak{O}_{\rho\tau,(D)}(N(D))$, *for all* $\rho, \tau \in G$,

(e) $\mathfrak{O}_{\tau,(D)}(N(D))^\pi = \mathfrak{O}_{\tau^\pi,(D)}(N(D))$, *for all* $\tau \in G$, $\pi \in N(D \text{ in } H)$.

Proof. Since $C(D \text{ in } H)$ is a normal subgroup of $N(D \text{ in } H)$, it is clear from (10.8a) and (2.16) that the suborder $\mathfrak{O}_{(D)}$ is $N(D \text{ in } H)$-invariant. Since $N(D) = N(D \text{ in } K)$ is also a normal subgroup of $N(D \text{ in } H)$, the suborder $\mathfrak{O}_{(D)}(N(D)) = C(N(D) \text{ in } \mathfrak{O}_{(D)})$ satisfies (a).

Clearly $\mathfrak{O}_{1,(D)} = \mathfrak{O}_1 \cap \mathfrak{O}_{(D)}$ is an $N(D)$-invariant suborder of $\mathfrak{O}$ containing $1_{\mathfrak{O}} \in \mathfrak{D}_1$. Hence so is $\mathfrak{O}_{1,(D)}(N(D))$. By (2.15) each element of

$$\mathfrak{O}(C(D \text{ in } K)) = \mathfrak{D}(C(D \text{ in } K))$$

centralizes

$$\mathfrak{O}_{1,(D)} = \oplus \sum_{\sigma \in C(D \text{ in } K)} \mathfrak{D}_\sigma .$$

Since $C(D \text{ in } K)$ is a (normal) subgroup of $N(D)$, we conclude that

$$\mathfrak{O}_{(D)}(N(D)) \subseteq \mathfrak{O}(N(D)) \subseteq \mathfrak{O}(C(D \text{ in } K))$$

centralizes

$$\mathfrak{O}_{1,(D)}(N(D)) \subseteq \mathfrak{O}_{1,(D)} .$$

Therefore (b) holds.

Evidently (10.8) implies $\mathfrak{O}_{(D)} = \oplus \sum_{\tau \in G} \mathfrak{O}_{\tau,(D)}$. Since all these $\mathfrak{R}$-lattices are $N(D)$-invariant, equation (c) is an immediate consequence of this.

The definition (10.8) also gives $\mathfrak{O}_{\rho,(D)} \mathfrak{O}_{\tau(D)} \subseteq \mathfrak{O}_{\rho\tau,(D)}$, for all $\rho, \tau \in G$. The inclusion (d) follows directly from this.

Finally, (e) is a consequence of (10.8b) and (2.16). So the proposition is proved.

From (10.1) and (10.3) it is clear that the image $\mathfrak{O}^I = \bar{\mathfrak{F}}[H^*]$ is the direct sum of the images $\mathfrak{O}_\tau^I = \oplus \sum_{\sigma \in \tau} \bar{\mathfrak{F}}[H^*]_\sigma$, for all $\tau \in G$. It follows from this and Proposition 10.9 (c) that the $\bar{\mathfrak{F}}$-algebra $\mathfrak{O}_{(D)}(N(D))^I$ is the direct sum of the images $\mathfrak{O}_{\tau,(D)}(N(D))^I$, for $\tau \in G$, of the $\mathfrak{O}_{\tau,(D)}(N(D))$. So Proposition

10.9 implies:

(10.10a) $\mathfrak{O}_{(D)}(N(D))^I$ *is a* $N(D$ in $H)$*-invariant* $\bar{\mathfrak{F}}$*-subalgebra of* $\bar{\mathfrak{F}}[H^*]$, *and is centralized by* $N(D)$.

(10.10b) *Each* $\mathfrak{O}_{\tau,(D)}(N(D))^I$, *for* $\tau \in G$, *is an* $\bar{\mathfrak{F}}$*-subspace of* $\mathfrak{O}_{(D)}(N(D))^I$.

(10 10c) $\mathfrak{O}_{1,(D)}(N(D))^I$ *is a central subalgebra of, and contains the identity of,* $\mathfrak{O}_{(D)}(N(D))^I$.

(10.10d) $\mathfrak{O}_{(D)}(N(D))^I = \oplus \sum_{\tau \in G} \mathfrak{O}_{\tau,(D)}(N(D))^I$ (*as* $\bar{\mathfrak{F}}$*-spaces*).

(10.10e) $\mathfrak{O}_{\rho,(D)}(N(D))^I \mathfrak{O}_{\tau,(D)}(N(D))^I \subseteq \mathfrak{O}_{\rho\tau,(D)}(N(D))^I$, *for all* $\rho, \tau \in G$.

(10.10f) $[\mathfrak{O}_{\tau,(D)}(N(D))^I]^\pi = \mathfrak{O}_{\tau^\pi,(D)}(N(D))^I$, *for all* $\tau \in G$, $\pi \in N(D$ in $H)$.

From (10.10c, e) it is clear that the maximal ideal $\mathfrak{N}^I$ of $\mathfrak{O}_{1,(D)}(N(D))^I$ satisfies

$$\mathfrak{O}_{\tau,(D)}(N(D))^I\mathfrak{N}^I = \mathfrak{N}^I\mathfrak{O}_{\tau,(D)}(N(D))^I \subseteq \mathfrak{O}_{\tau,(D)}(N(D))^I, \quad \textit{for all} \quad \tau \in G.$$

In view of (10.10d), this implies that it generates a graded two-sided ideal.

$$\mathfrak{O}_{(D)}(N(D))^I\mathfrak{N}^I = \mathfrak{N}^I\mathfrak{O}_{(D)}(N(D))^I = \oplus \sum_{\tau \in G} \mathfrak{O}_{\tau,(D)}(N(D))^I\mathfrak{N}^I$$

of $\mathfrak{O}_{(D)}(N(D))^I$. We note by $\mathfrak{B}$ the corresponding factor algebra. For each $\tau \in G$, we identify

$$\mathfrak{O}_{\tau,(D)}(N(D))^I/\mathfrak{O}_{\tau,(D)}(N(D))^I\mathfrak{N}^I$$

in the usual way with its image in $\mathfrak{B}$, and call the resulting $\bar{\mathfrak{F}}$-subspace $\mathfrak{B}_\tau$. Notice that $\mathfrak{B}_1 = \mathfrak{O}_{1,(D)}(N(D))^I/\mathfrak{N}^I \simeq \bar{\mathfrak{F}}$ by (10.10c), since $\mathfrak{N}^I$ is a maximal ideal in the commutative subalgebra $\mathfrak{O}_{1,(D)}(N(D))^I$ of $\mathfrak{O}_{(D)}(N(D))^I$. Hence (10.10) gives:

(10.11a) $\mathfrak{B}$ *is an* $\bar{\mathfrak{F}}$*-order.*

(10.11b) *Each* $\mathfrak{B}_\tau$, $\tau \in G$, *is an* $\bar{\mathfrak{F}}$*-subspace of* $\mathfrak{B}$.

(10.11c) $\mathfrak{B}_1 = \bar{\mathfrak{F}} \cdot 1_{\mathfrak{B}}$.

(10.11d) $\mathfrak{B} = \oplus \sum_{\tau \in G} \mathfrak{B}_\tau$ (*as* $\bar{\mathfrak{F}}$*-spaces*).

(10.11e) $\mathfrak{B}_\rho \mathfrak{B}_\tau \subseteq \mathfrak{B}_{\rho\tau}$, *for all* $\rho, \tau \in G$.

By (10.10f) the group $N(D$ in $H)$ leaves invariant the subalgebra $\mathfrak{O}_{1,(D)}(N(D))^I$, and hence permutes its maximal ideals among themselves. Let $N(D$ in $H)_{\mathfrak{N}}$ be the subgroup of all $\pi \in N(D$ in $H)$ fixing $\mathfrak{N}^I$. Evidently $N(D$ in $H)_{\mathfrak{N}}$ leaves invariant the ideal $\mathfrak{O}_{(D)}(N(D))^I\mathfrak{N}^I$, and hence acts naturally as automorphisms of the factor algebra $\mathfrak{B}$. From (10.10a, f) we have:

(10.12a) $N(D) \trianglelefteq N(D$ in $H)_{\mathfrak{N}}$ *and* $N(D)$ *centralizes* $\mathfrak{B}$,

(10.12b) $(\mathfrak{B}_\tau)^\pi = \mathfrak{B}_{\tau^\pi}$, *for all* $\tau \in G$, $\pi \in N(D$ in $H)_{\mathfrak{N}}$.

In view of (10.11c, e) and (10.10e) we can define a subset $G[\mathfrak{N}]$ of G by

$$(10.13) \qquad \begin{aligned} G[\mathfrak{N}] &= \{\tau \in G \mid \mathfrak{B}_\tau \mathfrak{B}_{\tau^{-1}} = \mathfrak{B}_1\} \\ &= \{\tau \in G \mid \mathfrak{O}_{\tau,(D)}(N(D))^I\mathfrak{O}_{\tau^{-1},(D)}(N(D))^I \nsubseteq \mathfrak{N}^I\}. \end{aligned}$$

This subset satisfies:

PROPOSITION 10.14. *The subset $G[\mathfrak{N}]$ is an $N(D \text{ in } H)_{\mathfrak{N}}$-invariant normal subgroup of $H(D)_{\mathfrak{N}}K/K$ (where $H(D)_{\mathfrak{N}} = H(D) \cap N(D \text{ in } H)_{\mathfrak{N}}$). There is a unique central extension $G[\mathfrak{N}]^*$ of $\bar{F}$ by $G[\mathfrak{N}]$ whose twisted group algebra is given by*:

(10.15a) $\quad \bar{\mathfrak{F}}[G[\mathfrak{N}]^*] = \oplus \sum_{\tau \in G[\mathfrak{N}]} \mathfrak{B}_\tau ,$
(10.15b) $\quad \bar{\mathfrak{F}}[G[\mathfrak{N}]^*]_\tau = \mathfrak{B}_\tau$, *for all* $\tau \in G[\mathfrak{N}]$.

Proof. Suppose that $\tau \in G[\mathfrak{N}]$. Then (10.13) gives us elements

$$y \in \mathfrak{O}_{\tau,(D)}(N(D)) \quad \text{and} \quad z \in \mathfrak{O}_{\tau^{-1},(D)}(N(D))$$

such that $y^I z^I \notin \mathfrak{N}^I$. In particular, $y^I \neq 0$. By (10.8b) there is a unique decomposition $y = \sum_{\sigma \in C(D \text{ in } \tau)} y_\sigma$, where $y_\sigma \in \mathfrak{D}_\sigma$ for each $\sigma \in C(D \text{ in } \tau)$. Since D fixes y and also fixes each $\sigma \in C(D \text{ in } \tau)$, it follows from (2.16) that D fixes each y_σ , i.e., that $y_\sigma \in \mathfrak{D}_\sigma(D)$, for all $\sigma \in C(D \text{ in } \tau)$. Obviously there is some $\sigma \in C(D \text{ in } \tau)$ such that $y_\sigma^I \neq 0$. By (10.2d), this σ lies in $H(D)$. So $\tau = \sigma K \in H(D)K/K$, and $G[\mathfrak{N}]$ is a subset of $H(D)K/K$.

Since $C(D \text{ in } \tau) \subseteq N(D \text{ in } H)$ is a coset of $C(D \text{ in } K) \subseteq N(D)$, Proposition 10.9 (a, e) tells us that each $\sigma \in C(D \text{ in } \tau)$ induces the same automorphism of the order $\mathfrak{O}_{1,(D)}(N(D))$. In particular, there is a fixed ideal

$$\mathfrak{N}_1 \in \text{Max } (\mathfrak{O}_{1,(D)}(N(D)))$$

such that $\mathfrak{N}^\sigma = \mathfrak{N}_1$, for all $\sigma \in C(D \text{ in } \tau)$. Equation (2.15) gives

$$y_\sigma \mathfrak{N}_1 = y_\sigma \mathfrak{N}^\sigma = \mathfrak{N} y_\sigma , \quad \textit{for all} \quad \sigma \in C(D \text{ in } \tau).$$

Since $y = \sum_{\sigma \in C(D \text{ in } \tau)} y_\sigma$, this implies that $y\mathfrak{N}_1 = \mathfrak{N}y$. If $\mathfrak{N}_1 \neq \mathfrak{N}$, then there is an element $w \in \mathfrak{N}_1$ such that $w \equiv 1 \pmod{\mathfrak{N}}$. By construction the product yz lies in $\mathfrak{O}_{1,(D)}(N(D)) - \mathfrak{N}$. Hence $yzw \in \mathfrak{N}_1 - \mathfrak{N}$. But $z \in \mathfrak{O}_{(D)}(N(D))$ centralizes $w \in \mathfrak{O}_{1,(D)}(N(D))$ by Proposition 10.9 (b). Hence

$$yzw = ywz \in y\mathfrak{N}_1 z = \mathfrak{N}yz \subseteq \mathfrak{N}\mathfrak{O}_{1,(D)}(N(D)) = \mathfrak{N}.$$

This contradicts the fact that $yzw \notin \mathfrak{N}$. Therefore $\mathfrak{N}_1 = \mathfrak{N} = \mathfrak{N}^\sigma$, for all $\sigma \in C(D \text{ in } \tau)$. Since some such σ can be chosen in $H(D)$ (by the preceding paragraph), we conclude that $G[\mathfrak{N}]$ is a subset of $H(D)_{\mathfrak{N}} K/K$.

If ρ is also an element of $G[\mathfrak{N}]$, then (10.11c) and (10.13) give

$$\mathfrak{B}_1 = \mathfrak{B}_\rho \mathfrak{B}_{\rho^{-1}} = \mathfrak{B}_\rho (\bar{\mathfrak{F}} \cdot 1) \mathfrak{B}_{\rho^{-1}} = \mathfrak{B}_\rho \mathfrak{B}_1 \mathfrak{B}_{\rho^{-1}} = \mathfrak{B}_\rho \mathfrak{B}_\tau \mathfrak{B}_{\tau^{-1}} \mathfrak{B}_{\rho^{-1}} .$$

From (10.11e) we get $\mathfrak{B}_\rho \mathfrak{B}_\tau \subseteq \mathfrak{B}_{\rho\tau}$ and $\mathfrak{B}_{\tau^{-1}} \mathfrak{B}_{\rho^{-1}} \subseteq \mathfrak{B}_{\tau^{-1}\rho^{-1}} = \mathfrak{B}_{(\rho\tau)^{-1}}$. Hence

$$\mathfrak{B}_1 = \mathfrak{B}_\rho \mathfrak{B}_\tau \mathfrak{B}_{\tau^{-1}} \mathfrak{B}_{\rho^{-1}} \subseteq \mathfrak{B}_\rho \mathfrak{B}_\tau \mathfrak{B}_{(\rho\tau)^{-1}} \subseteq \mathfrak{B}_{\rho\tau} \mathfrak{B}_{(\rho\tau)^{-1}} \subseteq \mathfrak{B}_1 .$$

So $\mathfrak{B}_{\rho\tau} \mathfrak{B}_{(\rho\tau)^{-1}} = \mathfrak{B}_1$ and $\rho\tau \in G[\mathfrak{N}]$. Since 1 lies in $G[\mathfrak{N}]$ by (10.11c), we conclude that $G[\mathfrak{N}]$ is a subgroup of the finite group G.

