ON A CLASS OF DOUBLY TRANSITIVE GROUPS

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The purpose of this paper is to prove the following theorem:

THEOREM. *Let G be a transitive group of permutations on the (finite) set of letters Ω. Let G_α be the subgroup of G fixing the letter α in Ω. Suppose G_α contains a normal subgroup Q of even order, which is regular on $\Omega - (\alpha)$. Then either*

(a) *G is a subgroup of the group of semi-linear transformations over a near field of odd characteristic or*

(b) *G is an extension of one of the groups $SL(2, q)$, $Sz(q)$ or $U(3, q)$ by a subgroup of its outer automorphism group. ($|\Omega| = 1 + q$, $1 + q^2$ or $1 + q^3$ in these three respective cases ($q = 2^n$).)*

Essentially "half" of this theorem was proved by Suzuki [8], under the assumption that the quotient group G_α/Q had odd order. We therefore consider only the case that G_α/Q has even order.

Since Q is regular on $\Omega - (\alpha)$, we may express G_α as a semidirect product $G_{\alpha\beta} Q$ where $G_{\alpha\beta} = G_\alpha \cap G_\beta$, the subgroup of permutations fixing both α and β.

For the rest of this paper, all groups considered are finite. We write $|X|$ for the cardinality of set X. If X is a subset of a group G, we write $X \subseteq G$, and if X is a subgroup of G, we write $X \leq G$. If $X \subseteq G$, $\langle X \rangle$ will denote the subgroup of G generated by X. If X is a subset of G, X^G denotes the set of all conjugate sets $\{g^1 X g \mid g \in G\}$. We will frequently write $\langle X^G \rangle$ instead of the more cumbersome $\langle \bigcup_{Y \in X^G} Y \rangle$. This is the normal closure of X in G and represents the smallest normal subgroup of G containing X. If M is a group of (right) operators of a group G it will frequently be convenient to proceed with computations in the semi-direct product GM and also to view GM as a group of right operators of G, the elements of G acting by conjugation. Action of these operators is indicated by exponential notation. Thus if $x \in G$, $g^{-1}xg$ may be written x^g and if σ is an automorphism of G, we may write

$$(x^g)^\sigma = x^{g\sigma} = x^{\sigma \cdot g^\sigma}.$$

The commutator $x^{-1}y^{-1}xy$ is written $[x, y]$. If σ is an automorphism of G and if $x \in G$, then the commutator $[x, \sigma]$ is assumed to be computed in the semidirect product $G\langle\sigma\rangle$, so $[x, \sigma] = x^{-1} \cdot x^\sigma$. If π is a set of primes, a π-group is a group whose order involves only primes in π. As usual, π' denotes the complement of π in the set of all primes. If π consists of a single prime p, the symbol p (rather than $\{p\}$) may replace the symbol π in the notation of

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the previous two sentences. Finally, $Z(G)$ denotes the center of G, $O_2(G)$ the maximal normal 2-subgroup of G, and $O_{2'}(G)$, the maximal normal 2′-subgroup of G.

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1. Some preliminary propositions

The proof of the theorem requires the use of the following propositions.

PROPOSITION 1. *Let G be a transitive permutation group on a set of letters Ω. Let G_α be the subgroup of G fixing the letter α in Ω. Suppose G_α contains a normal 2-subgroup A such that A is semi-regular on $\Omega - (\alpha)$. Then G contains a normal subgroup N such that either*

(i) *N is a Frobenius group with Frobenius complement A and Frobenius kernel N_1 which is abelian and regular on Ω, or*

(ii) *$N \simeq SL(2, q)$, $Sz(q)$ or $U(3, q)$, N is 2-transitive on Ω and $|\Omega| = 1 + q$, $1 + q^2$ or $1 + q^3$ respectively, where q is an appropriate power of 2.*

This is corollary 3 of [7].

The following proposition is only slightly more general than the corollary appearing in [6], but this generality is required, and the proof of it given in [1] is far more natural than the version in [6].

