A NEW RELATION ON THE STIEFEL-WHITNEY CLASSES OF SPIN MANIFOLDS

BY

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1. Introduction

Let all manifolds considered be n-dimensional, closed, compact, connected C^∞ manifolds. Let $\tau_M : M \to BO$ be the classifying map for the stable tangent bundle of M. Recall that $H^*(BO)$ is a polynomial algebra on the universal Stiefel-Whitney classes, $w_1, w_2, \cdots$, where all coefficients are Z_2. Let $S \subset H^*(BO)$, then define $I_n(S, \text{geom}) \subset H^*(BO)$ to be the ideal $\bigcap_M \ker \tau_M^*$ where the intersection is taken over all n-dimensional manifolds with $S \subset \ker \tau_M^*$. Let H be an n-dimensional Poincaré algebra. There is a unique right-left A-homomorphism $\tau_H : H^*(BO) \to H$, A the Steenrod algebra (see Brown-Peterson [5, Lemma 5.1, p. 44]). Define $I_n(S, \text{alg}) \subset H^*(BO)$ to be the ideal given by $\bigcap_H \ker \tau_H$ where H runs over all n-dimensional Poincaré algebras such that $\tau_H(S) = 0$.

For the cases $S = \emptyset$, $\{w_1\}$, $I_n(S, \text{geom})$ corresponds to the intersection of $\ker \tau_M^*$ taken over all manifolds, and respectively, all oriented manifolds. For these two cases, Brown and Peterson show that $I_n(S, \text{geom}) = I_n(S, \text{alg})$ [5, Theorems 5.2 and 5.4, p. 45]. Clearly, one has $I_n(S, \text{alg}) \subset I_n(S, \text{geom})$ for all S. [5, p. 45] gives an example to show that equality does not always hold.

In this paper, the case where $S = \{w_1, w_2\}$ will be considered. $I_n(\{w_1, w_2\}, \text{geom})$ corresponds to the intersection of $\ker \tau_M^*$ where M runs over all n-dimensional Spin manifolds.

Let $BO\langle k\rangle$ be the $k - 1$ connective covering over BO and $\mathbf{BO}\langle k\rangle$ the connected Ω-spectrum with 0^{th} term $BO\langle k\rangle$. For $k = 0$ and 2, the bottom cohomology classes of $\mathbf{BO}\langle 0\rangle$ and $\mathbf{BO}\langle 2\rangle$ induce maps

$$\eta : H_*(X, \mathbf{BO}\langle 0\rangle) \to H_*(X) \quad \text{and} \quad \gamma : H_*(X, \mathbf{BO}\langle 2\rangle) \to H_{*-2}(X)$$

on the generalized homology [12]. In Section 2, the computation of $I_n(\{w_1, w_2\}, \text{geom})^q$ is reduced to a problem about the image of the maps η and γ for the space $X = K(Z_2, n - q)$. This reduction is a generalization to Spin of Brown-Peterson results for SO [5]. The major part of this paper is devoted to obtaining certain information about the image of η for $X = K(Z_2, 2)$. These results are stated in Section 2 and used there to prove that $I_n(\{w_1, w_2\}, \text{geom})$ is not equal to $I_n(\{w_1, w_2\}, \text{alg})$ in general. In particular, it will be shown that

$$w_7 \in I_9(\{w_1, w_2\}, \text{geom})^7 \quad \text{but} \quad w_7 \notin I_9(\{w_1, w_2\}, \text{alg})^7.$$

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These results on the image of η give a partial computation of

$$I_n(\{w_1, w_2\}, \text{geom})^{n-2}.$$

The proof of the main results will be deferred until Section 5. In Section 3. generalities about η and γ are proven. These are used in Section 4 to determine $I_n(\{w_1, w_2\}, \text{geom})^q$ for $q = n, n-1$, and $q \leq n/2$ where it is shown to be equal to $I_n(\{w_1, w_2\}, \text{alg})^q$. Methods for obtaining more information about higher codimensions are given in Section 6.

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2. $I_n(Spin)$

Brown-Peterson [5] is the reference for all of Section 2.

Define $I_n(Spin) \subset H^*(BSpin)$ to be $\bigcap_M \ker \tau_M^*$ where the intersection is taken over all n-dimensional Spin manifolds and $\tau_M : M \to BSpin$ is the classifying map for the stable tangent bundle.

Let (M, f) represent an element in the Spin bordism group, $\Omega_*(X, Spin)$. M is a Spin manifold and $f : M \to X$. Define

$$\Psi_q : H^*(BSpin) \to \Omega_*(K(Z_2, q), Spin)^*$$

by $\Psi_q(u)(M, f) = (\tau_M^*(u) \cdot f^*(c^q))(M) \in Z_2$, where $c^q \in H^q(K(Z_2, q))$ is the generator. Ψ_q is well defined. $H^*(BSpin)$ is a right A-module where the action is given by

$$u \cdot a = \Phi^{-1}(\chi(a)\Phi(u)),$$

$\Phi : H^*(BSpin) \to H^*(\mathbf{MSpin})$ is the Thom isomorphism and χ is the canonical anti-automorphism. Define

$$\theta(X) : H^*(BSpin) \otimes_A H^*(X) \to \Omega_*(X, Spin)^*$$

by $\theta(X)(u \otimes_A v)(M, f) = (\tau_M^*(u) \cdot f^*(v))(M) \in Z_2$. We have then

$$\Psi_q = \theta(K(Z_2, q)) \cdot \lambda_q$$

where $\lambda_q : H^*(BSpin) \to H^*(BSpin) \otimes_A H^*(K(Z_2, q))$ is defined by

$$\lambda_q(u) = u \otimes_A c^q.$$

THEOREM 2.1 (Brown-Peterson, Lemma 2.2).