The above argument also shows that $\mathfrak{B}_\rho \mathfrak{B}_\tau \mathfrak{B}_{(\rho\tau)^{-1}} = \mathfrak{B}_1$. Since $(\rho\tau)^{-1}$

also lies in the subgroup $G[\mathfrak{N}]$, we have $\mathfrak{B}_{(\rho\tau)^{-1}}\,\mathfrak{B}_{\rho\tau} = \mathfrak{B}_1$. In view of (10.11c), this implies

$$\mathfrak{B}_\rho\,\mathfrak{B}_\tau = \mathfrak{B}_\rho\,\mathfrak{B}_\tau\,\mathfrak{B}_1 = \mathfrak{B}_\rho\,\mathfrak{B}_\tau\,\mathfrak{B}_{(\rho\tau)^{-1}}\,\mathfrak{B}_{\rho\tau} = \mathfrak{B}_1\,\mathfrak{B}_{\rho\tau} = \mathfrak{B}_{\rho\tau}.$$

This and (10.11a, b, c, d) tell us that $\oplus \sum_{\tau \in G[\mathfrak{N}]} \mathfrak{B}_\tau$, $\{\mathfrak{B}_\tau \mid \tau \in G[\mathfrak{N}]\}$ is a graded Clifford system over $\bar{\mathfrak{F}}$ satisfying [CCT, 14.1]. Then [CCT, §14] gives us the unique central extension $G[\mathfrak{N}]^*$ of $\bar{F}$ by $G[\mathfrak{N}]$ satisfying (10.15).

It is clear from (10.12b) and (10.13) that the subgroup $G[\mathfrak{N}]$ is $N(D \text{ in } H)_{\mathfrak{N}}$-invariant. Since $H(D)_{\mathfrak{N}}$ is a subgroup of $C(D \text{ in } H)_{\mathfrak{N}} \leq N(D \text{ in } H)_{\mathfrak{N}}$ (by Proposition 9.2), we conclude that $G[\mathfrak{N}]$ is a normal subgroup of $H(D)_{\mathfrak{N}}K/K$. So the proposition is proved.

Corollary 10.16. *The action of $N(D \text{ in } H)_{\mathfrak{N}}$ on $\mathfrak{B}$ leaves invariant the subalgebra $\bar{\mathfrak{F}}[G[\mathfrak{N}]^*]$. Hence $N(D \text{ in } H)_{\mathfrak{N}}$ acts as automorphisms of $G[\mathfrak{N}]^*$ so that the extension maps* pr : $G[\mathfrak{N}]^* \to G[\mathfrak{N}]$ *and* in : $\bar{F} \to G[\mathfrak{N}]^*$ *satisfy:*

(10.17a) $\mathrm{pr}\,(\rho^\pi) = \mathrm{pr}\,(\rho)^\pi \in G[\mathfrak{N}]$, *for all* $\rho \in G[\mathfrak{N}]^*$, $\pi \in N(D \text{ in } H)_{\mathfrak{N}}$,
(10.17b) $N(D \text{ in } H)_{\mathfrak{N}}$ *centralizes* $\mathrm{Ker}\,(\mathrm{pr}) = \mathrm{in}\,(\bar{F})$.

Furthermore, the normal subgroup $N(D)$ of $N(D \text{ in } H)_{\mathfrak{N}}$ centralizes $G[\mathfrak{N}]^*$.

Proof. This is an immediate consequence of the proposition and (10.12).

Propositions 8.1 and (10.4) tell us that the Brauer homomorphism S is an identity-preserving $\bar{\mathfrak{F}}$-homomorphism of the algebra $\mathfrak{C}^I = \mathfrak{D}(K)^I$ into

$$\mathfrak{O}_{(D)}(N(D))^I = \mathfrak{D}_{(D)}(N(D))^I.$$

Composing it with the natural maps of $\mathfrak{C}$ onto $\mathfrak{C}^I$ and $\mathfrak{O}_{(D)}(N(D))^I$ onto

$$\mathfrak{B} = \mathfrak{O}_{(D)}(N(D))^I/\mathfrak{O}_{(D)}(N(D))^I\mathfrak{N}^I,$$

we obtain an identity-preserving $\mathfrak{R}$-homomorphism S' of the order $\mathfrak{C}$ into the $\bar{\mathfrak{F}}$-algebra $\mathfrak{B}$. Equation (10.7) implies that

$$S(\mathfrak{M}^I) \subseteq \mathfrak{N}^I \subseteq \mathfrak{O}_{(D)}(N(D))^I\mathfrak{N}^I.$$

Hence $S'(\mathfrak{M}) = 0$. In view of (10.6), this says that $S'(1-e) = 0$ and $S'(J(e\mathfrak{C}_1)) = 0$. So $S'(e) = S'(1) = 1$, and the restriction of S' is an identity-preserving $\mathfrak{R}$-homomorphism of the suborder $\mathfrak{C}[B] \subseteq e\mathfrak{C}$ into $\mathfrak{B}$. Furthermore,

$$S'(\mathfrak{C}[B]J(\mathfrak{C}[B]_1)) = S'(\mathfrak{C}[B])S'(J(e\mathfrak{C}_1)) = \{0\}.$$

By (2.14a), this implies that S' induces a unique identity-preserving $\bar{\mathfrak{F}}$-homomorphism $\bar{S}$ of the algebra $\bar{\mathfrak{F}}[G[B]^*]$ into $\mathfrak{B}$ so that the following diagram commutes:

$$(10.18)\qquad \begin{array}{ccccccc} \mathfrak{C}[B] & \longrightarrow & \mathfrak{C} & \longrightarrow & \mathfrak{C}^I & \xrightarrow{S} & \mathfrak{O}_{(D)}(N(D))^I \\ \big\downarrow & & & & & & \big\downarrow \\ \bar{\mathfrak{F}}[G[B]^*] & & & \xrightarrow{\quad\bar{S}\quad} & & & \mathfrak{B}. \end{array}$$

Here all the unlabelled maps are either inclusions or natural projections.

The definition of S in Proposition 8.1 tells us that it carries $\mathfrak{C}_\tau^I \subseteq \bar{\mathfrak{F}}[H^*]_\tau$ into $\mathfrak{O}_{\tau,(D)}(N(D))^I = \mathfrak{O}_{(D)}(N(D))^I \cap \bar{\mathfrak{F}}[H^*]_\tau$, for any $\tau \in G$. It follows that S' sends $\mathfrak{C}[B]_\tau \subseteq \mathfrak{C}_\tau$ into $\mathfrak{B}_\tau$, for all $\tau \in G[B]$. In view of (2.14b), this implies that

$$\bar{S}(\bar{\mathfrak{F}}[G[B]^*]_\tau) \subseteq \mathfrak{B}_\tau, \quad \textit{for all} \quad \tau \in G[B]. \tag{10.19}$$

Now we can give the first half of the analysis of $G[B]^*$.

THEOREM 10.20. *The homomorphism $\bar{S}$ sends the algebra $\bar{\mathfrak{F}}[G[B]^*]$ isomorphically onto $\bar{\mathfrak{F}}[G[\mathfrak{N}]^*]$. Hence $G[B] = G[\mathfrak{N}]$, and the restriction of $\bar{S}$ is an isomorphism of $G[B]^*$ onto $G[\mathfrak{N}]^*$ as extensions of $\bar{F}$ by $G[B] = G[\mathfrak{N}]$. Furthermore, G_B is equal to $N(D \text{ in } H)_{\mathfrak{N}} K/K$, and the isomorphism $\bar{S}$ preserves the actions of $N(D \text{ in } H)_{\mathfrak{N}}/N(D) \simeq G_B$ in the sense that*

$$\bar{S}(y^{\pi K}) = \bar{S}(y)^\pi, \quad \textit{for all} \quad y \in \bar{\mathfrak{F}}[G[B]^*], \quad \pi \in N(D \,\mathrm{in}\, H)_{\mathfrak{N}}. \tag{10.21}$$

Proof. Since our operator groups D, $N(D)$, and K are all subgroups of the normal subgroup K of H, the ideals and submodules corresponding to those of §8 are all graded with respect to $G = H/K$. Thus, in addition to (10.10d) and Proposition 10.9(c), we easily verify that:

$$\begin{aligned}
&(10.22\text{a}) \quad \mathfrak{O}_{(D)}(N(D) \mid D)^I = \oplus \textstyle\sum_{\tau\in G} \mathfrak{O}_{\tau,(D)}(N(D) \mid D)^I,\\
&(10.22\text{ b}) \quad \mathfrak{O}_{(D)}(N(D) \parallel D)^I = \oplus \textstyle\sum_{\tau\in G} \mathfrak{O}_{\tau,(D)}(N(D) \parallel D)^I,\\
&(10.22\text{ c}) \quad \mathfrak{O}(K)^I = \oplus \textstyle\sum_{\tau\in G} \mathfrak{O}_\tau(K)^I,\\
&(10.22\text{ d}) \quad \mathfrak{O}(K \parallel D)^I = \oplus \textstyle\sum_{\tau\in G} \mathfrak{O}_\tau(K \parallel D)^I,
\end{aligned}$$

where, in each case, the τ^{th} term on the right is the intersection of the left side with $\mathfrak{O}_\tau^I$.

We shall use these gradings to show that $G[\mathfrak{N}]$ is a subgroup of $G[B]$. Let τ be any element of $G[\mathfrak{N}]$. Then (10.10e) and (10.13) imply that $\mathfrak{O}_{\tau,(D)}(N(D))^I\mathfrak{O}_{\tau^{-1},(D)}(N(D))^I$ is a two-sided ideal of $\mathfrak{O}_{1,(D)}(N(D))^I$ which is not contained in the maximal two-sided ideal $\mathfrak{N}^I$. Hence

$$\mathfrak{N}^I + \mathfrak{O}_{\tau,(D)}(N(D))^I\mathfrak{O}_{\tau^{-1},(D)}(N(D))^I = \mathfrak{O}_{1,(D)}(N(D))^I.$$

Since D is a defect group of $\mathfrak{N}$, we also have

$$\mathfrak{N}^I + \mathfrak{O}_{1,(D)}(N(D) \mid D)^I = \mathfrak{O}_{1,(D)}(N(D))^I.$$

Multiplying the former equation on the right and left by the latter, we obtain

$$\begin{aligned}\mathfrak{N}^I + \mathfrak{O}_{1,(D)}(N(D) \mid D)^I\mathfrak{O}_{\tau,(D)}(N(D))^I\mathfrak{O}_{\tau^{-1},(D)}(N(D))^I\mathfrak{O}_{1,(D)}(N(D) \mid D)^I&\\ = \mathfrak{O}_{1,(D)}(N(D))^I&.\end{aligned}$$

Because $\mathfrak{O}_{(D)}(N(D) \mid D)^I$ is a two-sided ideal of $\mathfrak{O}_{(D)}(N(D))^I$, the inclusions (10.10e) and the decomposition (10.22a) give

$$\mathfrak{O}_{1,(D)}(N(D) \mid D)^I\mathfrak{O}_{\tau,(D)}(N(D))^I \subseteq \mathfrak{O}_{\tau,(D)}(N(D) \mid D)^I,$$

$$\mathfrak{O}_{\tau^{-1},(D)}(N(D))^I\mathfrak{O}_{1,(D)}(N(D) \mid D)^I \subseteq \mathfrak{O}_{\tau^{-1},(D)}(N(D) \mid D)^I.$$

This and the preceding equation imply

$$\mathfrak{N}^I + \mathfrak{O}_{\tau,(D)}(N(D) \mid D)^I \mathfrak{O}_{\tau^{-1},(D)}(N(D) \mid D)^I = \mathfrak{O}_{1,(D)}(N(D))^I.$$

From the definitions (10.8a) of $\mathfrak{O}_{(D)} = \mathfrak{D}_{(D)}$ and (8.2) of $\mathfrak{O}_{(D)}(N(D) \parallel C)^I$ it is clear that $\mathfrak{O}_{(D)}(N(D) \parallel C)^I = 0$, for all subgroups $C < D$. Therefore

$$\mathfrak{O}_{(D)}(N(D) \mid D)^I = \mathfrak{O}_{(D)}(N(D) \parallel D)^I,$$

by (8.4). In view of the decompositions (10.22a, b), the preceding equation now becomes

$$\mathfrak{N}^I + \mathfrak{O}_{\tau,(D)}(N(D) \parallel D)^I \mathfrak{O}_{\tau^{-1},(D)}(N(D) \parallel D)^I = \mathfrak{O}_{1,(D)}(N(D))^I.$$

By Lemma 8.5, the homomorphism S sends $\mathfrak{O}(K \parallel D)^I$ one-to-one onto $\mathfrak{O}_{(D)}(N(D) \parallel D)^I$. Since it also carries $\mathfrak{O}_\tau(K)^I$ into $\mathfrak{O}_{\tau,(D)}(N(D))^I$, it must, in view of (10.22b, d), send $\mathfrak{O}_\tau(K \parallel D)^I$ onto $\mathfrak{O}_{\tau,(D)}(N(D) \parallel D)^I$. Similarly, it sends $\mathfrak{O}_{\tau^{-1}}(K \parallel D)^I$ onto $\mathfrak{O}_{\tau^{-1},(D)}(N(D) \parallel D)^I$. Therefore the preceding equation and (10.7) imply that

$$\mathfrak{M}^I + \mathfrak{O}_\tau(K \parallel D)^I \mathfrak{O}_{\tau^{-1}}(K \parallel D)^I = \mathfrak{O}_1(K)^I.$$

Since $\mathfrak{O}_\tau(K \parallel D)^I \subseteq \mathfrak{O}_\tau(K)^I$ and $\mathfrak{O}_{\tau^{-1}}(K \parallel D)^I \subseteq \mathfrak{O}_{\tau^{-1}}(K)^I$, this gives

$$\mathfrak{M}^I + \mathfrak{O}_\tau(K)^I \mathfrak{O}_{\tau^{-1}}(K)^I = \mathfrak{O}_1(K)^I.$$

Taking inverse images in $\mathfrak{O}_1(K) = \mathfrak{C}_1$, $\mathfrak{O}_\tau(K) = \mathfrak{C}_\tau$ and $\mathfrak{O}_{\tau^{-1}}(K) = \mathfrak{C}_{\tau^{-1}}$, we obtain

$$\mathfrak{M} + \mathfrak{C}_\tau \, \mathfrak{C}_{\tau^{-1}} = \mathfrak{C}_1 .$$

Multiplying this by the central idempotent e of $\mathfrak{C}$ and using (10.6), we get

$$J(e\mathfrak{C}_1) + (e\mathfrak{C}_\tau)(e\mathfrak{C}_{\tau^{-1}}) = e\mathfrak{C}_1 .$$

But $(e\mathfrak{C}_\tau)(e\mathfrak{C}_{\tau^{-1}})$ is a two-sided ideal of $e\mathfrak{C}_1$ by (2.8e). Hence

$$(e\mathfrak{C}_\tau)(e\mathfrak{C}_{\tau^{-1}}) = e\mathfrak{C}_1$$

and $\tau \in G[B]$ by (2.9a). Therefore $G[\mathfrak{N}]$ is a subgroup of $G[B]$.