PROPOSITION 2 (Alperin). *If V is an elementary subgroup of order 4 in a group G and if $V \cap O_2(G) = 1$, then there is an involution t of G conjugate to an element of V which commutes with no element of $V^\#$.*

We conclude this section with

PROPOSITION 3. *Let G be a group admitting an automorphism τ of order 2. Suppose the subgroup $C_G(\tau)$ contains a unique involution t. Then either $\langle t^G \rangle$ is elementary abelian or else $tO_{2'}(G)$ is the unique involution in $G/O_{2'}(G)$.*

Proof. Let S be a 2 Sylow subgroup of $C_G(\tau)$. Then by hypothesis t is the unique involution in S. If S were a full 2-Sylow subgroup of G, then, by a theorem of Brauer and Suzuki [3], $tO_{2'}(G)$ would be the unique involution in $G/O_{2'}(G)$ and we would be done. Thus we may assume that S is not a 2-Sylow subgroup of G. Then there exists a τ-invariant 2-subgroup $S_1 = \langle x, S \rangle$ containing S as a subgroup of index 2. Then $[x, \tau]$ is a non-identity element of S. Since $\tau^2 = 1$,

$$x = x^{\tau^2} = (x[x, \tau])^\tau = x[x, \tau]^2.$$

Thus $[x, \tau] = t$, the unique involution in S. Thus $\tau t = t\tau = x^{-1}\tau x$. Note that since x normalizes S, x centralizes t. Thus τ is conjugate to τt in $C_G(t)$.

Now the class t^G is a τ-invariant set with t as the unique element in the class fixed by τ. Thus we may write

$$t^G = \{t, t_1, t_1^\tau, t_2, t_2^\tau, \cdots, t_m, t_m^\tau\}$$

where $m = (|t^G| - 1)/2$. Set $u_i = t_i t_i^\tau$, $i = 1, \cdots, m$. The groups $\langle t_i, t_i^\tau \rangle$ are τ-invariant dihedral groups and the elements u_i are inverted by τ.

Suppose some u_i has odd order. Then there are an odd number of conjugates of t in $\langle t_i, t_i^\tau\rangle$, and since this set of involutions is invariant under τ, one of them is fixed by τ and hence is t. Thus $t \in \langle t_i, t_i^\tau\rangle$ and t inverts u_i. Then τt centralizes u_i. Since $\tau t = x^{-1}\tau x$, we see that $u_i^{x^{-1}} \in C_G(\tau)$ which contains t in its center. Thus t centralzes $u_i^{x^{-1}}$. On the other hand t centralizes x and inverts u_i; consequently t also inverts $u_i^{x^{-1}}$. Since t now both centralizes and inverts $u_i^{x^{-1}}$ it follows that the latter has order 1 or 2, contrary to our initial assumption that u_i had odd order and $u_i \neq 1$ (since $t_i \neq t_i^\tau$).

Hence we must suppose that each u_i has even order. Since τ inverts u_i, some power of u_i is an involution fixed by τ, as well as by t_i and t_i^τ. Clearly this involution is t, the unique involution in $C_G(\tau)$. Thus t commutes with t_i and t_i^τ, $i = 1, \cdots, m$. It follows that all members of t^G commute with one another and so $\langle t^G \rangle$ is a normal elementary 2-subgroup of G. This completes the proof of proposition 3.

2. Proof of the theorem

Let G be a transitive group of permutations on the set of letters Ω. Fix a letter α in Ω, and let G_α be the subgroup of G fixing α. By assumption, G_α contains a normal subgroup Q which is regular on $\Omega - (\alpha)$. We may then write $G_\alpha = G_{\alpha\beta} Q$ where $G_{\alpha\beta} \cap Q = 1$. Also by assumption, Q has even order, and so the number of letters $|\Omega|$ is odd. For the sake of consistency with the notation of [8] we write $K = G_{\alpha\beta}$. Also by the result in [8], we shall assume that K has even order. The proof of the theorem now proceeds by a series of short steps, (A) through (P) below. Induction on $|\Omega|$ and $|G|$ is utilized at steps (G), (H) and (J).

(A). $O_2(Q) = 1$.