$$I_n(Spin)^q = \ker (\Psi_{n-q} \mid H^q(BSpin)).$$

Before going further we need to recall some results on **MSpin** from Anderson-Brown-Peterson [3, p. 272]. Let $J = (j_1, \cdots, j_k)$ be a sequence of integers such that $k \geq 0$ and $j_i > 1$. Let $n(J) = \sum j_i$. Let **MSpin** be the Thom spectrum associated to the Spin groups. Let J be such that $n(J)$ is even, and J' for $n(J')$ odd. Then there are maps

$$f_J : \mathbf{MSpin} \to \mathbf{BO}\langle 4n(J)\rangle \quad \text{and} \quad f_{J'} : \mathbf{MSpin} \to \mathbf{BO}\langle 4n(J') - 2\rangle$$

defined in [3]. If $z \in H^n(\mathbf{MSpin})$, let $f_z : \mathbf{MSpin} \to \mathbf{K}(Z_2, n)$ denote the corresponding spectrum map. The main result we need is:

THEOREM 2.2 (Anderson-Brown-Peterson). *There is a collection of elements $z_i \in H^*(\mathbf{MSpin})$ such that the map*

$$F : \mathbf{MSpin} \to \textstyle\prod_J \mathbf{BO}\langle 4n(J)\rangle \times \prod_{J'} \mathbf{BO}\langle 4n(J') - 2\rangle \times \prod \mathbf{K}(Z_2, \dim z_i) = \mathbf{L},$$

given by $F = \prod f_J \times \prod f_{J'} \times \prod f_{z_i}$, induces an isomorphism on Z_2 cohomology.

Thus, there is a C_2 isomorphism

$$F(X) : H_*(X, \mathbf{MSpin}) \to H_*(X, \mathbf{L})$$

where C_2 is the class of groups of odd order. There is also an equivalence

$$\sigma(X) : \Omega_*(X, Spin) \to H_*(X, \mathbf{MSpin})$$

similar to the one for SO in Conner and Floyd [6].

LEMMA 2.3. *There are elements*

$$P_J \in H^{4n(J)}(BSpin), \quad Q_{J'} \in H^{4n(J')-2}(BSpin) \quad \text{and} \quad N_i = \Phi^{-1}(z_i)$$

such that $H^*(BSpin)$, as a right A-module, is generated by $\{P_{J'} Q_{J'}, N_i\}$ with

$$P_J Sq^1 A = P_J Sq^2 A = Q_{J'} Sq^2 Sq^1 A = 0$$

as the only relations. Also, $P_J = \Phi^{-1}(f_J^* \alpha)$ and $Q_{J'} = \Phi^{-1}(f_{J'}^* \alpha)$ where α is the bottom cohomology class of $\mathbf{BO}\langle n\rangle$ for $n = 0$ or $2 \bmod 8$.

Proof. This follows from 2.2 and the fact that

$$H^*(\mathbf{BO}\langle 8k\rangle) = A/A(Sq^1, Sq^2) \quad \text{and} \quad H^*(\mathbf{BO}\langle 8k + 2\rangle) = A/A(Sq^3) \quad [9].$$

Define $R(X) \subset H^*(X) = H_*(X)^*$ by $R(X) = \ker \eta^*$, and $S(X) \subset H^*(X)$ by $S(X) = \ker \gamma^*$.

LEMMA 2.4. *If X is a CW-complex with a finite number of cells in each dimension, then the kernel of $\theta(X)$ is given by*

$$\{P_J \otimes_A R(X), Q_{J'} \otimes_A S(X)\}.$$

Proof. With minor modifications, the proof is the same as for Lemma 4.2 of [5, p. 47].

Let c^s be the fundamental class of $K(Z_2, s)$. Define

$$T_s = \{a \in A \mid ac^s = 0\}, \quad R_s = \{a \in A \mid ac^s \in R(K(Z_2, s))\},$$

and

$$S_s = \{a \in A \mid ac^s \in S(K(Z_2, s))\}.$$

Let $P \subset H^*(BSpin)$ be the vector space with basis $P_{J'}$ and Q the vector space with basis $Q_{J'}$. Define

$$F_n = \sum_{2j>n-i} H^i(BSpin)Sq^j.$$

We have:

THEOREM 2.5. $I_n(Spin)^q = (F_n)^q + \sum_i P^i R_{n-q}^{q-i} + \sum_j Q^j S_{n-q}^{q-j}$.

Proof. Similar to 4.3 of [5, p. 43].

We can now state our main result and give its application. In Section 5 we prove the following:

THEOREM 2.6. *For I admissible, $I = (i_1, \cdots, i_k)$, if $e(I) \geq 2$, then all $Sq^I \in R_2$ except those with $i_k = 2$. If $e(I) = 1$, and $k \geq 5$, then $Sq^I \notin R_2$; if $k \leq 3$, then $Sq^I \in R_2$; for $k = 4$, not known.*

COROLLARY 2.7.

$$w_7 \in I_9(\{w_1, w_2\}, \text{geom})^7 \quad \text{but} \quad \notin I_9(\{w_1, w_2\}, \text{alg})^7.$$

Proof. For $J = \emptyset$, $P_J = 1 \in H^*(BSpin)$, so by Theorem 2.5,

$$(1)Sq^4Sq^2Sq^1 = w_7 \in I_9(Spin)^7 \Rightarrow w_7 \in I_9(\{w_1, w_2\}, \text{geom})^7.$$