By (10.19), the homomorphism $\bar{S}$ carries $\mathfrak{F}[G[B]^*]_1 = \mathfrak{F} \cdot 1$ into $\mathfrak{B}_1$. Since $\bar{S}$ is identity-preserving and $\mathfrak{F}$-linear, this and (10.11c) imply that it induces an isomorphism of $\mathfrak{F}[G[B]^*]_1$ onto $\mathfrak{B}_1$. If τ is now any element of $G[B]$, then (10.11e), (10.19), and the equation

$$\mathfrak{F}[G[B]^*]_1 = \mathfrak{F}[G[B]^*]_\tau \mathfrak{F}[G[B]^*]_{\tau^{-1}}$$

yield

$$\mathfrak{B}_1 = \bar{S}(\mathfrak{F}[G[B]^*]_1) = \bar{S}(\mathfrak{F}[G[B]^*]_\tau)\bar{S}(\mathfrak{F}[G[B]^*]_{\tau^{-1}}) \subseteq \mathfrak{B}_\tau \mathfrak{B}_{\tau^{-1}} \subseteq \mathfrak{B}_1 .$$

Therefore $\mathfrak{B}_\tau \, \mathfrak{B}_{\tau^{-1}} = \mathfrak{B}_1$ and $\tau \in G[\mathfrak{N}]$, by (10.13). Hence $G[B] = G[\mathfrak{N}]$. In view of (10.15) and (10.19), this implies that $\bar{S}$ is an $\mathfrak{F}$-homomorphism of $\mathfrak{F}[G[B]^*]$ into $\mathfrak{F}[G[\mathfrak{N}]^*]$ carrying $\mathfrak{F}[G[B]^*]_\tau$ into $\mathfrak{F}[G[\mathfrak{N}]^*]_\tau$, for all $\tau \in G[B] = G[\mathfrak{N}]$ and sending $\mathfrak{F}[G[B]^*]_1$ isomorphically onto $\mathfrak{F}[G[\mathfrak{N}]^*]_1$. The first two statements of the theorem follow directly from this and [CCT, 13.10].

The inverse image H_B of G_B in H fixes B and K, and hence permutes among themselves the defect groups of B in K. But these groups are just the K-conjugates of D. We conclude that $H_B = N(D \text{ in } H)_B \,.\, K$, where $N(D \text{ in } H)_B$ is the subgroup of $N(D \text{ in } H)$ fixing B. Evidently $N(D \text{ in } H)_B$ is also the subgroup of $N(D \text{ in } H)$ fixing the maximal ideal $\mathfrak{M}$ lying in B. From the definition of S in Proposition 8.1, it is clear that

$$(10.23) \qquad S(y^{\pi}) = S(y)^{\pi}, \quad \textit{for all} \quad y \in \mathfrak{O}(K)^{I} = \mathfrak{C}^{I} \quad \textit{and} \quad \pi \in N(D \text{ in } H).$$

In view of the unicity of the relation between $\mathfrak{M}$ and $\mathfrak{N}$ in Brauer's First Main Theorem 8.7, this implies that $N(D \text{ in } H)_B$ is also the subgroup $N(D \text{ in } H)_{\mathfrak{N}}$ of $N(D \text{ in } H)$ fixing $\mathfrak{N}$. So $G_B = H_B/K = N(D \text{ in } H)_{\mathfrak{N}}\, K/K$.

Finally, (10.21) follows directly from (10.23) and the definition of $\bar{S}$. So the theorem is proved.

11. Second half of the analysis of $G[B]^*$

We continue to use the notation and hypotheses of the preceding section.

Conditions (5.1) and (6.1) are clearly satisfied with K, $\mathfrak{O}_1$, K, and the $\mathfrak{D}_\sigma$, $\sigma \in K$, in place of E, $\mathfrak{C}$, G, and the $\mathfrak{C}_\sigma$, $\sigma \in G$, respectively. So (9.4) holds (see the remarks immediately after it) and Theorem 9.5 gives us a unique $N(D)$-orbit in Max $(\mathfrak{O}_{1,(D)}(D)^I)$ corresponding to

$$\mathfrak{N} \in \text{Max}\,(\mathfrak{O}_{1,(D)}(N(D)) \mid D).$$

Fix an ideal $\mathfrak{L}$ in this $N(D)$-orbit. Then (9.6) becomes

$$(11.1) \qquad [\mathfrak{O}_{1,(D)}(D)^I/\mathfrak{L}](N(D)_{\mathfrak{L}} \mid D) \neq \{0\}.$$

By Proposition 9.2, the image $\mathfrak{O}_{(D)}(D)^I$ is the twisted group algebra

$$(11.2a) \qquad \mathfrak{O}_{(D)}(D)^I = \bar{\mathfrak{F}}[H(D)^*] = \oplus \textstyle\sum_{\sigma \in H(D)} \bar{\mathfrak{F}}[H^*]_\sigma\,,$$

of the inverse image $H(D)^*$ in H^* of the subgroup $H(D)$ of $C(D \text{ in } H)$. It follows that

$$(11.2b) \qquad \mathfrak{O}_{\tau,(D)}(D)^I = \bar{\mathfrak{F}}[H(D)^*]_{\tau \cap H(D)} = \oplus \textstyle\sum_{\sigma \in \tau \cap H(D)} \bar{\mathfrak{F}}[H^*]_\sigma\,, \quad \textit{for all cosets} \quad \tau \in G = H/K.$$

Notice here that $\tau \cap H(D)$ is either empty (in which case $\bar{\mathfrak{F}}[H(D)^*]_{\tau \cap H(D)} = \{0\}$) or else is a coset of the normal subgroup $K(D) = K \cap H(D)$ in $H(D)$. Furthermore, $\tau \to \tau \cap H(D)$ is the natural isomorphism of $H(D)K/K$ onto $H(D)/K(D)$.

Equation (11.2b) tells us in particular that $\mathfrak{O}_{1,(D)}(D)^I$ is the twisted group algebra $\bar{\mathfrak{F}}[K(D)^*]$ of the inverse image $K(D)^*$ of $K(D)$ in H^*. So $\mathfrak{L} \in \text{Max}\,(\mathfrak{O}_{1,(D)}(D)^I)$ corresponds to a unique irreducible $\bar{\mathfrak{F}}$-character φ of the algebra $\bar{\mathfrak{F}}[K(D)^*]$.

We recall from [CCT] the construction of the Clifford extension $H(D)^*\langle\varphi\rangle$. Since $K(D)$ is a normal subgroup of $H(D)$, the natural conjugation action [CCT, 15.4] of $H(D)$ on $\bar{\mathfrak{F}}[H(D)^*]$ leaves invariant the subalgebra $\bar{\mathfrak{F}}[K(D)^*]$.

So $H(D)$ permutes among themselves the ideals in Max $(\bar{\mathfrak{F}}[K(D)^*])$. The intersection $\bigcap_{\sigma\in H(D)}\mathfrak{L}^\sigma$ is an $H(D)$-invariant two-sided ideal of $\bar{\mathfrak{F}}[K(D)^*]$. It follows (see [CCT, 2.4] and [CCT, 13.2]) that

$$\bar{\mathfrak{F}}[H(D)^*](\textstyle\bigcap_{\sigma\in H(D)}\mathfrak{L}^\sigma) = (\bigcap_{\sigma\in H(D)}\mathfrak{L}^\sigma)\bar{\mathfrak{F}}[H(D)^*]$$

is a graded two-sided ideal of $\bar{\mathfrak{F}}[H(D)^*]$, with:

$$\begin{aligned} &(11.3a) \quad \bar{\mathfrak{F}}[H(D)^*](\textstyle\bigcap_{\sigma\in H(D)}\mathfrak{L}^\sigma) = \oplus \sum_{\rho\in H(D)/K(D)}\bar{\mathfrak{F}}[H(D)^*]_\rho(\bigcap_{\sigma\in H(D)}\mathfrak{L}^\sigma),\\ &(11.3b) \quad \bar{\mathfrak{F}}[H(D)^*]_\rho(\textstyle\bigcap_{\sigma\in H(D)}\mathfrak{L}^\sigma) = (\bigcap_{\sigma\in H(D)}\mathfrak{L}^\sigma)\bar{\mathfrak{F}}[H(D)^*]_\rho \subseteq \bar{\mathfrak{F}}[H(D)^*]_\rho,\end{aligned}$$

for all $\rho \in H(D)/K(D)$.

As usual, we identify each factor space $\bar{\mathfrak{F}}[H(D)^*]_\rho/\bar{\mathfrak{F}}[H(D)^*]_\rho(\bigcap_{\sigma\in H(D)}\mathfrak{L}^\sigma)$ with its image in the factor ring, setting:

$$\begin{aligned} &(11.4a) \quad \mathfrak{A} = \bar{\mathfrak{F}}[H(D)^*]/\bar{\mathfrak{F}}[H(D)^*](\textstyle\bigcap_{\sigma\in H(D)}\mathfrak{L}^\sigma),\\ &(11.4b) \quad \mathfrak{A}_\rho = \bar{\mathfrak{F}}[H(D)^*]_\rho/\bar{\mathfrak{F}}[H(D)^*]_\rho(\textstyle\bigcap_{\sigma\in H(D)}\mathfrak{L}^\sigma), \textit{ for all } \rho\in H(D)/K(D).\end{aligned}$$

Then, by [CCT, 13.3], conditions (2.1) are satisfied with $\bar{\mathfrak{F}}$, $\mathfrak{A}$, $H(D)/K(D)$, and the $\mathfrak{A}_\rho$ in place of $\mathfrak{R}$, $\mathfrak{O}$, G, and the $\mathfrak{O}_\sigma$, respectively.

The subalgebra $\mathfrak{A}_1 = \bar{\mathfrak{F}}[K(D)^*]/(\bigcap_{\sigma\in H(D)}\mathfrak{L}^\sigma)$ is finite-dimensional and semisimple. Let d be the unique primitive central idempotent of $\mathfrak{A}_1$ corresponding to its maximal two-sided ideal $\mathfrak{L}/(\bigcap_{\sigma\in H(D)}\mathfrak{L}^\sigma)$, so that

$$(11.5) \qquad (1-d)\mathfrak{A}_1 = \mathfrak{A}_1(1-d) = \mathfrak{L}/(\textstyle\bigcap_{\sigma\in H(D)}\mathfrak{L}^\sigma).$$

We denote by $H(D)_\varphi$ the subgroup of $H(D)$ fixing $\mathfrak{L}$ (or, equivalently, φ or d) under conjugation. Then, by [CCT, 13.9], conditions (2.1) are also satisfied with $\bar{\mathfrak{F}}$, $d\mathfrak{A}d$, $H(D)_\varphi/K(D)$, and the $d\mathfrak{A}_\rho = \mathfrak{A}_\rho d$, for $\rho\in H(D)_\varphi/K(D)$, in place of $\mathfrak{R}$, $\mathfrak{O}$, G, and the $\mathfrak{O}_\sigma$, $\sigma\in G$, respectively. Finally, by [CCT, 8.7 and 15.14], $H(D)^*\langle\varphi\rangle$ is the unique central extension of $\bar{F}$ by $H(D)_\varphi/K(D)$ whose twisted group algebra is given by:

$$\begin{aligned} &(11.6a) \quad \bar{\mathfrak{F}}[H(D)^*\langle\varphi\rangle] = C(d\mathfrak{A}_1 \text{ in } d\mathfrak{A}d),\\ &(11.6b) \quad \bar{\mathfrak{F}}[H(D)^*\langle\varphi\rangle]_\rho = C(d\mathfrak{A}_1 \text{ in } d\mathfrak{A}_\rho), \textit{ for all } \rho\in H(D)_\varphi/K(D).\end{aligned}$$

From the definition (10.2d) of $H(D)$ it is clear that it is a normal subgroup of $N(D \text{ in } H)$. Hence so is its intersection $K(D)$ with K. It follows that $N(D \text{ in } H)$ acts by conjugation on $H(D)/K(D)$, and that the natural conjugation action of $N(D \text{ in } H)$ on $\bar{\mathfrak{F}}[H^*]$ leaves invariant the subalgebra $\bar{\mathfrak{F}}[H(D)^*]$ and satisfies

$$(11.7) \qquad (\bar{\mathfrak{F}}[H(D)^*]_\rho)^\pi = \bar{\mathfrak{F}}[H(D)^*]_{\rho^\pi}, \quad \textit{for all } \rho\in H(D)/K(D),\ \pi\in N(D \text{ in } H).$$

In particular, $N(D \text{ in } H)$ leaves $\bar{\mathfrak{F}}[K(D)^*]$ invariant. So it permutes among themselves the members of Max $(\bar{\mathfrak{F}}[K(D)^*])$. Let $N(D \text{ in } H)_\varphi$ be the subgroup of $N(D \text{ in } H)$ fixing $\mathfrak{L}\in$ Max $(\bar{\mathfrak{F}}[K(D)^*])$ (or, equivalently, fixing φ). Since $N(D \text{ in } H)_\varphi$ leaves invariant both $H(D)$ and $\mathfrak{L}$, it follows from (11.4) and (11.7) that there is an induced action of $N(D \text{ in } H)_\varphi$ as $\bar{\mathfrak{F}}$-automorphisms of the factor algebra $\mathfrak{A}$ satisfying

$$(11.8) \qquad (\mathfrak{A}_\rho)^\pi = \mathfrak{A}_{\rho^\pi}, \quad \textit{for all } \rho \in H(D)/K(D),\ \pi \in N(D \text{ in } H)_\varphi .$$

Because $N(D \text{ in } H)_\varphi$ fixes $\mathfrak{L}$, it fixes the corresponding primitive central idempotent d of $\mathfrak{A}_1$. From this, (11.8), and (11.6) we conclude that $\bar{\mathfrak{F}}[H(D)^*\langle\varphi\rangle]$ is a $N(D \text{ in } H)_\varphi$-invariant subalgebra of $\mathfrak{A}$, that $H(D)_\varphi/K(D)$ is a $N(D \text{ in } H)_\varphi$-invariant subgroup of $H(D)/K(D)$, and that

$$(11.9) \qquad \begin{aligned}(\bar{\mathfrak{F}}[H(D)^*\langle\varphi\rangle]_\rho)^\pi &= \bar{\mathfrak{F}}[H(D)^*\langle\varphi\rangle]_{\rho^\pi}, \\ &\textit{for all} \quad \rho \in H(D)_\varphi/K(D),\ \pi \in N(D \text{ in } H)_\varphi .\end{aligned}$$

Evidently the subgroup $N(D)_\varphi = N(D \text{ in } H)_\varphi \cap K$ centralizes $H(D)_\varphi/K(D)$. Because each $\bar{\mathfrak{F}}[H(D)^*\langle\varphi\rangle]_\rho$ is one dimensional over $\bar{\mathfrak{F}}$, this and (11.9) imply the existence of a unique map

$$\omega : N(D)_\varphi \times [H(D)_\varphi/K(D)] \to \bar{F}$$

satisfying

$$(11.10) \qquad \begin{aligned} y^\pi &= \omega(\pi, \rho)y, \\ &\textit{for all} \quad \pi \in N(D)_\varphi,\ \rho \in H(D)_\varphi/K(D),\ y \in \bar{\mathfrak{F}}[H(D)^*\langle\varphi\rangle]_\rho .\end{aligned}$$