Set $A = O_2(Q)$. By way of contradiction assume $|A| > 1$. Then A is normal in G_α and is semi regular on $\Omega - (\alpha)$. Then G and A satisfy the hypothesis of Proposition 1, and so either (i) or (ii) or Proposition 1 must hold. If (ii) holds, $N = \langle A^G \rangle$ is 2-transitive on Ω and so no permutation on Ω can centralize the group of permutations N. Thus G/N is faithfully represented on the automorphism group of N modulo the inner automorphism group of N and conclusion (b) of our theorem holds. If (i) holds, then QN_1 is a 2-transitive Frobenius group which is normal in G. Then there is a near-field corresponding to QN_1 and $G_{\alpha\beta}$ is a complement in G to QN_1 and faithfully acts on QN_1 so as to induce automorphisms on the corresponding near-field. The conclusion of (a) thus holds.

Thus if A is non-trivial we are done. Without loss of generality, then, we may assume $A = 1$, which is (A).

(B) *For each element $x \in K$ such that x has prime order, $C_Q(x)$ is non-trivial.*

If $C_Q(x) = 1$, when x has prime order, then Q is nilpotent by a fundamental theorem of Thompson [9]. In that case, since Q has even order, $O_2(Q) \neq 1$, and this contradicts (A).

At this point we introduce a "glossary" of subgroups. For each element x in K set

Ω_x = *points in* Ω *fixed by* x (*thus* $\Omega_x \supseteq \{\alpha, \beta\}$)
$L_x = \mathrm{Stab}_G(\Omega_x)$ (*clearly* $C_G(x) \leq N_G(\langle x \rangle) \leq L_x$)
$N_x = \langle C_Q(x)^{L_x} \rangle$ (*clearly* $N_x \trianglelefteq L_x$)
K_x = *point-wise stabilizer of* Ω_x (clearly $K_x \trianglelefteq L_x$, $K_x \leq K$).

(C) $\Omega_x = \{\alpha\} \cup \{\beta^{C_Q(x)}\}$ *for all* $x \in K$.

First, by our hypothesis on Q, $\Omega = \{\alpha\} \cup \{\beta^Q\}$ and $\{\alpha, \beta\} \subseteq \Omega_x$. If

$$\beta^a \in \Omega_x \qquad \beta^{ax} = \beta^{xa^x} = \beta^{a^x}$$

so $a = a^x$ from the regularity of Q. This asserts, $a \in C_Q(x)$. Thus

$$\Omega_x \subseteq \{\alpha\} \cup \{\beta^{C_Q(x)}\}.$$

The reverse inclusion is trivial.

(D) *If $x \in K$ either*
(i) $|\Omega_x| = 2$ *and* $|C_Q(x)| = 1$ *or*
(ii) $|\Omega_x| > 2$, *N_x is 2-transitive on Ω_x and $N_x \leq C_G(K_x)$. Moreover,* $L_x = (K \cap L_x)N_x$.

If $|\Omega_x| = 2$, then by (C), $|C_Q(x)| = 1$ and (i) holds.

If $|\Omega_x| > 2$, then also by (C), $|C_Q(x)| > 1$. Then $C_Q(x)$ fixes α and is regular on $\Omega_x - (\alpha)$. Since $x \in G_\beta$, x also normalizes Q_1, the unique conjugate of Q lying in G_β. Again by (C),

$$\Omega_x = \{\beta\} \cup \{\alpha^{C_{Q_1}(x)}\}$$

and $C_{Q_1}(x)$ lies in L_x, fixes β, and is transitive on $\Omega_x - (\beta)$. It follows that $\langle C_Q(x), C_{Q_1}(x) \rangle$ is 2-transitive on Ω_x and so contains every conjugate of $C_Q(x)$ lying in L_x (there is exactly one conjugate for each point in Ω_x). Thus $\langle C_Q(x), C_{Q_1}(x) \rangle = N_x$ which is 2-transitive. Since $K_x \trianglelefteq L_x$, $[K_x, C_Q(x)] \leq K_x \cap Q = 1$. Similarly $[K_x, C_{Q_1}(x)] = 1$ and so $N_x \leq C_G(K_x)$. Since N_x is a normal 2-transitive subgroup of L_x it follows that the section L_x/N_x is covered by K, the subgroup fixing 2 letters. Thus $L_x = (L_x \cap K)N_x$. All conclusions in (ii) are now proved.