Now we will construct a 9-dimensional Poincaré algebra with $w_1 = w_2 = 0$ and $w_7 \neq 0$. Then, by definition,

$$w_7 \notin I_9(\{w_1, w_2\}, \text{alg})^7$$

and this will conclude the proof. Let x_2 be a 2-dim class, and let

$$Sq^1x_2. \quad Sq^2Sq^1x_2, \quad Sq^4Sq^2Sq^1x_2, \quad w_4, \quad w_6, \quad \text{and} \quad w_7$$

be non-zero classes with w_4, w_6, and w_7 the Stiefel-Whitney classes. Then the action of A and the cup products are given by the following equalities:

$$1 \cdot Sq^4Sq^2Sq^1x_2 = (1)Sq^4 \cdot Sq^2Sq^1x_2 = w_4 \cdot Sq^2Sq^1x_2 = w_4 Sq^2 \cdot Sq^1x_2$$
$$= w_6 \cdot Sq^1x_2 = w_6 Sq^1 \cdot x_2 = w_7 \cdot x_2 \neq 0.$$

This gives the desired Poincaré algebra.

3. R_s and S_s

To study R_s we will consider the map

$$\eta : H_*(K(Z_2, s), \mathbf{BO}\langle 0\rangle) \to H_*(K(Z_2, s)), \quad R(K(Z_2, s)) = \ker \eta^*,$$

$$\eta^* : H^*(K(Z_2, s)) \to H_*(K(Z_2, s), \mathbf{BO}\langle 0\rangle)^*.$$

So for $a \in A$, $ac^s(\text{image } \eta) = 0$ iff $a \in R_s$. $H_*(K(Z_2, s), \mathbf{BO}\langle 0\rangle)$ is just the stable homotopy group of $K(Z_2, s) \wedge \mathbf{BO}\langle 0\rangle$. We will study the Adams spectral sequence for this [1]. The E_2 term is

$$\text{Ext}_A(H^*(K(Z_2, s)) \otimes H^*(\mathbf{BO}\langle 0\rangle), Z_2).$$

Now $H^*(\mathbf{BO}\langle 0\rangle)$ is $A/A(Sq^1, Sq^2) = A \otimes_{A_1} Z_2$ where A_1 is the sub-Hopf algebra generated by Sq^1 and Sq^2. So

$$E_2 = \mathrm{Ext}_A\ (H^*(K(Z_2, s)) \otimes (A \otimes_{A_1} Z_2), Z_2)$$

and by Anderson-Brown-Peterson [2, p. 464] or Peterson [8], we have this is isomorphic to

$$\mathrm{Ext}_A\ (A \otimes_{A_1} H^*(K(Z_2, 2)), Z_2).$$

By a change of rings theorem, this is $\mathrm{Ext}_{A_1}\ (H^*(K(Z_2, s)), Z_2)$, from [7].

Now, in the Adams spectral sequence for $H_*(K(Z_2, s))$, $E_2 = E_\infty$ is

$$\mathrm{Hom}_{Z_2}\ (H^*(K(Z_2, s)), Z_2)$$

and is in $E^{*,0}$. So, since we are only interested in the image of η, we need only look at the $E^{*,0}$ part of

$$\mathrm{Ext}_{A_1}\ (H^*(K(Z_2, s)), Z_2)$$

which is just $\mathrm{Hom}_{A_1}\ (H^*(K(Z_2, s)), Z_2)$. It induces a map

$$\eta' : \mathrm{Hom}_{A_1}\ (H^*(K(Z_2, s)), Z_2) \to \mathrm{Hom}_{Z_2}\ (H^*(K(Z_2, s)), Z_2);$$

thus, we have:

PROPOSITION 3.1. *$a \in A$ is in R_s iff there is no $f \in \mathrm{Hom}_{A_1}\ (H^*(K(Z_2, s)), Z_2)$ such that*

$$\eta'(f)(ac^s) \neq 0 \quad \textit{and} \quad d_r f = 0 \quad \textit{for all } r.$$

PROPOSITION 3.2. *If $\eta'(f)(Sq^I c^s) = 0\ \forall f\ (\Rightarrow Sq^I \in R_s)$, then*

$$P_J\, Sq^I \in I_n(\{w_1, w_2\}, \mathrm{alg})^{n-s}$$

where $s + n(I) + 4n(J) = n$.

Proof.

$$\eta'(f)(Sq^I c^s) = 0\ \forall f \quad \Leftrightarrow Sq^I c^s = Sq^1\theta(c^s) + Sq^2\theta'(c^s)$$

where θ and θ' are possibly unstable operations. Then for any n-dimensional Poincaré algebra H, with

$$\tau_H(\{w_1, w_2\}) = 0,$$

$$\tau_H(P_J\, Sq^I)\cdot z = \tau_H(P_J)\cdot Sq^I z = \tau_H(P_J\, Sq^1)\cdot \theta(z) + \tau_H(P_J\, Sq^2)\cdot \theta'(z) = 0$$

for any z of dim s. Therefore, $\tau_H(P_J\, Sq^I) = 0$.