Since $N(D)_\varphi$ acts as algebra automorphisms of $\bar{\mathfrak{F}}[H(D)^*\langle\varphi\rangle]$, this map is bilinear in the sense that:

$$(11.11a) \qquad \begin{aligned}\omega(\pi_1\pi_2, \rho) &= \omega(\pi_1, \rho)\omega(\pi_2, \rho), \\ &\textit{for all} \quad \pi_1, \pi_2 \in N(D)_\varphi,\ \rho \in H(D)_\varphi/K(D),\end{aligned}$$

$$(11.11b) \qquad \begin{aligned}\omega(\pi, \rho_1\rho_2) &= \omega(\pi, \rho_1)\omega(\pi, \rho_2), \\ &\textit{for all} \quad \pi \in N(D)_\varphi,\ \rho_1, \rho_2 \in H(D)_\varphi/K(D).\end{aligned}$$

So we can define normal subgroups $N(D)_\omega$ of $N(D)_\varphi$ and $H(D)_\omega$ of $H(D)_\varphi$ by:

$$(11.12a) \qquad N(D)_\omega = \{\pi \in N(D)_\varphi \mid \omega(\pi, \rho) = 1, \textit{ for all } \rho \in H(D)_\varphi/K(D)\},$$
$$(11.12b) \qquad H(D)_\omega = \{\rho \in H(D)_\varphi \mid \omega(\pi, \rho K(D)) = 1, \textit{ for all } \pi \in N(D)_\varphi\}.$$

Because $\bar{F}$ is the multiplicative group of a field $\bar{\mathfrak{F}}$ of characteristic p, we have:

(11.13) *Both $N(D)_\varphi/N(D)_\omega$ and $H(D)_\varphi/H(D)_\omega$ are abelian p'-groups. The map ω induces a non-singular bilinear pairing of these groups into the cyclic subgroup of $|N(D)_\varphi|^{\text{th}}$ roots of unity in $\bar{F}$. Hence they are naturally dual to each other.*

We denote by $H(D)^*\langle\varphi\rangle_\omega$ the inverse image of $H(D)_\omega/K(D)$ in $H(D)^*\langle\varphi\rangle$. Then (11.10) implies that the twisted group algebra of the extension $H(D)^*\langle\varphi\rangle_\omega$ satisfies

$$(11.14) \qquad \begin{aligned}\bar{\mathfrak{F}}[H(D)^*\langle\varphi\rangle_\omega] &= \oplus \textstyle\sum_{\rho \in H(D)_\omega/K(D)} \bar{\mathfrak{F}}[H(D)^*\langle\varphi\rangle]_\rho \\ &= \bar{\mathfrak{F}}[H(D)^*\langle\varphi\rangle](N(D)_\varphi).\end{aligned}$$

Any element $\sigma \in K(D)$ acts on $\mathfrak{F}[H^*]$ as conjugation by any non-zero element y_σ of $\mathfrak{F}[H^*]_\sigma$. Hence σ acts on $\mathfrak{A}$ as conjugation by the image $\bar{y}_\sigma$ of y_σ in $\mathfrak{A}_1$. By (11.2b) and (11.4b) these images $\bar{y}_\sigma$, for $\sigma \in K(D)$, span $\mathfrak{A}_1$. It follows that

$$(11.15) \qquad \mathfrak{A}(K(D)) = C(\mathfrak{A}_1 \text{ in } \mathfrak{A}).$$

Since d is a central idempotent of $\mathfrak{A}_1$, this and (11.6a) imply that

$$\mathfrak{F}[H(D)^*\langle\varphi\rangle] = dC(\mathfrak{A}_1 \text{ in } \mathfrak{A})d = d\mathfrak{A}d(K(D)).$$

Combined with (11.6) and (11.14), this gives:

$$(11.16\text{a}) \quad \mathfrak{F}[H(D)^*\langle\varphi\rangle_\omega] = d\mathfrak{A}d(N(D)_\varphi),$$
$$(11.16\text{b}) \quad \mathfrak{F}[H(D)^*\langle\varphi\rangle]_\rho = d\mathfrak{A}_\rho(N(D)_\varphi), \textit{ for all } \rho \in H(D)_\omega/K(D).$$

We have just seen that the normal subgroup $K(D)$ of $N(D \text{ in } H)_\varphi$ centralizes $\mathfrak{F}[H(D)^*\langle\varphi\rangle]$. Since D centralizes $\mathfrak{O}_{(D)}(D)^I = \mathfrak{F}[H(D)^*]$, it is also a normal subgroup of $N(D \text{ in } H)_\varphi$ centralizing $\mathfrak{F}[H(D)^*\langle\varphi\rangle]$. By (11.10), this implies that

$$(11.17) \qquad K(D)D \trianglelefteq N(D \text{ in } H) \quad \textit{and} \quad K(D)D \trianglelefteq N(D)_\omega.$$

The natural projection $y \to \bar{y}$ of $\mathfrak{O}_{(D)}(D)^I = \mathfrak{F}[H(D)^*]$ onto its factor algebra $\mathfrak{A}$ clearly sends the subalgebra $\mathfrak{F}[H(D)^*](N(D)) \subseteq \mathfrak{F}[H(D)^*](N(D)_\varphi)$ into $\mathfrak{A}(N(D)_\varphi)$. In view of (11.15), the $N(D)_\varphi$-invariant central idempotent d of $\mathfrak{A}_1$ centralizes $\mathfrak{A}(N(D)_\varphi) \subseteq \mathfrak{A}(K(D))$. It follows that the map $y \to dy = yd$ is an algebra homomorphism of $\mathfrak{A}(N(D)_\varphi)$ into $d\mathfrak{A}d(N(D)_\varphi) = \mathfrak{F}[H(D)^*\langle\varphi\rangle_\omega]$. Hence

$$T : y \to d\bar{y} = \bar{y}d$$

is an identity-preserving homomorphism of the algebra $\mathfrak{F}[H(D)^*](N(D))$ into $\mathfrak{F}[H(D)^*\langle\varphi\rangle_\omega]$.

Theorem 9.5 tells us that

$$\mathfrak{N}^I = \mathfrak{L} \cap \mathfrak{O}_{1,(D)}(N(D))^I \subseteq \mathfrak{L} \cap \mathfrak{F}[H(D)^*]_{K(D)}(N(D)).$$

Hence the image of $\mathfrak{N}^I$ in $\mathfrak{A}_1$ is contained in that of $\mathfrak{L}$, which is $(1 - d)\mathfrak{A}_1$ by (11.5). Since

$$d(1 - d) = d - d^2 = 0,$$

we conclude that $T(\mathfrak{N}^I) = \{0\}$. So the restriction of T is a homomorphism of the subalgebra $\mathfrak{O}_{(D)}(N(D))^I \subseteq \mathfrak{F}[H(D)^*](N(D))$ into $\mathfrak{F}[H(D)^*\langle\varphi\rangle_\omega]$ sending the two-sided ideal $\mathfrak{O}_{(D)}(N(D))^I\mathfrak{N}^I$ into zero. Hence it induces a unique identity-preserving homomorphism $\bar{T}$ of the factor algebra

$$\mathfrak{B} = \mathfrak{O}_{(D)}(N(D))^I/\mathfrak{O}_{(D)}(N(D))^I\mathfrak{N}^I$$

into

$$\mathfrak{F}[H(D)^*\langle\varphi\rangle_\omega]$$

so that the following diagram commutes:

$$
(11.18)\qquad
\begin{tikzcd}
\mathfrak{O}_{(D)}(N(D))^I \arrow[r] \arrow[d] & \bar{\mathfrak{F}}[H(D)^*](N(D)) \arrow[r] \arrow[dr, "T"] & \mathfrak{A}(N(D)_\phi) \arrow[d, "y \to yd = dy"] \\
\mathfrak{B} \arrow[rr, "\bar{T}"] & & \bar{\mathfrak{F}}[H(D)^*\langle\varphi\rangle_\omega].
\end{tikzcd}
$$

Here the unlabelled maps are either inclusions or (restrictions of) natural projections.

In view of (11.2b) and (11.4b), the natural projection sends $\mathfrak{O}_{\tau,(D)}(N(D))^I$ into $\mathfrak{A}_{\tau\cap H(D)}(N(D)_\varphi)$, for all $\tau \in G = H/K$. Because d lies in $\mathfrak{A}_1$, multiplication by it sends the image of $\mathfrak{O}_{\tau,(D)}(N(D))^I$ into

$$\bar{\mathfrak{F}}[H(D)^*\langle\varphi\rangle_\omega] \cap \mathfrak{A}_{\tau\cap H(D)}\,.$$

By (11.6) and (11.14), this intersection is $\bar{\mathfrak{F}}[H(D)^*\langle\varphi\rangle_\omega]_{\tau\cap H(D)}$ if $\tau \cap H(D) \subseteq H(D)_\omega$ (i.e., if $\tau \in H(D)_\omega K/K$), and $\{0\}$ otherwise. Since $\mathfrak{B}_\tau$ is the image of $\mathfrak{O}_{\tau,(D)}(N(D))^I$ in $\mathfrak{B}$, we conclude that

$$
(11.19)\qquad
\begin{aligned}
\bar{T}(\mathfrak{B}_\tau) &\subseteq \bar{\mathfrak{F}}[H(D)^*\langle\varphi\rangle]_{\tau\cap H(D)}\,, && \textit{if} \quad \tau \in H(D)_\omega K/K,\\
&= \{0\} \,, && \textit{if} \quad \tau \in G - (H(D)_\omega K/K).
\end{aligned}
$$

Now we can complete the analysis of $G[B]^*$.

THEOREM 11.20. *Group $G[\mathfrak{N}]$ is equal to $H(D)_\omega K/K$. The homomorphism $\bar{T}$ sends the subalgebra $\bar{\mathfrak{F}}[G[\mathfrak{N}]^*]$ of $\mathfrak{B}$ isomorphically onto $\bar{\mathfrak{F}}[H(D)^*\langle\varphi\rangle_\omega]$. Hence its restriction is an isomorphism of $G[\mathfrak{N}]^*$ onto $H(D)^*\langle\varphi\rangle_\omega$ as extensions of $\bar{F}$, which is compatible with the natural isomorphism $\tau \to \tau \cap H(D)_\omega$ of $G[\mathfrak{N}]$ onto $H(D)_\omega/K(D)$.*

Furthermore, the group $N(D \text{ in } H)_\mathfrak{N}$ is equal to $N(D \text{ in } H)_\varphi N(D)$, and the above isomorphism preserves the actions of

$$N(D \text{ in } H)_\mathfrak{N}/N(D) \simeq N(D \text{ in } H)_\varphi/N(D)_\varphi\,,$$

in the sense that

$$(11.21)\qquad \bar{T}(y^\pi) = \bar{T}(y)^\pi, \quad \textit{for all} \quad y \in \bar{\mathfrak{F}}[G[\mathfrak{N}]^*],\ \pi \in N(D \text{ in } H)_\varphi\,.$$

Proof. The semi-simple algebra $\mathfrak{A}_1$ is the direct sum of its sub-algebras $d\mathfrak{A}_1$ and $(1-d)\mathfrak{A}_1$, both of which are $N(D)_\varphi = N(D)_\mathfrak{k}$-invariant. In view of (11.2b), (11.4b), and (11.5), this implies that (11.1) is equivalent to

$$d\mathfrak{A}_1(N(D)_\varphi \mid D) \neq \{0\}.$$

By (11.16b) the space $d\mathfrak{A}_1(N(D)_\varphi) = \bar{\mathfrak{F}}[H(D)^*\langle\varphi\rangle]_1 = \bar{\mathfrak{F}}\cdot 1$ is one dimensional over $\bar{\mathfrak{F}}$. Hence its non-zero subspace $d\mathfrak{A}_1(N(D)_\varphi \mid D)$ is equal to itself. In particular, the identity 1 lies in the two-sided ideal $d\mathfrak{A}d(N(D)_\varphi \mid D)$ of $d\mathfrak{A}d(N(D)_\varphi)$. So $d\mathfrak{A}d(N(D)_\varphi) = d\mathfrak{A}d(N(D)_\varphi \mid D)$. In view of (11.16), this implies that

$$\bar{\mathfrak{F}}[H(D)^*\langle\varphi\rangle]_\rho = d\mathfrak{A}_\rho(N(D)_\varphi \mid D), \quad \textit{for all} \quad \rho \in H(D)_\omega/K(D).$$

Now fix an element $\rho \in H(D)_\omega/K(D)$. Choose any non-zero element

$$y \in \bar{\mathfrak{F}}[H(D)^*\langle\varphi\rangle]_\rho .$$

Then $y \in H(D)^*\langle\varphi\rangle$ is a unit of $\bar{\mathfrak{F}}[H(D)^*\langle\varphi\rangle]$, and its inverse $x = y^{-1}$ lies in $\bar{\mathfrak{F}}[H(D)^*\langle\varphi\rangle]_{\rho^{-1}}$. So the above equation gives us two elements $z \in d\mathfrak{A}_\rho = \mathfrak{A}_\rho d$ and $w \in d\mathfrak{A}_{\rho^{-1}} = \mathfrak{A}_{\rho^{-1}}\, d$ such that

$$(11.22) \qquad y = \mathrm{tr}_{D\to N(D)_\varphi}(z), \quad x = y^{-1} = \mathrm{tr}_{D\to N(D)_\varphi}(w).$$

By (11.2b) and (11.4b) we can choose elements $z_0 \in \mathfrak{O}_{\rho K,(D)}(D)$ and $w_0 \in \mathfrak{O}_{\rho^{-1}K,(D)}(D)$ having z and w, respectively, as images in $\mathfrak{A}$. Since both $\mathfrak{O}_{\rho K,(D)}(D)$ and $\mathfrak{O}_{\rho^{-1}K,(D)}(D)$ are $N(D)$-invariant, the product $z_0^\sigma w_0^\tau$ lies in $\mathfrak{O}_{1,(D)}(D)$, for any $\sigma,\tau \in N(D)$. Evidently we can choose an element $d_0 \in \mathfrak{O}_{1,(D)}(D)$ so that its image d_0^I in $\mathfrak{O}_{1,(D)}(D)^I = \bar{\mathfrak{F}}[K(D)^*]$ satisfies

$$\begin{aligned} d_0^I &\equiv 1 \pmod{\mathfrak{P}} \\ &\equiv 0 \pmod{\mathfrak{P}'}, \quad \textit{for all} \quad \mathfrak{P}' \neq \mathfrak{P} \textit{ in } \mathrm{Max}(\bar{\mathfrak{F}}[K(D)^*]). \end{aligned}$$

Then d_0^σ, d_0^τ also lie in $\mathfrak{O}_{1,(D)}(D)$, for any $\sigma, \tau \in N(D)$. Furthermore, the above conditions on d_0^I imply that

$$(d_0^\sigma z_0^\sigma w_0^\tau d_0^\tau)^I = (d_0^I)^\sigma (z_0^\sigma w_0^\tau)^I (d_0^I)^\tau \equiv 0 \pmod{\mathfrak{P}},$$

$$\textit{unless} \quad \sigma, \tau \in N(D)_\varphi = N(D)_\mathfrak{P} .$$

We conclude that

$$y_0 = \mathrm{tr}_{D\to N(D)}(d_0 z_0) \in \mathfrak{O}_{\rho K,(D)}(N(D))$$

and

$$x_0 = \mathrm{tr}_{D\to N(D)}(w_0 d_0) \in \mathfrak{O}_{\rho^{-1}K,(D)}(N(D))$$

satisfy

$$\begin{aligned} (y_0 x_0)^I = \textstyle\sum_{\sigma,\tau\in N(D)/D} (d_0^\sigma z_0^\sigma w_0^\tau d_0^\tau)^I &\equiv \textstyle\sum_{\sigma,\tau\in N(D)_\varphi/D} (d_0^\sigma z_0^\sigma w_0^\tau d_0^\tau)^I \pmod{\mathfrak{P}} \\ &\equiv \mathrm{tr}_{D\to N(D)_\varphi}(d_0 z_0)\ \mathrm{tr}_{D\to N(D)_\varphi}(w_0 d_0) \pmod{\mathfrak{P}}. \end{aligned}$$

By construction, the image in $\mathfrak{A}$ of d_0 is the identity d of $\bar{\mathfrak{F}}[H(D)^*\langle\varphi\rangle]$ (see (11.6a)). Hence the image of $d_0 z_0$ is $dz = z$, and that of $w_0 d_0$ is $wd = w$. Now (11.22) and the above congruences tell us that the image in $\mathfrak{A}_1$ of the product of $y_0^I \in \mathfrak{O}_{\rho K,(D)}(N(D))^I$ and $x_0^I \in \mathfrak{O}_{\rho^{-1}K,(D)}(N(D))^I$ is congruent to d modulo the image of $\mathfrak{P}$. Hence $y_0^I\, x_0^I \notin \mathfrak{P} \cap \mathfrak{O}_{1,(D)}(N(D))^I$, which is $\mathfrak{N}^I$ by Theorem 9.5. From (10.13) we conclude that $\rho K \in G[\mathfrak{N}]$. Therefore $H(D)_\omega K/K \leq G[\mathfrak{N}]$.