(E) *G has no non-trivial normal solvable subgroups.*

If N is a minimal normal solvable subgroup of the primitive group G, it easily follows that N is elementary abelian and is regular on Ω. Then QN is a normal 2-transitive Frobenius subgroup, and so Q has a central involution inverting N. Then $O_2(Q)$ is non-trivial against (A).

(F) *A 2-Sylow subgroup of Q contains more than one involution.*

Let Q_2 denote a 2-Sylow subgroup of Q and suppose s were the unique involution in Q_2. Suppose a conjugate s^g of s commutes with s. Then s^g fixes α, the unique letter left fixed by s, and g also fixes α. By the Brauer-Suzuki theorem [3], $sO_{2'}(Q)$ is the unique involution in $Q/O_{2'}(Q)$. Thus, since $g \in G_\alpha \leq N(Q)$, g leaves the coset $sO_{2'}(Q)$ invariant, and so $s^g = sn$ where $n \in O_{2'}(Q)$. Since s^g commutes with s, n also commutes with s. On the other hand $sns = n^{-1}$ since $sn = s^g$ is an involution. Since n has odd order this forces $n = 1$ and so $s^g = s$. We have just proved that s is not fused in G to any further involution in $C_G(s)$. Thus Glauberman's Z^* theorem [5] may be applied, and so $C_G(s)O_{2'}(G) = G$. By the Feit Thompson theorem [4], $O_{2'}(G)$ is solvable and so by (E), $O_{2'}(G) = 1$. Then $G = C_G(s) \leq G_\alpha$, which contradicts the assumption that G is transitive on Ω and $|\Omega| \geq 3$ (since Q is assumed to be non-trivial).

(G) *A 2 Sylow subgroup of K is not cyclic.*

Let S denote a 2-Sylow subgroup of K. Then SQ has odd index in G. Assume for the remainder of this paragraph that S is cyclic. If $y \in S^g \cap Q$ for any $g \in G$, then y fixes $\{\alpha^g, \beta^g\}$ since $y \in S^g \cap K^g$. On the other hand, as a member of Q, y is either the identity element or fixes exactly one letter, because of the regularity of Q on $\Omega - (\alpha)$. Thus $y = 1$ and so $S^g \cap Q = 1$ for all $g \in G$. Now we represent G as a permutation group on the cosets of Q. A generator of the cyclic group S is then represented as $[G:SQ]$ cycles of length $|S|$ since $S^g \cap Q = 1$ for all $g \in G$. This is an odd permutation. Now observe that $Q^x \cap G_\alpha = Q^x \cap N_G(Q) \neq 1$ implies $x \in G_\alpha$ and $Q^x = Q$. Thus Q acts on its own cosets by fixing all cosets of Q in $N_G(Q)$ and acting semiregularly on the remaining cosets in $[N_G(Q):Q]$ orbits of length $|Q|$. Since $[N_G(Q):Q] \equiv 0 \bmod |S|$ and S is non-trivial by assumption, every 2-element in Q is represented by an even permutation in this representation. Thus we see that G contains a normal subgroup G_1 of index 2 in G, namely the elements represented by even permutations in the representation of G on the cosets of Q. Thus $SG_1 = G$ and $Q \leq G_1$. Since $[G:G_1] = 2$ and $|\Omega|$ is odd, G_1 is transitive on Ω. Since $Q \leq G_1$ it follows that G_1 is a 2-transitive group obeying the same hypotheses as G. By induction, either G_1 contains a normal abelian transitive subgroup (as in conclusion (a)) or G_1 contains a normal simple 2-transitive subgroup N_1 of "Bender type". The former case contradicts step (E). In the latter case, since G_1/N_1 is solvable, $N_1 \trianglelefteq G$. Then $G_{\alpha\beta}N_1 = G$ and it is clear that $G_{\alpha\beta}/(G_{\alpha\beta} \cap N_1)$ must be isomorphic to a subgroup of the outer automorphism group of N_1. In this way case (b) of the conclusion of the theorem is obtained.

We may thus assume S is non-cyclic.