Conjecture. If for $Sq^I c^s$, I admissible, there is an

$$f \in \mathrm{Hom}_{A_1}\ (H^*(K(Z_2, s)), Z_2)$$

such that $f(Sq^I c^s) \neq 0$, that is, $Sq^I c^s$ is not in the image of $Sq^1 + Sq^2$, then

$$P_J\, Sq^I \notin I_n(\{w_1, w_2\}, \mathrm{alg})^{n-2}, \quad n = n(I) + 4n(J) + s.$$

For S_s, the same type of analysis can be made. E_2 is

$$\mathrm{Ext}_{A_1}\ (H^*(K(Z_2, s)) \otimes A_1/A_1(Sq^3), Z_2).$$

The corresponding propositions are:

PROPOSITION 3.3. *$a \in A$ is in S_s iff there is no $f \in E_2^{*,0}$ such that*

$$\gamma'(f)(Sq^I c^s) \neq 0 \quad \textit{and} \quad d_r f = 0 \quad \textit{for all } r.$$

PROPOSITION 3.4. *If $\gamma'(f)(Sq^I c^s) = 0\ \forall f$, ($\Rightarrow Sq^I \in S_s$) then*

$$Q_{J'} Sq^I \in I_n(\{w_1, w_2\}, \text{alg})^{n-s}, \quad n = s + n(I) + 4n(J') - 2.$$

(*Note*: $\gamma'(f)(Sq^I c^s) = 0\ \forall f \quad \Leftrightarrow \quad Sq^I c^s = Sq^2 Sq^1 \theta c^s$.)

Conjecture. If for $Sq^I c^s$, I admissible, there is an $f \in E_2^{*,0}$ such that $f(Sq^I c^s) \neq 0$, that is, $Sq^I c^s$ is not in the image of $Sq^2 Sq^1$, then

$$Q_{J'} Sq^I \notin I_n(\{w_1, w_2\}, \text{alg})^{n-s}, \quad n = s + n(I) + n(J') - 2.$$

4. $I_n(Spin)^q$, $q = n, n-1$, and $q \leq n/2$.

PROPOSITION 4.1.

$$I_n(Spin)^n = (F_n)^n = \textstyle\sum_{j>0} H^{n-j}(BSpin)Sq^j.$$

Proof. This follows from 2.5.

$$I_n(\{w_1, w_2\}, \text{geom})^n = (F_n)^n + \langle\{w_1, w_2\}\rangle$$

where $\langle\{w_1, w_2\}\rangle$ denotes the ideal over A generated by $\{w_1, w_2\}$. So we have the following well-known result [3, p. 273]:

PROPOSITION 4.2.

$$I_n(\{w_1, w_2\}, \text{geom})^n = I_n(\{w_1, w_2\}, \text{alg})^n.$$

Proof. $F_n \subset I_n(\{w_1, w_2\}, \text{alg})$ [5, p. 42].

THEOREM 4.3. *$I_n(Spin)^q = 0$ for $q \leq n/2$.*

Proof. For dimensions $\leq 2s$, $\tilde{H}^*(K(Z_2, s))$ is isomorphic to A where 1 goes to c^s. A is a free A_1-module, so $\tilde{H}^*(K(Z_2, s))$ is free up to dim $2s$. Therefore, E_2 is in $E_2^{*,0}$ for dim $\leq 2s$ and all differentials are zero. For $n(I) \leq s$, by 3.3 and 3.1, $Sq^I = Sq^1 a + Sq^2 a'$ for $Sq^I \in R_s$, and $Sq^I = Sq^2 Sq^1 b$ for $Sq^I \in S_s$, a, a', and $b \in A$, since $n(I) \leq s$. Therefore, $P_J Sq^I$ or $Q_{J'} Sq^I = 0$. In particular, if dimension $\{P_J \text{ or } Q_{J'}\} + n(I) \leq s$, this is always true. So if $n - q = s \geq n/2$, the $n(I) \leq n/2 \leq s$ and all terms of this type are zero. Now for $i + j = q \leq n/2$,

$$(F_n)^q = \textstyle\sum_{j+q>n} H^i(BSpin)Sq^j = \sum_{j>n/2\geq q\geq j} H^i(BSpin)Sq^j = 0.$$

For $s = 1$, the only Sq^I not in T_1 are those with $e(I) = 1$. So for $Sq^I \neq Sq^1$ (Sq^1 is handled by 4.3),

$$Sq^I c^1 = (c^1)^{2k} = Sq^2 Sq^1 (c^1)^{2k-3},$$

so by 3.2 and 3.4 we have:

THEOREM 4.4.

$$I_n(\{w_1, w_2\}, \text{geom})^{n-1} = I_n(\{w_1, w_2\}, \text{alg})^{n-1}.$$

5. R_2

The proof of Theorem 2.6 is divided up into the three lemmas of this section.

All Sq^I with $e(I) > 2$ are in R_2, but since $Sq^I c^2 = 0 \in H^*(K(Z_2, 2))$, all relations given by these Sq^I are in $I_n(\{w_1, w_2\}, \text{alg})^{n-2}$ by 3.2. This leaves Sq^I with $e(I) = 1$ or 2. For $I = (i_1, \cdots, i_r, i_{r+1}, \cdots, i_k)$ with I admissible, $i_k = 1$, $i_r = 2i_{r+1} + 1$, $e(I) = 2$, then

$$\begin{aligned} Sq^I c^2 &= (Sq^{i_r} Sq^{i_{r+1}} \cdots Sq^{i_k} c^2)^{2^{r-1}} \\ &= Sq^1\{(Sq^{i_r} \cdots Sq^{i_k} c^2)^{2^{r-1}-1} (Sq^{i_r - 1} Sq^{i_{r+1}} \cdots Sq^{i_k} c^2)\}. \end{aligned}$$