Since the algebra homomorphism $\bar{T}$ is identity-preserving, it sends $\mathfrak{B}_1 = \bar{\mathfrak{F}}\cdot 1$ isomorphically onto $\bar{\mathfrak{F}}[H(D)^*\langle\varphi\rangle]_1 = \bar{\mathfrak{F}}\cdot 1$. Now let τ be any element of $G[\mathfrak{N}]$. By (10.13) we have $\mathfrak{B}_\tau \mathfrak{B}_{\tau^{-1}} = \mathfrak{B}_1$. Hence

$$\bar{T}(\mathfrak{B}_\tau)\,\bar{T}(\mathfrak{B}_{\tau^{-1}}) = \bar{T}(\mathfrak{B}_1) = \bar{\mathfrak{F}}\cdot 1 \neq \{0\}.$$

In particular, $\bar{T}(\mathfrak{B}_\tau) \neq \{0\}$. So (11.19) tells us that $\tau \in H(D)_\omega K/K$. Therefore $G[\mathfrak{N}] = H(D)_\omega K/K$, which is the first statement of the theorem. Because $\bar{T}$ sends $\mathfrak{F}[G(\mathfrak{N})^*]_1 = \mathfrak{B}_1$ isomorphically onto $\mathfrak{F}[H(D)^*\langle\varphi\rangle]_1$, the second and third statements of the theorem follow directly from this, (11.19), and [CCT, 13.10].

The unicity of the correspondence in Theorem 9.5 between $\mathfrak{N}$ and the $N(D)$-conjugacy class of $\mathfrak{L}$ tells us that

$$N(D \text{ in } H)_{\mathfrak{N}} = N(D \text{ in } H)_{\mathfrak{L}} N(D) = N(D \text{ in } H)_\varphi N(D).$$

Finally, (11.21) is an immediate consequence of the definition (11.18) of $\bar{T}$. So the theorem is proved.

12. Computing Clifford extensions for blocks

Before putting together the two halves of the above analysis to obtain the final result, we recall what all the notation means. We assume that (7.1) holds, and denote by H^* the central extension of $\bar{F}$ by H satisfying (10.1). We choose a normal subgroup K of H, denoting the factor group H/K by G. We also choose a block B of $\mathfrak{O}_1 = \oplus \sum_{\sigma \in K} \mathfrak{D}_\sigma$. We fix a defect group D of B in K.

Define the normal subgroup $H(D)$ of $N(D \text{ in } H)$ by (10.2d). Then the image of $\mathfrak{D}_{(D)}(D)$ in $\mathfrak{F}[H^*]$ is (by (9.3)) the twisted group algebra $\mathfrak{F}[H(D)^*]$ of the inverse image $H(D)^*$ of $H(D)$ in H^*. Let $K(D)^*$ be the inverse image in $H(D)^*$ of the normal subgroup $K(D) = K \cap H(D)$ of $N(D \text{ in } H)$. Then the Brauer analysis of Theorems 8.7 and 9.5 gives us a unique $N(D \text{ in } K)$-orbit of irreducible characters φ of $\mathfrak{F}[K(D)^*]$ corresponding to the block B.

Choose such a character φ. Since $K(D)^*$ is a normal subextension of $H(D)^*$, [CCT] gives us a Clifford extension $H(D)^*\langle\varphi\rangle$ of $\bar{F}$ by $H(D)_\varphi/K(D)$, where $H(D)_\varphi$ is the subgroup of $H(D)$ fixing φ under conjugation (see (11.4) and (11.6)). The similar subgroup $N(D \text{ in } H)_\varphi$ of $N(D \text{ in } H)$ acts naturally by conjugation on $H(D)^*\langle\varphi\rangle$, centralizing the image of $\bar{F}$ and leaving invariant the projection onto $H(D)_\varphi/K(D)$. Its normal subgroup $N(D \text{ in } K)_\varphi$ centralizes $H(D)_\varphi/K(D)$. Hence there is a unique bilinear map

$$\omega : N(D \text{ in } K)_\varphi \times [H(D)_\varphi/K(D)] \to \bar{F}$$

such that

$$(12.1) \quad \rho^\pi = \omega(\pi, \mathrm{pr}(\rho))\rho, \quad \textit{for all} \quad \rho \in H(D)^*\langle\varphi\rangle,\ \pi \in N(D \text{ in } K)_\varphi .$$

We define the normal subgroup $H(D)_\omega$ of $H(D)_\varphi$ to be the inverse image of the "right kernel" of ω (see (11.12b)). Then the inverse image $H(D)^*\langle\varphi\rangle_\omega$ in $H(D)^*\langle\varphi\rangle$ of $H(D)_\omega/K(D)$ is precisely the centralizer of $N(D \text{ in } K)_\varphi$ in $H(D)^*\langle\varphi\rangle$. So it is acted upon naturally by $N(D \text{ in } H)_\varphi/N(D \text{ in } K)_\varphi$.

We apply the analysis of §2 to $\mathfrak{O} = \mathfrak{D}$, G, B, and the $\mathfrak{O}_\tau = \oplus \sum_{\sigma\in\tau} \mathfrak{D}_\sigma$, for $\tau \in G = H/K$. The suborder $\mathfrak{C}$ is now just $\mathfrak{D}(K)$ (by (10.4a)). So the Brauer homomorphism S defined in Proposition 8.1 sends the image

$\mathfrak{C}^I = \mathfrak{D}(K)^I$ of $\mathfrak{C}$ in $\bar{\mathfrak{F}}[H^*]$ into the subalgebra

$$\bar{\mathfrak{F}}[H(D)^*]\ (N(D \text{ in } K))$$

of $\bar{\mathfrak{F}}[H(D)^*](K(D))$. The natural map of $\bar{\mathfrak{F}}[H(D)^*](K(D))$ into $\bar{\mathfrak{F}}[H(D)^*\langle\varphi\rangle]$ defines a homomorphism T of the subalgebra

$$\bar{\mathfrak{F}}[H(D)^*](N(D \text{ in } K))$$

into $\bar{\mathfrak{F}}[H(D)^*\langle\varphi\rangle_\omega]$ (see diagram (11.18)). Combining these maps with the inclusion $\mathfrak{C}[B] \subseteq \mathfrak{C}$, we obtain an $\mathfrak{R}$-homomorphism of $\mathfrak{C}$[B] into $\bar{\mathfrak{F}}[H(D)^*\langle\varphi\rangle_\omega]$. By (10.18) and (11.18) this map induces a unique homomorphism $\bar{R}$ of the factor algebra $\bar{\mathfrak{F}}[G[B]^*]$ of $\mathfrak{C}[B]$ into $\bar{\mathfrak{F}}[H(D)^*\langle\varphi\rangle_\omega]$ so that the following diagram is commutative:

$$\text{(12.2)}\qquad \begin{array}{ccc} \mathfrak{C}[B] \longrightarrow \mathfrak{C}^I & \xrightarrow{\ S\ } & \bar{\mathfrak{F}}[H(D)^*](N(D \text{ in } K)) \\ \downarrow & & \downarrow T \\ \bar{\mathfrak{F}}[G[B]^*] & \xrightarrow{\quad \bar{R} \quad} & \bar{\mathfrak{F}}[H(D)^*\langle\varphi\rangle_\omega]. \end{array}$$

Now we have:

THEOREM 12.3. *The group $G[B]$ is equal to $H(D)_\omega K/K$. The map $\bar{R}$ is an isomorphism of the algebra $\bar{\mathfrak{F}}[G[B]^*]$ onto $\bar{\mathfrak{F}}[H(D)^*\langle\varphi\rangle_\omega]$ which preserves the gradings in the sense that*

$$\text{(12.4)}\qquad \bar{R}(\bar{\mathfrak{F}}[G[B]^*]_\tau) = \bar{\mathfrak{F}}[H(D)^*\langle\varphi\rangle]_{\tau\cap H(D)}, \quad \textit{for all} \quad \tau \in G[B].$$

Hence its restriction is an isomorphism of $G[B]^*$ onto $H(D)^*\langle\varphi\rangle_\omega$ as extensions of $\bar{F}$ which is compatible with the natural isomorphism

$$\tau \to \tau \cap H(D) = \tau \cap H(D)_\omega$$

of $G[B] = H(D)_\omega K/K$ onto $H(D)_\omega/K(D)$.

Furthermore, the group G_B is equal to $N(D$ in $H)_\varphi K/K$, and the above isomorphism preserves the actions of $G_B \simeq N(D$ in $H)_\varphi/N(D$ in $K)_\varphi$ in the sense that

$$\text{(12.5)}\quad \bar{R}(y^{\pi K}) = \bar{R}(y)^\pi, \quad \textit{for all} \quad y \in \bar{\mathfrak{F}}[G[B]^*], \pi \in N(D \text{ in } H)_\varphi/N(D \text{ in } K)_\varphi .$$

Proof. This is just the logical union of Theorems 10.20 and 11.20, plus the grading conditions (10.19) and (11.19).

Perhaps we should note a few of the special properties of the case of blocks of groups, i.e., the case in which $\mathfrak{D}$ is the group ring $\mathfrak{R}H$ of the finite group H. and $\mathfrak{D}_\sigma = \mathfrak{R}\sigma$, for all $\sigma \in H$. It is evident from (10.1) that $\bar{\mathfrak{F}}[H^*]$ is, in this case, the modular group algebra $\bar{\mathfrak{F}}H$, with $\bar{\mathfrak{F}}[H^*]_\sigma = \bar{\mathfrak{F}}\sigma$, for all $\sigma \in H$. The group D is now an ordinary defect group of the p-block B of the normal subgroup K of H. By (10.2d), the subgroup $H(D)$ coincides with $C(D$ in $H)$, and $K(D)$ with $C(D$ in $K)$. The irreducible character φ of $\bar{\mathfrak{F}}C(D$ in $K) =$

$\mathfrak{F}[K(D)^*]$ is just one of the modular irreducible characters in the blocks of $C(D \text{ in } K)$ corresponding to the block B in §11 of [1]. Its Clifford extension $H(D)^*\langle\varphi\rangle$ is that, $C(D \text{ in } H)\langle\varphi\rangle$, of φ in $C(D \text{ in } H)$, and the bilinear map ω of (12.1) sends

$$N(D \text{ in } K)_\varphi \times [C(D \text{ in } H)_\varphi/C(D \text{ in } K)]$$

into $\bar{F}$. Its "right kernel" $H(D)_\omega/K(D)$ is now $C(D \text{ in } H)_\omega/C(D \text{ in } K)$, whose inverse image

$$C(D \text{ in } H)\langle\varphi\rangle_\omega$$

in $C(D \text{ in } H)\langle\varphi\rangle$ is canonically isomorphic to the Clifford extension

$$C(D \text{ in } H)_\omega\langle\varphi\rangle$$

of φ in $C(D \text{ in } H)_\omega$ (see [CCT, 16.1]).

Setting $G = H/K$, and defining G_B, $G[B]$, and $G[B]^*$ as in §2, we obtain the following special case of the above theorem for blocks of groups:

Corollary 12.6. *The group $G[B]$ equals $C(D \text{ in } H)_\omega K/K$, while the map $\bar{R}$ of the theorem defines an isomorphism of $G[B]^*$ onto $C(D \text{ in } H)_\omega\langle\varphi\rangle$ as extensions of $\bar{F}$ which is compatible with the natural isomorphism*

$$\tau \to \tau \cap C(D \text{ in } H)_\omega$$

of $G[B]$ onto $C(D \text{ in } H)_\omega/C(D \text{ in } K)$. Furthermore, G_B equals $N(D \text{ in } H)_\varphi K/K$, and the above isomorphism preserves the conjugacy actions of $G_B \simeq N(D \text{ in } H)_\varphi/N(D \text{ in } K)_\varphi$ on the two extensions.

13. Miscellanea

In the situation of §2, the group G acts on the family $\mathrm{Id}(\mathfrak{O}_1)$ of two-sided ideals $\mathfrak{I}$ of $\mathfrak{O}_1$ by (2.3). We shall say that such an ideal $\mathfrak{I}$ *lies in* the block B if

$$e \equiv 1 \pmod{\mathfrak{I}}. \tag{13.1}$$

Since e is a central idempotent of $\mathfrak{O}_1$, this occurs if and only if

$$\mathfrak{I} = e\mathfrak{I} \oplus (1 - e)\mathfrak{O}_1. \tag{13.2}$$

Furthermore $\mathfrak{I} \to e\mathfrak{I}$ is a one-to-one correspondence between two-sided ideals $\mathfrak{I}$ of $\mathfrak{O}_1$ lying in B and two-sided ideals $e\mathfrak{I} = \mathfrak{I}e$ of the suborder $e\mathfrak{O}_1$.

Proposition 13.3. *The subgroup $G[B]$ of G fixes every two-sided ideal $\mathfrak{I}$ of $\mathfrak{O}_1$ lying in B.*

Proof. If $\sigma \in G[B]$, then Proposition 2.7(b), (2.9a), the inclusions $e\mathfrak{C}_{\sigma^{-1}} \subseteq \mathfrak{O}_{\sigma^{-1}}$, $\mathfrak{C}_\sigma \subseteq \mathfrak{O}_\sigma$, and (2.1e) give

$$e\mathfrak{O}_\sigma \subseteq e\mathfrak{C}_1\mathfrak{O}_\sigma = (e\mathfrak{C}_\sigma)(e\mathfrak{C}_{\sigma^{-1}})\mathfrak{O}_\sigma \subseteq (e\mathfrak{C}_\sigma)\mathfrak{O}_{\sigma^{-1}}\mathfrak{O}_\sigma = (e\mathfrak{C}_\sigma)\mathfrak{O}_1 \subseteq e\mathfrak{O}_\sigma\mathfrak{O}_1 = e\mathfrak{O}_\sigma.$$

Hence $e\mathfrak{O}_\sigma = (e\mathfrak{C}_\sigma)\mathfrak{O}_1$. Since σ^{-1} also lies in the subgroup $G[B]$, a symmetric argument shows that $\mathfrak{O}_\sigma e = \mathfrak{O}_1(e\mathfrak{C}_\sigma)$.