(H) *Let τ be any involution in K. Then $|\Omega_\tau|$ is $1 + q$, $1 + q^2$ or $1 + q^3$ where $q = 2^n > 2$ and*

$$\bar{N}_\tau = N_\tau/(N_\tau \cap K_\tau) \simeq SL(2, q),\ Sz(q) \text{ or } U(3, q),$$

respectively.

We will let "bar" denote application of the homomorphism $L_\tau \to L_\tau/K_\tau = \bar{L}_\tau$, the group of permutations of Ω_τ induced by L_τ, and by restriction apply this mapping to subgroups of L_τ.

By (B), since τ is an involution in K, $|C_Q(\tau)| > 1$ and so case (ii) of (D) holds. Thus $\bar{N}_\tau$ is a 2-transitive group of permutations on Ω_τ. Since τ normalizes a 2-Sylow subgroup of Q, necessarily $C_Q(\tau)$ has even order. Since $C_Q(\tau)$ is regular on $\Omega_\tau - (\alpha)$, $|\Omega_\tau|$ is odd. Indeed $C_Q(\tau) = Q \cap N_\tau\, C_Q(\tau)^- \simeq C_Q(\tau)$ so that a point stabilizer $(G_\alpha \cap N_\tau)^-$ in $\bar{N}_\tau$ restricted to Ω_τ contains a normal subgroup $C_Q(\tau)^-$ of even order which is regular on $\Omega_\tau - (\alpha)$. Thus the hypotheses of the theorem are satisfied with $\bar{N}_\tau$, $C_Q(\tau)^-$ and Ω_τ in the roles of G, Q and Ω respectively. Since $\tau \neq 1$ implies $|\Omega_\tau| < |\Omega|$, we may apply induction to assert that either (a) $\bar{N}_\tau$ is a group of semilinear transformations over a near field, or (b) $\bar{N}_\tau$ is an extension of $SL(2, q)$, $Sz(q)$ or $U(3, q)$ by its outer automorphism group.

Consider the former case (a). The subgroup of translations $\bar{M}$ is normalized by $C_Q(\tau)^-$ and is therefore transitive and regular on Ω_τ and so $C_Q(\tau)^-\bar{M}$ is a Frobenius group. It follows that $C_Q(\tau)^- \simeq C_Q(\tau)$ contains a unique involution s.

At this point we can apply Proposition 3, for Q is a group admitting τ as an automorphism of order 2, and such that $C_Q(\tau)$ has a unique involution s. Thus by Proposition 3, either $\langle s^Q\rangle$ is a normal 2-subgroup of Q, or else $sO_{2'}(Q)$ is the unique involution in $G/O_{2'}(Q)$. In the former case, $|O_2(Q)| > 1$ and this contradicts (A). In the latter case, a 2-Sylow subgroup of Q contains a unique involution, and this contradicts (*F*).

Thus we must assume case (b) holds for $\bar{N}_\tau$ and Ω_τ. Thus $\bar{N}_\tau$ contains a normal 2-transitive subgroup $\bar{M}_\tau$ isomorphic to $SL(2, q)$, $Sz(q)$ or $U(3, q)$. Thus $(M_\tau \cap Q)^-$ is regular on $\Omega_\tau - (\alpha)$ and so coincides with $C_Q(\tau)$. Thus, since M_τ is transitive, $M_\tau \geq \langle C_Q(\tau)^{L_\tau}\rangle = N_\tau$ whence $\bar{M}_\tau = \bar{N}_\tau$ is itself simple. The conclusion of (H) now holds.

(I) *Fix τ as in* (H). *Choose an involution $t = (\alpha\beta)\cdots$ in N_τ transposing α and β. Set $V = K \cap N_\tau$, the subgroup of N_τ fixing α and β. The following hold*:

(i) *V is abelian, and is normalized by t.*
(ii) *$V = U \times C_V(t)$, where $U \simeq Z_{q-1}$, U is inverted by t.*
(iii) *U is normal in $L_\tau \cap K$.*

Since $t = (\alpha\beta)\cdots$ normalizes $G_{\alpha\beta} = K$ and lies in N_τ, t normalizes $V = N_\tau \cap K$. Then $V/(K_\tau \cap N_\tau)$ corresponds to the subgroup fixing 2 letters in

$$N_\tau/(K_\tau \cap N_\tau) = \bar{N}_\tau \simeq SL(2, q),\ Sz(q) \text{ or } U(3, q).$$