So such an I is contained in R_2 but gives rise to relations in $I_n(\{w_1, w_2\}, \text{alg})^{n-2}$ by Theorem 3.2. The only other I with $e(I) = 2$ are those with $i_k = 2$. In this case $Sq^I c^2 = (c^2)^{2^k}$, excluding $k = 1$ which is taken care of by 4.3. It is easily seen that this is not in the image of $Sq^1 + Sq^2$, likewise, in $H^*(BSO)$, $w_2^{2^k}$ is not in the image of $Sq^1 + Sq^2$. Consider the map $w_2 : BSO \to K(Z_2, 2)$ realizing w_2. It induces a map w_2' on the E_2 term of the Adams spectral sequences for

$$H_*(BSO, \mathbf{BO}\langle 0\rangle) \to H_*(K(Z_2, 2), \mathbf{BO}\langle 0\rangle).$$

Now since $(w_2)^{2^k}$ is not in the image of $Sq^1 + Sq^2$ there is a map

$$f \in \text{Hom}_{A_1}(H^*(BSO), Z_2) = E_2^{*,0}$$

with $f((w_2)^{2^k}) \neq 0$. From Anderson-Brown-Peterson [2, p. 468] or Peterson [8], we know that $d_r f = 0 \ \forall r$. Therefore

$$(w_2' f)(c^2)^{2^k} \neq 0 \quad \text{and} \quad d_r(w_2'(f)) = w_2'(d_r f) = 0 \ \forall r.$$

Thus $Sq^I \notin R_2$ by 3.1.

The only I left are those with $e(I) = 1$. Summarizing, thus far we have:

LEMMA 5.1. *For I admissible $I = (i_1, i_2, \cdots, i_k)$, $e(I) \geq 2$, all $Sq^I \in R_2$ except those with $e(I) = 2$ and $i_k = 2$.*

The main results of this section will depend on an explicit calculation of part of $\text{Ext}_{A_1}(H^*(K(Z_2, 2)), Z_2)$. The major part of this task is to calculate the A_1 structure of $H^*(K(Z_2, 2))$ as far as we need it, which is up to about dim 40. Below dim 40, $H^*(K(Z_2, 2))$, denoted by M, is a polynomial algebra with generators in dimensions 2, 3, 5, 9, 17, and 33 which are

$$c^2 = u_2, \quad Sq^1 c^2 = u_3, \quad Sq^2 Sq^1 c^2 = u_5, \quad Sq^4 Sq^2 Sq^1 c^2 = u_9,$$

etc. respectively (Serre [9]). Our approach to the problem of finding the A_1

structure of M will be to break it up as an A_1 module into the tensor product of two A_1 modules B and C.

B is defined as $P(u_2, u_3, u_5) \otimes E(u_9)$ where E denotes the exterior algebra and P the polynomial algebra. As a Z_2 module,

$$M = B \otimes P((u_9)^2, u_{17}, u_{33}).$$

We wish to replace $(u_9)^2$, u_{17}, u_{33} with equivalent polynomial generators u'_{18}, u'_{17}, u'_{33} such that

$$C = P(u'_{18}, u'_{17}, u'_{33}) \quad \text{and} \quad M = B \otimes C$$

as A_1 modules. To do this, let

$$u'_{17} = u_{17} + u_5\{(u_2)^6 + (u_3)^2(u_2)^3 + (u_5)^2 u_2 + u_5 u_3 (u_2)^2\} = u_{17} + X,$$

$$u'_{18} = (u_9)^2 + Sq^1 X,$$

$$\begin{aligned} u'_{33} = {} & u_{33} + u_9(u_5)^4(u_2)^2 + u_9(u_5)^3(u_3)^3 + u_9(u_5)^2(u_3)^4 u_2 + u_9 u_5 (u_3)^5 (u_2)^2 \\ & + u_9(u_3)^6(u_2)^3 + u_9(u_3)^4(u_2)^6 + u_9(u_2)^{12} + (u_5)^5(u_3)^2 u_2 + (u_5)^5(u_2)^4 \\ & + (u_5)^4(u_3)^3(u_2)^2 + (u_5)^3(u_3)^6 + (u_5)^3(u_3)^4(u_2)^3 + (u_5)^2(u_3)^7 u_2 \\ & + (u_5)^2(u_3)^5(u_2)^4 + u_5(u_3)^8(u_2)^2 + (u_3)^9(u_2)^3. \end{aligned}$$

These are obviously polynomial generators, so all that needs to be shown is that B and C are closed under A_1 action. The A_1 action on M is described by the Cartan formula and the following relations:

$$Sq^1 u_2 = u_3, \quad Sq^2 u_2 = (u_2)^2, \quad Sq^1 u_3 = 0,$$

$$Sq^2 u_3 = u_5, \quad Sq^1 u_5 = (u_3)^2,$$

$$Sq^2 u_5 = Sq^2 u_9 = Sq^2 u_{17} = Sq^2 u_{33} = 0,$$

$$Sq^1 u_9 = (u_5)^2, \quad Sq^1 u_{17} = (u_9)^2, \quad Sq^1 u_{33} = (u_{17})^2.$$

Thus, it is merely a calculation to show

$$Sq^1 u'_{17} = u'_{18}, \quad Sq^1 u'_{18} = 0, \quad Sq^1 u'_{33} = (u'_{17})^2$$

and that Sq^2 is zero on u'_{17}, u'_{18}, and u'_{33}, giving us that C is closed under A_1 action; thus, $M = B \otimes C$ as an A_1 module.

We now turn our attention to B. It is easily verified that an element of the form u_2^{4k} is not in the image of $Sq^1 + Sq^2$ and can therefore be used as an A_1 generator which is trivial. This fact implies that B is isomorphic as an A_1 module to $D \otimes T$ where

$$T = P(u_2^4) \quad \text{and} \quad D = P(u_3, u_5) \otimes E(u_2, u_2^2, u_9)$$

where when u_2^4 occurs in D from the A_1 action it can be set $= 0$.