Because e is a central idempotent of $\mathfrak{O}_1$, the product $e\mathfrak{J} = \mathfrak{J}e$ is also a two-sided ideal of $\mathfrak{O}_1$. Using (2.3) and the above formulas (for the elements σ, 1, and σ^{-1} of $G[B]$), we compute

$$\begin{aligned}(e\mathfrak{J})^\sigma = \mathfrak{O}_{\sigma^{-1}}e\mathfrak{J}e\mathfrak{O}_\sigma &= \mathfrak{O}_1(e\mathfrak{C}_{\sigma^{-1}})\mathfrak{J}(e\mathfrak{C}_\sigma)\mathfrak{O}_1 = \mathfrak{O}_1(e\mathfrak{C}_{\sigma^{-1}})(e\mathfrak{C}_\sigma)\mathfrak{J}\mathfrak{O}_1\\ &= \mathfrak{O}_1(e\mathfrak{C}_1)\mathfrak{J}\mathfrak{O}_1 = \mathfrak{O}_1e\mathfrak{J}\mathfrak{O}_1 = e\mathfrak{J}.\end{aligned}$$

Here we used the fact that $e\mathfrak{C}_\sigma \subseteq \mathfrak{C} = C(\mathfrak{O}_1 \text{ in } \mathfrak{O})$ centralizes $\mathfrak{J} \subseteq \mathfrak{O}_1$, and the definition (2.9) of $G[B]$.

We know from Proposition 2.17 that $\sigma^{-1} \in G[B] \leq G_B$ centralizes e, and hence $1 - e$. In view of (2.15), this implies that $\mathfrak{O}_{\sigma^{-1}}(1 - e) = (1 - e)\mathfrak{O}_{\sigma^{-1}}$. Hence the two-sided ideal $(1 - e)\mathfrak{O}_1$ satisfies

$$[(1 - e)\mathfrak{O}_1]^\sigma = \mathfrak{O}_{\sigma^{-1}}(1 - e)\mathfrak{O}_1\mathfrak{O}_\sigma = (1 - e)\mathfrak{O}_{\sigma^{-1}}\mathfrak{O}_1\mathfrak{O}_\sigma = (1 - e)\mathfrak{O}_1.$$

By (13.2) we have $\mathfrak{J}^\sigma = (e\mathfrak{J})^\sigma + [(1 - e)\mathfrak{O}_1]^\sigma = e\mathfrak{J} + (1 - e)\mathfrak{O}_1 = \mathfrak{J}$. So the proposition is proved.

To each maximal two-sided ideal $\mathfrak{M}$ of $\mathfrak{O}_1$ lying in the block B we can now assign a Clifford extension $G[B]\langle\mathfrak{M}\rangle$ of $\bar{F}$ by $G[B]$ as follows: the $G[B]$-invariance of $\mathfrak{M}$ implies that $\mathfrak{O}_\sigma\mathfrak{M} = \mathfrak{M}\mathfrak{O}_\sigma$, for all $\sigma \in G[B]$, and hence that

$$\mathfrak{O}_{G[B]}\mathfrak{M} = \oplus \textstyle\sum_{\sigma \in G[B]} \mathfrak{O}_\sigma \mathfrak{M}$$

is a graded two-sided ideal of the suborder $\mathfrak{O}_{G[B]} = \oplus \sum_{\sigma \in G[B]} \mathfrak{O}_\sigma$ of $\mathfrak{O}$. We identify each $\mathfrak{O}_\sigma/\mathfrak{O}_\sigma \mathfrak{M}$, $\sigma \in G[B]$, naturally with its image in the factor algebra $\mathfrak{O}_{G[B]}/\mathfrak{O}_{G[B]} \mathfrak{M}$. Then conditions (2.1) are satisfied with $\bar{\mathfrak{F}}$, $\mathfrak{O}_{G[B]}/\mathfrak{O}_{G[B]} \mathfrak{M}$, $G[B]$, and the $\mathfrak{O}_\sigma/\mathfrak{O}_\sigma \mathfrak{M}$, in place of $\mathfrak{R}$, $\mathfrak{O}$, G, and the $\mathfrak{O}_\sigma$, respectively. In addition, $\mathfrak{O}_1/\mathfrak{O}_1 \mathfrak{M} \simeq \mathfrak{O}_1/\mathfrak{M}$ is a finite-dimensional simple algebra over the algebraically closed field $\bar{\mathfrak{F}}$. It follows (see [CCT, §§8, 14]) that there is a unique central extension $G[B]\langle\mathfrak{M}\rangle$ of $\bar{F}$ by $G[B]$ whose twisted group algebra is given by:

(13.4a) $\bar{\mathfrak{F}}[G[B]\langle\mathfrak{M}\rangle] = C(\mathfrak{O}_1/\mathfrak{M} \text{ in } \mathfrak{O}_{G[B]}/\mathfrak{O}_{G[B]} \mathfrak{M})$,
(13.4b) $\bar{\mathfrak{F}}[G[B]\langle\mathfrak{M}\rangle]_\sigma = C(\mathfrak{O}_1/\mathfrak{M} \text{ in } \mathfrak{O}_\sigma/\mathfrak{O}_\sigma\mathfrak{M})$, *for all* $\sigma \in G[B]$.

This Clifford extension for $\mathfrak{M}$ is related to the Clifford extension $G[B]^*$ for B by:

PROPOSITION 13.5. *For any maximal two-sided ideal* $\mathfrak{M}$ *of* $\mathfrak{O}_1$ *lying in the block* B, *the natural map of* $\mathfrak{C}[B] \subseteq \mathfrak{O}_{G[B]}$ *into* $\mathfrak{O}_{G[B]}/\mathfrak{O}_{G[B]} \mathfrak{M}$ *induces an isomorphism* φ *of the algebra* $\bar{\mathfrak{F}}[G[B]^*]$ *onto* $\bar{\mathfrak{F}}[G[B]\langle\mathfrak{M}\rangle]$ *sending* $\bar{\mathfrak{F}}[G[B]^*]_\sigma$ *onto* $\bar{\mathfrak{F}}[G[B]\langle\mathfrak{M}\rangle]_\sigma$, *for all* $\sigma \in G[B]$. *Hence the restriction of* φ *is an isomorphism of* $G[B]^*$ *onto* $G[B]\langle\mathfrak{M}\rangle$ *as extensions of* $\bar{F}$ *by* $G[B]$.

Proof. Evidently the natural map is an $\mathfrak{R}$-homomorphism ψ of

$$\mathfrak{C}[B] \subseteq \mathfrak{C}(\mathfrak{O}_1 \text{ in } \mathfrak{O}_{G[B]})$$

into

$$\bar{\mathfrak{F}}[G[B]\langle\mathfrak{M}\rangle] = C(\mathfrak{O}_1/\mathfrak{M} \text{ in } \mathfrak{O}_{G[B]}/\mathfrak{O}_{G[B]} \mathfrak{M}).$$

Since the identity e of $\mathfrak{C}[B]$ is congruent to 1 modulo $\mathfrak{M}$, it is mapped onto the identity of $\bar{\mathfrak{F}}[G[B]\langle\mathfrak{M}\rangle]$. Furthermore ψ sends $\mathfrak{C}[B]_\sigma \subseteq \mathfrak{O}_\sigma$ into

$$\bar{\mathfrak{F}}[G[B]\langle\mathfrak{M}\rangle]_\sigma = C(\mathfrak{O}_1/\mathfrak{M} \quad \text{in} \quad \mathfrak{O}_\sigma/\mathfrak{O}_\sigma\mathfrak{M}),$$

for all $\sigma \in G[B]$. Hence $\psi(\mathfrak{C}[B]_1) = \bar{\mathfrak{F}}[G[B]\langle\mathfrak{M}\rangle]_1 \simeq \bar{\mathfrak{F}}$. In view of (2.11), this implies that $J(\mathfrak{C}[B]_1)$, and therefore $\mathfrak{C}[B]J(\mathfrak{C}[B]_1)$, are contained in the kernel of ψ. This and (2.14) tell us that ψ induces an $\bar{\mathfrak{F}}$-homomorphism φ of the algebra $\bar{\mathfrak{F}}[G[B]^*]$ into $\bar{\mathfrak{F}}[G[B]\langle\mathfrak{M}\rangle]$ sending $\bar{\mathfrak{F}}[G[B]^*]_\sigma$ into $\bar{\mathfrak{F}}[G[B]\langle\mathfrak{M}\rangle]_\sigma$, for all $\sigma \in G[B]$. Since φ maps $\bar{\mathfrak{F}}[G[B]^*]_1 = \bar{\mathfrak{F}}\cdot 1$ isomorphically onto $\bar{\mathfrak{F}}[G[B]\langle\mathfrak{M}\rangle]_1 = \bar{\mathfrak{F}}\cdot 1$, the proposition follows from this and [CCT, 13.10].

Besides the above "modular" maximal two-sided ideals $\mathfrak{M}$, the block B also contains some "ordinary" maximal two-sided ideals $\mathfrak{N}$, which also have Clifford extensions. The $\mathfrak{R}$-order $\mathfrak{O}$ defines the finite-dimensional algebra $\mathfrak{F} \otimes \mathfrak{O} = \mathfrak{F} \otimes_{\mathfrak{R}} \mathfrak{O}$ over the field of fractions $\mathfrak{F}$ of $\mathfrak{R}$. Conditions (2.1) imply the same conditions for $\mathfrak{F} \otimes \mathfrak{O}$ and its $\mathfrak{F}$-subspaces $\mathfrak{F} \otimes \mathfrak{O}_\sigma = \mathfrak{F} \otimes_{\mathfrak{R}} \mathfrak{O}_\sigma$, for $\sigma \in G$. We say that a two-sided ideal $\mathfrak{K}$ of $\mathfrak{F} \otimes \mathfrak{O}_1$ *lies in* the block B if the image $1 \otimes e$ of e is congruent to 1 modulo $\mathfrak{K}$. Since $\mathfrak{R}$ is a valuation ring and $\mathfrak{K}$ is an $\mathfrak{F}$-subspace of $\mathfrak{F} \otimes \mathfrak{O}_1$, we have $\mathfrak{K} = \mathfrak{F} \otimes_{\mathfrak{R}} \mathfrak{J}$, where

$$\mathfrak{J} = \{y \in \mathfrak{O}_1 \mid 1 \otimes y \in \mathfrak{K}\}.$$

Evidently $\mathfrak{J}$ is a two-sided ideal of $\mathfrak{O}_1$ lying in B. Because

$$\mathfrak{K}^\sigma = (\mathfrak{F} \otimes \mathfrak{O}_{\sigma^{-1}})(\mathfrak{F} \otimes \mathfrak{J})(\mathfrak{F} \otimes \mathfrak{O}_\sigma) = \mathfrak{F} \otimes \mathfrak{O}_{\sigma^{-1}}\mathfrak{J}\mathfrak{O}_\sigma = \mathfrak{F} \otimes \mathfrak{J}^\sigma,$$

for all $\sigma \in G$, Proposition 13.3 implies that

(13.6) *$G[B]$ fixes every two-sided ideal $\mathfrak{K}$ of $\mathfrak{F} \otimes \mathfrak{O}_1$ lying in B.*

As in (13.4), the Clifford extension $G[B]\langle\mathfrak{N}\rangle$ for a maximal two-sided ideal $\mathfrak{N}$ of $\mathfrak{F} \otimes \mathfrak{O}_1$ lying in B is the central extension of the multiplicative group F of $\mathfrak{F}$ by $G[B]$ whose twisted group algebra is given by:

(13.7a) $\mathfrak{F}[G[B]\langle\mathfrak{N}\rangle] = C((\mathfrak{F} \otimes \mathfrak{O}_1)/\mathfrak{N} \text{ in } (\mathfrak{F} \otimes \mathfrak{O}_{G[B]})/(\mathfrak{F} \otimes \mathfrak{O}_{G[B]})\mathfrak{N})$,
(13.7b) $\mathfrak{F}[G[B]\langle\mathfrak{N}\rangle]_\sigma = C((\mathfrak{F} \otimes \mathfrak{O}_1)/\mathfrak{N} \text{ in } (\mathfrak{F} \otimes \mathfrak{O}_\sigma)/(\mathfrak{F} \otimes \mathfrak{O}_\sigma)\mathfrak{N})$, *for all* $\sigma \in G[B]$.

The Clifford extension $G[B]\langle\mathfrak{N}\rangle$ of F is not directly comparable with the Clifford extension $G[B]^*$ of $\bar{F}$. However, the former extension determines a "residue class extension" $G[B]\langle\mathfrak{N}\rangle^-$ (see [ICE, 4.8–4.13]) of $\bar{F}$ by $G[B]$ which turns out to be naturally isomorphic to $G[B]^*$. We recall the construction of this new extension.

Since $\mathfrak{F}$ is an algebraically closed field, the subset $\operatorname{tor}(G[B]\langle\mathfrak{N}\rangle)$ of all torsion elements of $G[B]\langle\mathfrak{N}\rangle$ is actually a subgroup, and $G[B]\langle\mathfrak{N}\rangle$ is the product of this subgroup and the image of F (see [ICE, 4.8]). Hence

$$\operatorname{tor}(G[B]\langle\mathfrak{N}\rangle)_\sigma = \operatorname{tor}(G[B]\langle\mathfrak{N}\rangle) \cap \operatorname{pr}^{-1}(\sigma)$$

is a coset of $\operatorname{tor}(G[B]\langle\mathfrak{N}\rangle)_1 = U\cdot 1$, for any $\sigma \in G[B]$, where U is the subgroup of all roots of unity in $\mathfrak{F}$. Evidently U is contained in the unit group of the integrally closed subring $\mathfrak{R}$, from which we conclude that the $\mathfrak{R}$-submodule $\mathfrak{R}[G[B]\langle\mathfrak{N}\rangle]_\sigma$ of $\mathfrak{F}[G[B]\langle\mathfrak{N}\rangle]_\sigma$ generated by $\mathrm{tor}(G[B]\langle\mathfrak{N}\rangle)_\sigma$ is an $\mathfrak{R}$-lattice of rank one generated by any element of $\mathrm{tor}(G[B]\langle\mathfrak{N}\rangle)_\sigma$. Clearly

$$\mathrm{tor}(G[B]\langle\mathfrak{N}\rangle)_\sigma \mathrm{tor}(G[B]\langle\mathfrak{N}\rangle)_\tau = \mathrm{tor}(G[B]\langle\mathfrak{N}\rangle)_{\sigma\tau},$$

for all $\sigma, \tau \in G[B]$, which implies that $\mathfrak{R}[G[B]\langle\mathfrak{N}\rangle] = \oplus \sum_{\sigma\in G[B]} \mathfrak{R}[G[B]\langle\mathfrak{N}\rangle]_\sigma$, $G[B]$, and the $\mathfrak{R}[G[B]\langle\mathfrak{N}\rangle]_\sigma$ satisfy (2.1) in place of $\mathfrak{O}$, G, and the $\mathfrak{O}_\sigma$, respectively.