Thus $[t, V](K_\tau \cap N_\tau)/(K_\tau \cap N_\tau)$ is cyclic of order $q - 1$, and $V/(K_\tau \cap N_\tau)$ is also cyclic of order $q - 1$ or $(q^2 - 1)/(3, q + 1)$. By (D)(ii), $N_\tau \leq C(K_\tau)$ and so $K_\tau \cap N_\tau$ is central in N_τ. Thus V is a cyclic extension of $K_\tau \cap N_\tau$, which lies in its center. It follows that V is abelian. Let W be the $2'$-Hall subgroup of V. Then W covers $V/(K_\tau \cap N_\tau)$ and

$$W = [t, W] \times C_W(t).$$

Set $U = [t, W]$. Since t centralizes $K_\tau \cap N_\tau$ and $V = W(K_\tau \cap N_\tau)$ it follows that $[t, V] = [t, U] = U$, and that $U \cap (K_\tau \cap N_\tau) = 1$. Thus

$$U \simeq [t, V](K_\tau \cap N_\tau)/(K_\tau \cap N_\tau) \simeq Z_{q-1}.$$

Now

$$V = W(K_\tau \cap N_\tau) = (U \times C_W(t))(K_\tau \cap N_\tau).$$

Since $U(K_\tau \cap N_\tau)/(K_\tau \cap N_\tau)$ is a direct factor of $V/(K_\tau \cap N_\tau)$ with $C_{V/(K_\tau \cap N_\tau)}(t)$ as a complement (the section $V/(K_\tau \cap N_\tau)$ is t-isomorphic to $W/(W \cap K_\tau)$), it follows that

$$U \cap C_W(t)(K_\tau \cap N_\tau) \leq K_\tau \cap N_\tau.$$

But $U \cap (K_\tau \cap N_\tau) = 1$, thus $C_W(t)(K_\tau \cap N_\tau)$ is a t-invariant direct complement of U in V and it easily follows that $C_V(t) = C_W(t)(K_\tau \cap N_\tau)$ and so $V = U \times C_V(t)$. Thus (i) and (ii) are established.

Now $[t, L_\tau \cap K] \leq N_\tau \cap K$ since t normalizes K and since $t \in N_\tau \trianglelefteq L_\tau$. If $x \in L_\tau \cap K$, then $x^t = xk$ where $k \in N_\tau \cap K = V$. Since V is normal in $L_\tau \cap K$, and W is characteristic in V, W is normal in $L_\tau \cap K$. Thus U^x is a subgroup of the abelian group W, and thus is centralized by $k = [x, t] \in V$. Thus for each element u in U,

$$(u^x)^t = (u^t)^{xk} = (u^{-1})^{xk} = ((u^x)^{-1})^k = (u^x)^{-1}$$

since k centralizes U^x. Thus U^x is a subgroup of W which is inverted by t. It follows that $U^x = [t, W] = U$. Since x was an arbitrary element in $L_\tau \cap K$ we see that U is normal in $L_\tau \cap K$ and (iii) is proved.

(J) *Let u_0 represent any element of prime order in U. Let x be any element* *of* $L_\tau \cap K$ *such that* $C_Q(x)$ *contains an elementary subgroup of order* 4. *Then*:

(i) $C_Q(x)$ *is a* 2-*group*

(ii) x *fixes precisely* 2 *elements in* Ω_{u_0}, *namely* $\{\alpha, \beta\}$.

(iii) x *has fixed point free action on the group* $C_Q(u_0)$ *which is abelian.*

(iv) $|\Omega_{u_0}|$ *is even.*

Suppose x is an element of $L_\tau \cap K$ such that $C_Q(x)$ contains an elementary subgroup of order four. Then $|\Omega_x|$ is odd, and by (D)(ii), $\bar{N}_x$ is doubly transitive on Ω_x, its subgroup $C_Q(x)$ being a normal subgroup of $(N_x \cap G_\alpha)^-$ having even order and regular on $\Omega - (\alpha)$. By induction and the definition of N_x, either $\bar{N}_x$ is a Frobenius group $C_Q(x)^-\bar{N}_x'$ with Frobenius kernel $\bar{N}_x'$ regular on Ω_x or $\bar{N}_x \simeq SL(2, q_x)$, $Sz(q_x)$, or $U(3, q_x)$. The former case is excluded since $C_Q(\mathrm{a})$ contains an elementary subgroup of order four. Thus $C_Q(x)$ is a 2-Sylow subgroup of the simple group $\bar{N}_x$. (i) follows at once.