To summarize the progress thus far, we have broken M up into $M = D \otimes T \otimes C$ as an A_1 module where T is a trivial module. The only nice property C has is that it contains very few elements below dimension forty. D has the property that all but a finite part of it is free. We will now show this and give the A_1 structure of the non-free part.

To do this, calculate the Q_0 and Q_1 homology of D. The Q_0 homology is $E(u_2^2, u_5 + u_3u_2)$. The Q_1 homology is $E(u_2^2, u_9 + u_3^3)$. Now, by a calculation, the low dimensional A_1 structure of D can be determined. A part that contains these elements splits off as a direct sum from the rest of D and therefore by a theorem of Wall the rest is free [11, p. 253]. We give a tabulation of the low dimensional part of D. D_1 is $A_1/A_1(Sq^3)$ with generator u_2. D_2 is $A_1/A_1(Sq^1)$ with generator $u_5 + u_3u_2 = v_1$ of dim 5 and Z_2 with generator $u_9 + u_3^3 + u_5u_2^2 = v_2$ of dim 9 with the non-trivial extension $Sq^1v_2 = Sq^2Sq^1Sq^2v_1$. D_3 is $A_1/A_1(Sq^1)$ with generator $v_3 = u_3u_2^3 + u_5u_2^2 + u_3^3$ of dim 9 and $A_1/A_1(Sq^3)$ with generator $u_9u_2 + u_5u_3^2 + u_3^3u_2 = v_4$ of dim 11 with non-trivial extension $Sq^2Sq^1Sq^2v_3 = Sq^3v_4$. Also we have free generators

$$u_2^3, \quad u_5u_3u_2, \quad u_3^2u_2^3, \quad u_9u_3u_2 \quad \text{and} \quad u_5u_3u_2^3.$$

We now have an explicit calculation of the non-free A_1 structure of D and T. We need the same for C. By construction this is easy. In dim 17 with generator u'_{17}, we have an $A_1/A_1(Sq^2, Sq^2Sq^1)$ module C_1. In dim 33 with generator u'_{33}, we have an $A_1/A_1(Sq^2)$ and in dim 35 with generator $u'_{17}u'_{18}$ a Z_2 with non-trivial extension $Sq^1u'_{18}u'_{17} = Sq^2Sq^1u'_{33}$. Call this C_2.

Now to get the A_1 structure as far as we need it we must determine the non-free parts of the following modules: $C_1 \otimes D_i$ for $i = 1, 2, 3$. These are straightforward calculations. $C_1 \otimes D_1$ is $A_1/A_1(Sq^3)$ in dimensions 19 and 20, generators v_{19} and v_{20} with $Sq^3v_{19} = Sq^2v_{20}$. $C_1 \otimes D_3$ is the same but with generators in dimensions 28 and 29. $C_1 \otimes D_2 = A_1/A_1(Sq^2, Sq^2Sq^1)$ in dim 26.

We can now write down the entire A_1 structure of M as far as we need it. All we need now is Ext of this. We need $\mathrm{Ext}_{A_1}(\cdot, Z_2)$ of the following modules:

$$Z_2, A_1/A_1(Sq^3), A_1/A_1(Sq^2, Sq^2Sq^1), D_2, D_3, C_1 \otimes D_1, \quad \text{and} \quad C_2.$$

For Z_2 and $A_1/A_1(Sq^3)$, Ext is of period 8 and is as in Figures 1 and 2 where arrows denote towers. (See Anderson-Brown-Peterson [2].) Ext of $A_1/A_1(Sq^1)$ is just a tower, so Ext for D_2 is the sum for $A_1/A_1(Sq^1)$ and Z_2. For D_3, again Ext is the sum for $A_1/A_1(Sq^1)$ and $A_1/A_1(Sq^3)$.

For the last three cases, since the Q_0 homology of each is zero, we can apply a theorem of Adams that says there are no towers in Ext. We can find E_1 without considering the extensions and then kill off the towers. For example consider $A_1/A_1(Sq^2, Sq^2Sq^1)$ as $Z_2 + Z_2$ with non-trivial Sq^1. Then E_1 is given in Figure 3 and Ext in Figure 4, both of period 8. (See Peterson [8].) Similarly for E_1 and Ext for $C_1 \otimes D_1$, see Figures 5 and 6, for C_2, 7 and 8.

$\mathrm{Ext}_{A_1}(M, Z_2)$ can now be written out as far as we need it. To calculate the tower killing differentials, map the entire spectral sequence into the