The suborder $\mathfrak{R}[G[B]\langle\mathfrak{N}\rangle]_1 = \mathfrak{R}\cdot 1$ is a local ring whose radical $J(\mathfrak{R}[G[B]\langle\mathfrak{N}\rangle]_1)$ is just $\mathfrak{p}\cdot 1$. As in (2.12–2.14), this determines a unique central extension $G[B]\langle\mathfrak{N}\rangle^-$ of $\bar{F}$ by $G[B]$ whose twisted group algebra is given by:

(13.8a) $\bar{\mathfrak{F}}[G[B]\langle\mathfrak{N}\rangle^-] = \mathfrak{R}[G[B]\langle\mathfrak{N}\rangle]/\mathfrak{p}\mathfrak{R}[G[B]\langle\mathfrak{N}\rangle]$,
(13.8b) $\bar{\mathfrak{F}}[G[B]\langle\mathfrak{N}\rangle^-]_\sigma = \mathfrak{R}[G[B]\langle\mathfrak{N}\rangle]_\sigma/\mathfrak{p}\mathfrak{R}[G[B]\langle\mathfrak{N}\rangle]_\sigma$, *for all* $\sigma \in G[B]$.
(Compare [ICE, 4.12].)

Before describing the isomorphism of $G[B]^*$ onto $G[B]\langle\mathfrak{N}\rangle^-$, we note the following useful result:

LEMMA 13.9. *Let* $\mathfrak{A}_\sigma \subseteq \mathfrak{F}[G[B]\langle\mathfrak{N}\rangle]_\sigma$, *for* $\sigma \in G[B]$, *be any* $\mathfrak{R}$-*sublattices such that* (2.1) *holds with* $\mathfrak{A} = \oplus \sum_{\sigma\in G[B]} \mathfrak{A}_\sigma$, $G[B]$, *and the* $\mathfrak{A}_\sigma$ *in place of* $\mathfrak{O}$, G, *and the* $\mathfrak{O}_\sigma$, *respectively. Then* $\mathfrak{A} = \mathfrak{R}[G[B]\langle\mathfrak{N}\rangle]$ *and* $\mathfrak{A}_\sigma = \mathfrak{R}[G[B]\langle\mathfrak{N}\rangle]_\sigma$, *for all* $\sigma \in G[B]$.

Proof. Fix elements $\sigma \in G[B]$ and $y \in \mathrm{tor}(G[B]\langle\mathfrak{N}\rangle)_\sigma$. Then the one-dimensional subspace $\mathfrak{F}[G[B]\langle\mathfrak{N}\rangle]_\sigma$ is equal to $\mathfrak{F}y$. In particular, any element $z \in \mathfrak{A}_\sigma$ has the form $z = fy$, with $f \in \mathfrak{F}$.

If $f \notin \mathfrak{R}$, then $f^n \notin \mathfrak{R}$, for all $n > 0$, since $\mathfrak{R}$ is a valuation ring. There exists an $n > 0$ such that $\sigma^n = 1$ (by Proposition 2.2(d)!). Then

$$y^n \in \mathrm{tor}(G[B]\langle\mathfrak{N}\rangle)_{\sigma^n} = U\cdot 1$$

is a generator for $\mathfrak{R}[G[B]\langle\mathfrak{N}\rangle]_1 = \mathfrak{R}\cdot 1$. Since $\mathfrak{R}$ is a valuation ring, it is the only $\mathfrak{R}$-order in $\mathfrak{F}$. Hence $\mathfrak{A}_1 = \mathfrak{R}\cdot 1 = \mathfrak{R}y^n$. But $\mathfrak{A}_1 = \mathfrak{A}_{\sigma^n}$ contains $z^n = f^n y^n$ which does not lie in $\mathfrak{R}y^n$. This contradiction proves that f always lies in $\mathfrak{R}$, and hence that $\mathfrak{A}_\sigma$ is an $\mathfrak{R}$-sublattice of $\mathfrak{F}[G[B]\langle\mathfrak{N}\rangle]_\sigma = \mathfrak{R}y$.

If $\mathfrak{A}_\sigma \neq \mathfrak{R}y$, then we must have $\mathfrak{A}_\sigma \subseteq \mathfrak{p}y$. But then

$$\mathfrak{A}_{\sigma^{-1}} \subseteq \mathfrak{F}[G[B]\langle\mathfrak{N}\rangle]_{\sigma^{-1}} = \mathfrak{R}y^{-1}$$

and $\mathfrak{R}\cdot 1 = \mathfrak{A}_1 = \mathfrak{A}_\sigma\mathfrak{A}_{\sigma^{-1}} \subseteq \mathfrak{p}y\mathfrak{R}y^{-1} = \mathfrak{p}\cdot 1$, which is impossible. Therefore $\mathfrak{A}_\sigma = \mathfrak{R}y$, and the lemma is proved.

Now we have:

PROPOSITION 13.10. *For any maximal two-sided ideal* $\mathfrak{N}$ *of* $\mathfrak{F} \otimes \mathfrak{O}_1$ *lying in* B, *the natural map of* $\mathfrak{O}_{G[B]}$ *into*

$$(\mathfrak{F} \otimes \mathfrak{O}_{G[B]})/(\mathfrak{F} \otimes \mathfrak{O}_{G[B]})\mathfrak{N}$$

sends $\mathfrak{C}[B]$ *onto* $\mathfrak{R}[G[B]\langle\mathfrak{N}\rangle]$, *and induces an algebra isomorphism* ψ *of* $\bar{\mathfrak{F}}[G[B]^*]$ *onto* $\bar{\mathfrak{F}}[G[B]\langle\mathfrak{N}\rangle^-]$ *sending* $\bar{\mathfrak{F}}[G[B]^*]_\sigma$ *onto* $\bar{\mathfrak{F}}[G[B]\langle\mathfrak{N}\rangle^-]_\sigma$, *for all* $\sigma \in G[B]$. *Hence the restriction of* ψ *is an isomorphism of* $G[B]^*$ *onto* $G[B]\langle\mathfrak{N}\rangle^-$ *as extensions of* $\bar{F}$ *by* $G[B]$.

Proof. From (13.7) it is clear that the restriction φ of the natural map of $\mathfrak{O}_{G[B]}$ into $(\mathfrak{F} \otimes \mathfrak{O}_{G[B]})/(\mathfrak{F} \otimes \mathfrak{O}_{G[B]})\,\mathfrak{N}$ sends $\mathfrak{C}[B] \subseteq C(\mathfrak{O}_1 \text{ in } \mathfrak{O}_{G[B]})$ into $\mathfrak{F}[G[B]\langle\mathfrak{N}\rangle]$ and $\mathfrak{C}[B]_\sigma$ into $\mathfrak{F}[G[B]\langle\mathfrak{N}\rangle]_\sigma$, for all $\sigma \in G[B]$. Because the image $1 \otimes e$ of the identity e of $\mathfrak{C}[B]$ is congruent to 1 modulo $\mathfrak{N}$, the map φ is identity-preserving. It follows from this and Proposition 2.10 that (2.1) holds with $\varphi(\mathfrak{C}[B])$, $G[B]$, and the

$$\varphi(\mathfrak{C}[B]_\sigma) \subseteq \mathfrak{F}[G[B]\langle\mathfrak{N}\rangle]_\sigma$$

in place of $\mathfrak{O}$, G, and the $\mathfrak{O}_\sigma$, respectively. So Lemma 13.9 tells us that $\varphi(\mathfrak{C}[B]) = \mathfrak{R}[G[B]\langle\mathfrak{N}\rangle]$ and $\varphi(\mathfrak{C}[B]_\sigma) = \mathfrak{R}[G[B]\langle\mathfrak{N}\rangle]_\sigma$, for all $\sigma \in G[B]$. In view of (2.11), the composition of φ with the natural epimorphism (in (13.8a)) of $\mathfrak{R}[G[B]\langle\mathfrak{N}\rangle]$ onto $\bar{\mathfrak{F}}[G[B]\langle\mathfrak{N}\rangle^-]$ is an $\mathfrak{R}$-epimorphism having

$$\mathfrak{C}[B]J(\mathfrak{C}[B]_1)$$

in its kernel. By (2.14) it induces an $\mathfrak{F}$-epimorphism ψ of $\bar{\mathfrak{F}}[G[B]^*]$ onto $\bar{\mathfrak{F}}[G[B]\langle\mathfrak{N}\rangle^-]$ sending $\bar{\mathfrak{F}}[G[B]^*]_\sigma$ into $\bar{\mathfrak{F}}[G[B]\langle\mathfrak{N}\rangle^-]_\sigma$, for all $\sigma \in G[B]$. Since the restriction of ψ is an isomorphism of $\bar{\mathfrak{F}}[G[B]^*]_1 = \bar{\mathfrak{F}}\cdot 1$ onto $\bar{\mathfrak{F}}[G[B]\langle\mathfrak{N}\rangle^-]_1 = \bar{\mathfrak{F}}\cdot 1$, the proposition follows directly from this and [CCT, 13.10].

Even in the situation where both (5.1) and (6.1) hold, the image $\mathfrak{C}[B](E_B)^J$ of §5 can be strictly smaller than $\bar{\mathfrak{F}}[G[B]^*](E_B)$, while the group $G[B](D)$ of Proposition 9.2 can be strictly smaller than $C(D \text{ in } G[B])$. Both these phenomena are illustrated by the following example:

Example 13.11. Let $H = \langle\rho\rangle \times \langle\pi\rangle$ be the direct product of two cyclic groups $\langle\rho\rangle$, $\langle\pi\rangle$ of prime order p. Assume that $\mathfrak{F}$ has characteristic zero, while $\bar{\mathfrak{F}}$ has characteristic p. Let H^* be the unique central extension of $\bar{F}$ by H generated over $\bar{F}$ by two elements ρ^*, π^* satisfying:

(13.12a) $\operatorname{pr}(\rho^*) = \rho$, $\operatorname{pr}(\pi^*) = \pi$,
(13.12b) $(\rho^*)^p = (\pi^*)^p = 1$,
(13.12c) $\rho^*\pi^* = \zeta\pi^*\rho^*$,

where ζ is a primitive p^{th} root of unit in $\mathfrak{F}$. Then, as above, $\mathfrak{R}[H^*]$, H, and the $\mathfrak{R}[H^*]_\sigma$, $\sigma \in H$, satisfy (2.1) in place of $\mathfrak{O}$, G, and the $\mathfrak{O}_\sigma$, respectively.

Evidently $\mathfrak{R}[H^*]_1 = \mathfrak{R}\cdot 1$ has only one block B_0. Since $\mathfrak{R}[H^*]_1$ is central in $\mathfrak{R}[H^*]$, the suborders corresponding to $\mathfrak{C}$, $e\mathfrak{C}$, and $\mathfrak{C}[B]$ all coincide with $\mathfrak{R}[H^*]$, while $H_{B_0} = H[B_0] = H$.

Since $J(\mathfrak{R}) = \mathfrak{p}$ contains $\zeta - 1$, it is clear from (13.12) and (2.14) that $\bar{\mathfrak{F}}[H[B_0]^*] = \mathfrak{R}[H^*]/\mathfrak{p}\mathfrak{R}[H^*]$ is just the group algebra $\bar{\mathfrak{F}}H$, and hence that $H[B_0]^*$ is the split extension $H \times \bar{F}$ of $\bar{F}$.

We set $E = H$, operating on $\mathfrak{R}[H^*]$ as usual via (2.15). Then (5.1) and

(6.1) hold, while $E_{B_0} = E$. From (13.12) we see that

$$\mathfrak{R}[H^*](E_{B_0}) = Z(\mathfrak{R}[H^*]) = \mathfrak{R}[H^*]_1 .$$

Because $\bar{\mathfrak{F}}[H[B_0]^*] = \bar{\mathfrak{F}}H$ is abelian, this gives

$$\mathfrak{R}[H^*](E_{B_0})^J = \bar{\mathfrak{F}}\cdot 1 \subset \bar{\mathfrak{F}}H = \bar{\mathfrak{F}}[H[B_0]^*](E_{B_0}). \tag{13.13}$$

Like its residue class algebra $\bar{\mathfrak{F}}H$, the order $\mathfrak{R}[H^*]$ has but one block B, whose defect group (in E or in H) is $D = E = H$. Since D centralizes $H[B_0] = H$, equations (9.1) and (13.12) give

$$H[B_0](D) = \{1\} < C(D \text{ in } H[B_0]) = H. \tag{13.14}$$

When (2.1) holds, the residue class algebra $\bar{\mathfrak{O}} = \mathfrak{O}/\mathfrak{p}\mathfrak{O}$, the group G, and the subspaces $\bar{\mathfrak{O}}_\sigma = \mathfrak{O}_\sigma/\mathfrak{p}\mathfrak{O}_\sigma$ (identified with their images in $\bar{\mathfrak{O}}$) also satisfy (2.1) with $\bar{\mathfrak{F}}$ in place of $\mathfrak{R}$. Proposition 1.12 gives us a unique block $\bar{B}$ of $\bar{\mathfrak{O}}_1$ corresponding to the block B of $\mathfrak{O}_1$, the primitive central idempotent $\bar{e}$ in $\bar{B}$ being the image of e. So we can define Clifford extensions $G[B]^*$ and $G[\bar{B}]^*$ of $\bar{F}$ by subgroups $G[B]$ and $G[\bar{B}]$ of G. The order $\mathfrak{R}[H^*]$ of Example 13.11 can be used again to show that $G[\bar{B}]^*$ need not be isomorphic to $G[B]^*$.

Example 13.15. Evidently $\mathfrak{O} = \mathfrak{R}[H^*]$, $G = H/\langle\pi\rangle$, and $\mathfrak{O}_\tau = \oplus \sum_{\sigma\in\tau} \mathfrak{R}[H^*]_\sigma$, for $\tau \in G$, satisfy (2.1). Condition (13.12b) implies that $\mathfrak{O}_1 = \mathfrak{R}[\pi^*]$ is isomorphic to the group ring $\mathfrak{R}\langle\pi\rangle$. Hence $\mathfrak{O}_1$ has only one block B and $e = 1$. From (13.12c) we get $\mathfrak{C} = C(\mathfrak{O}_1 \text{ in } \mathfrak{O}) = \mathfrak{O}_1$. Hence $\mathfrak{C}[B] = e\mathfrak{C} = \mathfrak{C} = \mathfrak{O}_1$, $G[B] = 1$, and $G[B]^*$ is the split extension $\langle 1\rangle \times \bar{F}$ of $\bar{F}$.

On the other hand $\bar{\mathfrak{O}} \simeq \bar{\mathfrak{F}}H$ is commutative. Hence

$$\bar{\mathfrak{C}} = C(\bar{\mathfrak{O}}_1 \text{ in } \bar{\mathfrak{O}}) = \bar{\mathfrak{O}},$$

which implies that $\bar{\mathfrak{C}}[\bar{B}] = \bar{e}\bar{\mathfrak{C}} = \bar{\mathfrak{C}} = \bar{\mathfrak{O}}$, that $G[\bar{B}] = G$, and that $G[\bar{B}]^*$ is the split extension $G \times \bar{F}$ of $\bar{F}$. Evidently $G[\bar{B}]^*$ is not isomorphic to $G[B]^*$.