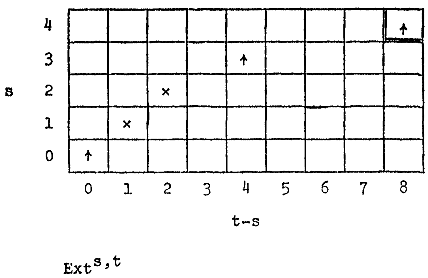

FIGURE 1

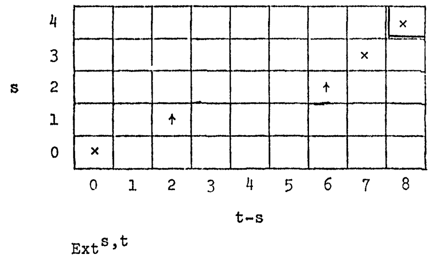

FIGURE 2

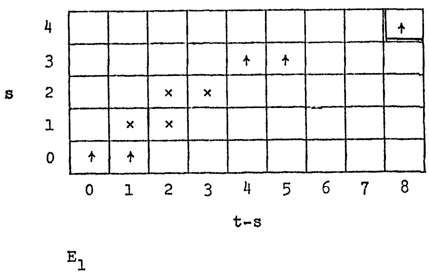

FIGURE 3

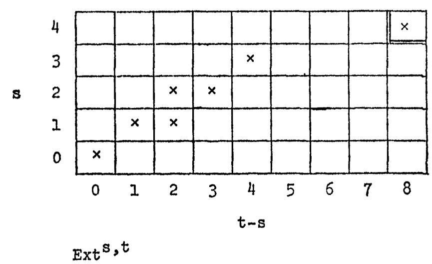

FIGURE 4

Adams spectral sequence for the integer homology of $K(Z_2, 2)$ by mapping the bottom class of $\mathbf{BO}\langle 0\rangle$ to $\mathbf{K}(Z, 0)$. Here we can observe which elements in E_2 correspond. The differentials in the $\mathbf{K}(Z, 0)$ spectral sequence are known (Browder [4]) and they are just the higher order Bocksteins. These impose the same tower killing differentials in our spectral sequence.

The actual calculation gives: Towers formed by D_2 kill off those of D_1 by a d_2. Likewise for towers formed by $D_2 \otimes u_2^{4k}$ and $D_1 \otimes u_2^{4k}$. Towers from D_3 kill off those from u_2^4 by a d_3. $D_3 \otimes u_2^4$ kill u_2^8 by a d_4. $D_3 \otimes u_2^8$ kill u_2^{12} by d_3 and $D_3 \otimes u_2^{12}$ kill u_2^{16} by d_5.

Now one can write doen the entire spectral sequence after the tower killing differentials and then stare, very hard. The main lemma follows:

LEMMA 5.2.

$$Sq^4Sq^2Sq^1 \in R_2, \quad Sq^8Sq^4Sq^2Sq^1 \text{ ? } R_2,$$

and

$$Sq^{16}Sq^8Sq^4Sq^2Sq^1 \notin R_2.$$

Proof. The first is easy because $E^{9,0}$ in the final version of our spectral sequence is zero and so it follows from 3.1. For the second, we look at the corresponding element in $E^{17,0}$ and we see that there may be a non-trivial d_4 on it. If $d_4 = 0$ then it is not in R_2, if $d_4 \neq 0$, then it is in R_2. Let $c \in E^{33,0}$ coming from Ext of C_2 correspond to the final case. The only elements in $E^{32,*}$ which could possibly be hit by a non-trivial differential from c are truncated towers, all of which have a non-trivial product with τ from

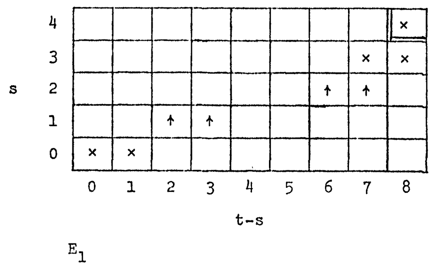

FIGURE 5

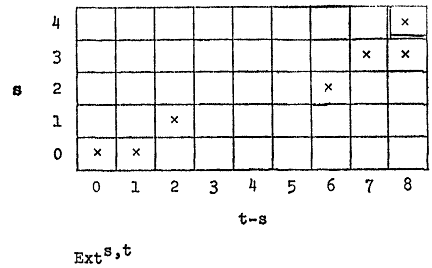

FIGURE 6

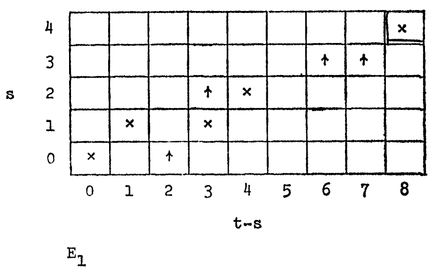

FIGURE 7

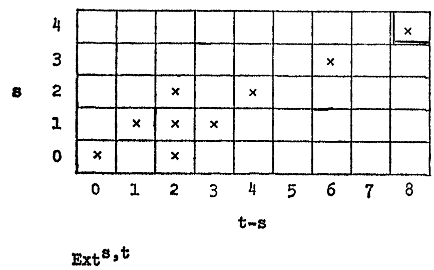

FIGURE 8

$\mathrm{Ext}_{A_1}(Z_2, Z_2)$. (See [2].) We can exclude the truncated tower based in $E^{32,0}$ because it is created by a d_5 and would have to be hit by a d_4, which is impossible. But note, τc (τ is of degree (4, 3)) must be zero because there is no element in the Ext of C_2 corresponding to that dimension. So if $d_r c \neq 0$ then $d_r(\tau c) = \tau(d_r c) \neq 0$, but since $\tau c = 0$, $d_r c$ must be zero. Thus by 3.1, we have the lemma.

If $J = \emptyset$, then $P_J = 1 \in H^*(BSpin)$. By 2.5 and the above lemma, we know there is a 33-dimensional Spin manifold, M_{33}, with

$$(1)\, Sq^{16}Sq^{8}Sq^{4}Sq^{2}Sq^{1} \neq 0.$$

Denote the 2^i-dimensional manifold, quaternionic projective 2^{i-2} space by QP_i. Using formula 8.1 and Lemma 2.3 of Brown-Peterson [5], which describe the right action, it is easily verified that

$$M_{33} \times QP_5 \times QP_6 \times \cdots \times QP_i, \quad i \geq 5,$$

has

$$(1)\, Sq^{2^i}Sq^{2^{i-1}} \cdots Sq^{8}Sq^{4}Sq^{2}Sq^{1} \neq 0.$$

So from Theorem 2.5 we have:

LEMMA 5.3 *$Sq^{2^i}Sq^{2^{i-1}} \cdots Sq^{8}Sq^{4}Sq^{2}Sq^{1} \notin R_2$ for $i \geq 4$.*

This concludes the proof of Theorem 2.6.