In general, the relations between $G[B]$, $G[B]^*$ and $G[\bar{B}]$, $G[\bar{B}]^*$ are given by:

PROPOSITION 13.16. *The group $G[B]$ is a normal subgroup of $G[\bar{B}]$, and $G[\bar{B}]/G[B]$ is a p-group, if $\bar{\mathfrak{F}}$ has prime characteristic p, and is $\{1\}$ otherwise. The natural map of $\mathfrak{O}$ onto $\bar{\mathfrak{O}}$ induces a monomorphism ξ of $G[B]^*$ into $G[\bar{B}]^*$ as extensions of $\bar{F}$ compatible with the inclusion map of $G[B]$ into $G[\bar{B}]$.*

The group G_B is equal to $G_{\bar{B}}$, and the above monomorphism ξ is $G_{\bar{B}}$-invariant.

If $\mathfrak{O}$, G, and the $\mathfrak{O}_\sigma$ come from a finite group H and its normal subgroup K as at the beginning of §2, then $G[B] = G[\bar{B}]$ and ξ is an isomorphism of $G[B]^*$ onto $G[\bar{B}]^*$.

Proof. The natural map of $\mathfrak{O}$ onto $\bar{\mathfrak{O}}$ sends $\mathfrak{C} = C(\mathfrak{O}_1 \text{ in } \mathfrak{O})$ into $\bar{\mathfrak{C}} = C(\bar{\mathfrak{O}}_1 \text{ in } \bar{\mathfrak{O}})$ and $\mathfrak{C}_\sigma$ into $\bar{\mathfrak{C}}_\sigma = \bar{\mathfrak{C}} \cap \bar{\mathfrak{O}}_\sigma$, for all $\sigma \in G$. By (2.15) it preserves the actions of G on $\mathfrak{C}$ and on $\bar{\mathfrak{C}}$. This and the uniqueness of the relation between e and $\bar{e}$ in Proposition 1.12 imply that $G_B = G_{\bar{B}}$. Proposition 2.17 tells us that both $G[B]$ and $G[\bar{B}]$ are normal subgroups of $G_B = G_{\bar{B}}$. Hence $G[\bar{B}]$ normalizes $G[B]$.

If $\sigma \in G[B]$, then $(e\mathfrak{C}_\sigma)(e\mathfrak{C}_{\sigma^{-1}}) = e\mathfrak{C}_1$ by (2.9a). Since the images of $e\mathfrak{C}_\sigma$, $e\mathfrak{C}_{\sigma^{-1}}$ are contained in $\bar{e}\bar{\mathfrak{C}}_\sigma$, $\bar{e}\bar{\mathfrak{C}}_{\sigma^{-1}}$, respectively, this implies that $\bar{e} \in (\bar{e}\bar{\mathfrak{C}}_\sigma)(\bar{e}\bar{\mathfrak{C}}_{\sigma^{-1}})$. In view of (2.8 e, f), we conclude that $(\bar{e}\bar{\mathfrak{C}}_\sigma)(\bar{e}\bar{\mathfrak{C}}_{\sigma^{-1}}) = \bar{e}\bar{\mathfrak{C}}_1$, i.e., that $\sigma \in G[\bar{B}]$. Hence $G[B]$ is a normal subgroup of $G[\bar{B}]$.

By (2.14) the natural maps of $\mathfrak{O}$ onto $\bar{\mathfrak{O}}$ and $\bar{\mathfrak{C}}[\bar{B}]$ onto $\bar{\mathfrak{F}}[G[\bar{B}]^*]$ induce an $\mathfrak{R}$-homomorphism of $\mathfrak{C}[B]$ into $\bar{\mathfrak{F}}[G[\bar{B}]^*]$ sending $\mathfrak{C}[B]_\sigma$ into $\bar{\mathfrak{F}}[G[\bar{B}]^*]_\sigma$, for all $\sigma \in G[B] \leq G[\bar{B}]$. Since $\bar{e}$ is the image of e, this $\mathfrak{R}$-homomorphism is identity-preserving. From (2.11) and the fact that $\bar{\mathfrak{F}}[G[\bar{B}]^*]_1 = \bar{\mathfrak{F}}\cdot 1$, we conclude that $J(\mathfrak{C}[B]_1)$ is in its kernel, and hence that it induces an identity-preserving homomorphism ξ^* of the algebra $\bar{\mathfrak{F}}[G[B]^*]$ into $\bar{\mathfrak{F}}[G[\bar{B}]^*]$ sending $\bar{\mathfrak{F}}[G[B]^*]_\sigma$ into $\bar{\mathfrak{F}}[G[\bar{B}]^*]_\sigma$, for all $\sigma \in G[B]$, and sending $\bar{\mathfrak{F}}[G[B]^*]_1 = \bar{\mathfrak{F}}\cdot 1$ isomorphically onto $\bar{\mathfrak{F}}[G[\bar{B}]^*]_1 = \bar{\mathfrak{F}}\cdot 1$. An application of [CCT, 13.10] now gives the second statement of the proposition with the restriction of ξ^* as ξ.

We have already seen that $G_B = G_{\bar{B}}$. From its definition ξ is clearly G_B-invariant. So the third statement holds.

When $\mathfrak{O}$, G, and the $\mathfrak{O}_\sigma$ come from H and K, then (2.6) implies that $\bar{\mathfrak{C}}_\sigma$ is the image of $\mathfrak{C}_\sigma$, for all $\sigma \in G$. If $\sigma \in G[\bar{B}]$, then this and

$$(\bar{e}\bar{\mathfrak{C}}_\sigma)(\bar{e}\bar{\mathfrak{C}}_{\sigma^{-1}}) = \bar{e}\bar{\mathfrak{C}}_1$$

imply that $(e\mathfrak{C}_\sigma)(e\mathfrak{C}_{\sigma^{-1}})$ is a two-sided ideal of $e\mathfrak{C}_1$ generating that ring modulo the kernel $e\mathfrak{C}_1 \cap \mathfrak{p}\mathfrak{O}_1$ of the epimorphism onto $\bar{e}\bar{\mathfrak{C}}_1$. But this kernel is contained in $e\mathfrak{C}_1 \cap J(\mathfrak{O}_1) \subseteq J(e\mathfrak{C}_1)$ by (1.2) and Proposition 1.9. Hence $(e\mathfrak{C}_\sigma)(e\mathfrak{C}_{\sigma^{-1}}) = e\mathfrak{C}_1$ and $\sigma \in G[B]$. This is enough to prove the last statement of the proposition.

It remains to be shown that $G[\bar{B}]/G[B]$ is either a p-group or trivial, depending on the characteristic of $\bar{\mathfrak{F}}$. For this it suffices to prove that any element $\sigma \in G[\bar{B}]$ whose order n is not divisible by the characteristic of $\bar{\mathfrak{F}}$ is also an element of $G[B]$.

We can choose an element $\bar{y} \in \bar{e}\bar{\mathfrak{C}}_\sigma = \bar{\mathfrak{C}}[\bar{B}]_\sigma$ whose image $\tilde{y}$ is a non-zero element of $\bar{\mathfrak{F}}[G[\bar{B}]^*]_\sigma$. Then $\tilde{y}^n$ is a non-zero element of $\bar{\mathfrak{F}}[G[\bar{B}]^*]_{\sigma^n} = \bar{\mathfrak{F}}\cdot 1$. Since the field $\bar{\mathfrak{F}}$ is algebraically closed, it contains an element f so that $f^n\tilde{y}^n = 1$. Replacing $\bar{y}$ by $f\bar{y}$, we can assume that $\tilde{y}^n = 1$, i.e., that

$$\bar{y}^n \in \bar{e} + J(\bar{\mathfrak{C}}[\bar{B}]_1).$$

By Proposition 1.16 there exists an element $z \in \bar{e} + J(\bar{\mathfrak{C}}[\bar{B}]_1)$ such that $\bar{y}^n z^n = \bar{e}$. Since $\bar{\mathfrak{C}}[\bar{B}]_1 = \bar{e}\bar{\mathfrak{C}}_1$ centralizes $\bar{y}$ (by (2.8b)), we have $(\bar{y}z)^n = \bar{y}^n z^n = \bar{e}$. So we can replace $\bar{y}$ by $\bar{y}z$ and assume that $\bar{y}^n = \bar{e}$.

Choose an element $y \in \mathfrak{O}_\sigma$ having $\bar{y}$ as its image. Since $\bar{y}\bar{e} = \bar{e}\bar{y} = \bar{y}$, we can replace y by eye and assume that $y \in e\mathfrak{O}_\sigma = \mathfrak{O}_\sigma e$. The power $y^n \in e\mathfrak{O}_1$ has $\bar{y}^n = \bar{e}$ as image, and hence lies in $e + \mathfrak{p}e\mathfrak{O}_1 \subseteq e + J(e\mathfrak{O}_1)$. In view of (1.7) and Proposition 1.9, this implies that $y^n \in e + J(\mathfrak{R}[y^n])$. Now Proposition 1.16 gives us an element $w \in e + J(\mathfrak{R}[y^n])$ such that $y^n w^n = e$. Since w is a polynomial in y^n (with coefficients in $\mathfrak{R}$), it commutes with y and its image in $\bar{\mathfrak{O}}_1$ is contained in $\bar{\mathfrak{F}}[\bar{y}^n] = \bar{\mathfrak{F}} \cdot \bar{e}$. It follows that we may replace y by yw to obtain an element y of $e\mathfrak{O}_\sigma$ satisfying $y^n = e$ whose image $\bar{y}$ lies in $\bar{e}\bar{\mathfrak{C}}_\sigma$.

Clearly y is a unit of $e\mathfrak{O}e$ whose inverse $y^{-1} = y^{n-1}$ lies in $e\mathfrak{O}_{\sigma^{-1}} = e\mathfrak{O}_{\sigma^{n-1}}$. From (2.1e) we see that $y^{-1}(e\mathfrak{O}_1)y = e\mathfrak{O}_1$. This gives $e\mathfrak{O}_1$ the structure of a lattice over the group ring $\mathfrak{R}\langle y\rangle$ of the cyclic subgroup $\langle y\rangle$ of order n generated by y. Because n is not divisible by the characteristic of $\bar{\mathfrak{F}}$, this group ring is the direct sum

$$\mathfrak{R}\langle y\rangle = \mathfrak{R} \oplus \cdots \oplus \mathfrak{R}$$

of n copies of $\mathfrak{R}$, with the projection onto the i^{th} copy sending y onto ζ^i, for a fixed primitive n^{th} root ζ of unity in $\mathfrak{R}$. It follows that $e\mathfrak{O}_1$ has a decomposition

$$e\mathfrak{O}_1 = \mathfrak{L}_1 \oplus \cdots \oplus \mathfrak{L}_n \quad (\textit{as } \mathfrak{R}\langle y\rangle\textit{-lattices}),$$

where y acts on the i^{th} sublattice $\mathfrak{L}_i$ by

$$(13.17) \qquad y^{-1}xy = \zeta^i x, \quad \textit{for all} \quad x \in \mathfrak{L}_i, \quad i = 1, \cdots, n.$$

Taking residues, we see that $\bar{e}\bar{\mathfrak{O}}_1$ is the direct sum of the images $\bar{\mathfrak{L}}_i = \mathfrak{L}_i / \mathfrak{p}\mathfrak{L}_i$ of the $\mathfrak{L}_i$. But the image $\bar{y} \in \bar{e}\bar{\mathfrak{C}}_\sigma$ centralizes $\bar{e}\bar{\mathfrak{O}}_1$. So conjugation by y is trivial on each $\bar{\mathfrak{L}}_i$. Since the image of ζ is a primitive n^{th} root of unity in $\bar{\mathfrak{F}}$, this and (13.17) imply that $\bar{\mathfrak{L}}_1 = \bar{\mathfrak{L}}_2 = \cdots = \bar{\mathfrak{L}}_{n-1} = \{0\}$. By the Nakayama Lemma this forces $\mathfrak{L}_1 = \cdots = \mathfrak{L}_{n-1} = \{0\}$. Hence $e\mathfrak{O}_1 = \mathfrak{L}_n$. Therefore y centralizes $e\mathfrak{O}_1$, i.e., $y \in C(e\mathfrak{O}_1 \text{ in } e\mathfrak{O}_\sigma) = e\mathfrak{C}_\sigma$.

Now it is clear that $y^{-1} = y^{n-1} \in e\mathfrak{C}_{\sigma^{-1}}$ and $e = yy^{-1} \in (e\mathfrak{C}_\sigma)\ (e\mathfrak{C}_{\sigma^{-1}})$. In view of (2.8e, f) this implies that $(e\mathfrak{C}_\sigma)(e\mathfrak{C}_{\sigma^{-1}}) = e\mathfrak{C}_1$. Therefore $\sigma \in G[B]$, and the proof of the proposition is complete.

Index of notation

We list here certain symbols and definitions used throughout the paper followed by the number of the statement in which they are defined, or the number of the statement next following their definition. The reader should notice that certain symbols which have a general meaning in one section may have a more restricted one in others. Thus $\mathfrak{O}$, which denotes a general $\mathfrak{R}$-order in §1, is fixed from (2.1) on as a specific $\mathfrak{R}$-order. In §7 and §§10–12 it has even more restricted meanings, while in the intervening §8 and §9 it reverts to its definition in (2.1). In these cases the notation used in a given section is indicated at or near the beginning of that section. Furthermore, we list in this index the places at which the meaning of the symbol in question changes as well as that at which it was originally fixed.

Finally, there are certain notational conventions used repeatedly in the paper which are summarized at the end of this index.

In general the centralizer of X in Y, however defined, is denoted by $C(X \text{ in } Y)$. Similarly the normalizer of X in Y is denoted by $N(X \text{ in } Y)$, except in a few special cases noted above when it is just called $N(X)$. There is one exception to this notation when X is a group acting on a ring Y. In that case the centralizer $C(X \text{ in } Y)$ is usually denoted by $Y(X)$, a notation introduced in §4. In this situation the expression $Y(X \mid Z)$ stands for the image of the trace map $\mathrm{tr}_{Z\to X} : Y(Z) \to Y(X)$, where Z is some subgroup of X (see §4, particularly (4.1)–(4.4)). The same notation is often used for the centralizer of X and the image of the trace map when Y is just some X-invariant submodule of a ring on which X acts.

When a group X acts on a set Y the stabilizer in X of a point $y \in Y$ is usually denoted by X_y. This should not be confused with the special notation $H(D)_\omega$, $N(D)_\omega$ and $H(D)^*\langle\varphi\rangle_\omega$ concerning the bilinear form ω of §11. Nor should it be confused with the notation X_y for the y^{th} component, $y \in Y$, of the grading of a ring X with respect to a group Y.

In general the expression X^* is reserved for a central extension of $\bar{F}$ by a group X. In that case $\bar{\mathfrak{F}}[X^*]$ denotes the corresponding twisted group algebra of X over $\bar{\mathfrak{F}}$.

Throughout the paper the superscript J is used to denote the images in the factor ring $\bar{\mathfrak{F}}[G[B]^*] = \mathfrak{C}[B]/\mathfrak{C}[B]J(\mathfrak{C}[B]_1)$ of elements or subsets of $\mathfrak{C}[B]$. In §§10–12 the superscript I is similarly used to denote the images in $\bar{\mathfrak{F}}[H^*] = \mathfrak{D}/\mathfrak{D}J(\mathfrak{D}_1)$ of elements or subsets of $\mathfrak{D}$. One should note that the superscript J enters into certain expressions, such as $\mathfrak{C}[B](E_B \parallel D)^J$ and $\mathfrak{C}[B]_{(D)}(N(D) \parallel D)^J$ listed above, which are defined directly in $\bar{\mathfrak{F}}[G[B]^*]$ and not as images (although, of course, they could have been so defined).

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