6. Products

There is a map $\Psi : K(Z_2, i) \wedge K(Z_2, j) \to K(Z_2, i + j)$ representing the bottom cohomology class, which gives a pairing on the Adams spectral sequence (see Adams [1]) which corresponds to

$$\Psi_* : H_*(K(Z_2, i), \mathbf{BO}\langle 0\rangle) \otimes H_*(K(Z_2, j), \mathbf{BO}\langle 0\rangle) \to H_*(K(Z_2, i + j), \mathbf{BO}\langle 0\rangle).$$

On the E_2 terms, it is a pairing

$$\Phi : \mathrm{Ext}_{A_1}(H^*(K(Z_2, i)), Z_2) \otimes \mathrm{Ext}_{A_1}(H^*(K(Z_2, j)), Z_2) \to \mathrm{Ext}_{A_1}(H^*(K(Z_2, i + j)), Z_2)$$

and d_r acts as a differential on products. In particular, on $E_2^{*,0}$, this pairing is just the map

$$\Phi : \mathrm{Hom}_{A_1}(H^*(K(Z_2, i)), Z_2) \otimes \mathrm{Hom}_{A_1}(H^*(K(Z_2, j)), Z_2) \to \mathrm{Hom}_{A_1}(H^*(K(Z_2, i + j)), Z_2)$$

given by $\Phi(f \otimes g)(x) = \Sigma f(x') \otimes g(x'')$ where

$$\Phi^*(x) = \Sigma x' \otimes x'',$$

$$\Psi^* : H^*(K(Z_2, i + j)) \to H^*(K(Z_2, i)) \otimes H^*(K(Z_2, j)).$$

The use of products can give considerable information, for example: if $e(I) < n - 1$, then for

$$\Psi : K(Z_2, n - 1) \wedge K(Z_2, 1) \to K(Z_2, n)$$

we have

$$\Psi^*(Sq^I c^n) = Sq^I(c^{n-1} \otimes c^1) = Sq^I c^{n-1} \otimes c^1 + \textstyle\sum_{r>0}(\cdots) \otimes (c^1)^{2^r}.$$

As shown earlier, $g((c^1)^{2^r}) = 0 \ \forall g \in \mathrm{Hom}_{A_1}(H^*(K(Z_2, 1)), Z_2)$. Now there is a map g such that $g(c^1) \neq 0$ and $d_r g = 0 \ \forall r$. So in this case $d_r \Phi(f \otimes g) = \Phi((d_r f) \otimes g)$. By understanding Φ and $d_r f$ we can decide whether $Sq^I \in R_n$ or not. For example, using the above method, we have:

Proposition 6.1. *If $e(I) \leq n - 1$ and $Sq^I \notin R_{n-1}$, then $Sq^I \notin R_n$.*

For the cases $e(I) = n - 1$ and n, one cannot always determine if $Sq^I \in R_n$ from lower dimensional information and products.

By studying products, one can verify the following using 5.4, 2.5, 6.1, and constructing simple Poincaré algebras:

Proposition 6.2.

$$w_7 \in I_n(\{w_1, w_2\}, \mathrm{geom})^7 \quad \text{and} \quad w_7 \notin I_n(\{w_1, w_2\}, \mathrm{alg})^7$$

for $n = 10, 11,$ and 12, but not for 13, where

$$w_7 \notin I_{13}(\{w_1, w_2\}, \mathrm{geom})^7.$$

For R_3, the first two cases which cannot be decided by products are $I = (12, 6, 3, 1)$ and $(20, 10, 4, 2, 1)$.

Bibliography

1. J. F. Adams, *On the structure and applications of the Steenrod algebra*, Comm. Math. Helv., vol. 32 (1958), pp. 180–214.
2. D. W. Anderson, E. H. Brown, Jr., and F. P. Peterson, *Pin cobordism and related topics*, Comm. Math. Helv., vol. 44 (1969), pp. 462–468.
3. ———, *The structure of the spin cobordism ring*, Ann. of Math., vol. 86 (1967), pp. 271-298.
4. W. Browder, *Torsion in H-spaces*, Ann. of Math., vol. 74 (1961), pp. 24–51.
5. E. H. Brown, Jr. and F. P. Peterson, *Relations among characteristic classes I*, Topology, vol. 3 (1964), pp. 39–52.
6. P. E. Conner and E. E. Floyd, *Differential periodic maps*, Springer, Berlin, 1964.
7. A. Liulevicius, *Notes on homotopy theory of Thom spectra*, Amer. J. Math., vol. 86 (1964), pp. 1–16.
8. F. P. Peterson, *Lectures on cobordism theory*, Kinokuniya Book Store, Tokyo, 1968.
9. J. -P. Serre, *Cohomologie modulo 2 des complexes d'Eilenberg-MacLane*, Comm. Math. Helv., vol. 27 (1953), pp. 198–231.
10. R. Stong, *Determination of $H^*(BO(k, \cdots, \infty), Z_2)$ and $H^*(BU(k, \cdots, \infty), Z_2)$*, Trans. Amer. Math. Soc., vol. 107 (1963), pp. 526–544.
11. C. T. C. Wall, *A characterization of simple modules over the Steenrod algebra mod* 2, Topology, vol. 1 (1962), pp. 249–254.
12. G. W. Whitehead *Generalized homology theories*, Trans. Amer. Math. Soc., vol. 102 (1962), pp. 227–283.

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