Next observe that since x is in $L_\tau \cap K$, that x normalizes U by (I). Since U is cyclic, x also normalizes $\langle u_0\rangle$ and thus stabilizes Ωu_0. That is, $L_\tau \cap K \leq L_{u_0} \cap K$.

Next we argue that $C_Q(u_0)$ has odd order. First, $UC_Q(\tau)$ is a Frobenius group, and so U fixes α and β and is semiregular on $\Omega, - \{\alpha, \beta\}$. From this it follows that $\Omega_\tau \cap \Omega_{u_0} = \{\alpha, \beta\}$. Thus τ fixes none of the letters $\{\beta^a \mid a \in C_Q(u_0)\}$ which make up $\Omega_{u_0} - \{\alpha, \beta\}$. Thus τ (being an element of L_{u_0}) normalizes $Q \cap N_{u_0} = C_Q(u_0)$ and acts without fixed points on $C_Q(u_0)$. Thus $C_Q(u_0)$ is abelian and has odd order. Thus (iv) holds.

Similarly, for each $x \in K \cap L_\tau$, x normalizes $Q \cap N_{u_0} = C_Q(u_0)$. Since $C_Q(x)$ is a 2-group by (i) and since $C_Q(u_0)$ has odd order it follows that x has fixed point free action on $C_Q(u_0)$. Thus (iii) holds.

Statement (ii) follows immediately from the fact that x fixes β^a, $a \in C_Q(u_0)$ if and only if x centralizes a. In that case $a = 1$ from (iii) and so $\Omega_{u_0} \cap \Omega_x = \{\alpha, \beta\}$, proving (ii).

(K) *A* 2-*Sylow subgroup of* K *is a generalized quaternion group.*

Assume S is not quaternion. Since, by step (G) S is also not cyclic, we may find involutions $\tau_1 \neq \tau_2$ in S with τ_1 central in S. Setting $\tau = \tau_1$, the groups L_τ, N_τ, K_τ, V, U and $\langle u_0\rangle$ of steps (H), (I) and (J) are then defined in terms of the involution τ, central in S. Then

$$S \leq C(\tau) \cap K_\tau \leq L_\tau \cap K \leq L_{u_0} \cap K.$$

This last containment follows from $\langle u_0\rangle$ being characteristic in U and step (I)(iii). Now any non-identity element in the fours group $\langle \tau_1, \tau_2\rangle$ satisfies the hypotheses of the element x in step (J). By (J) (iii) it follows that $\langle \tau_1, \tau_2\rangle C_Q(u_0)$ is a Frobenius group with Frobenius complement $\langle \tau_1, \tau_2\rangle$. This is clearly impossible since $\langle \tau_1, \tau_2\rangle$ is a fours-group.

(L) *For each element* u_0 *of prime order in* U, *there exists an element* $v = v(u_0)$ *in* K *which inverts* u_0, *that is* $v^{-1}u_0 v = u_0^{-1}$.

By step (B), $C_Q(u_0) > \{1\}$. Then by (D)(ii) N_{u_0} is 2-transitive on Ω_{u_0} and N_{u_0} centralizes K_{u_0} which contains u_0. Thus $C_G(u_0)$ is 2-transitive on Ω_{u_0}. In particular we know that $C_G(u_0)$ is not contained in G_α. Now

$$G = G_\alpha \cup QKtQ,$$

where $t = (\alpha\beta)\cdots$ is the involution of step (I) lying in N_τ and normalizing K. From the regularity of Q, elements in $QKtQ$ have a unique expression in the form $xvty$, $x \in Q$, $v \in K$, $y \in Q$. Since $C_G(u_0)$ is not contained in G_α, we can find such an element $xvty$ in $C_G(u_0)$. Then $xvty$ can be written as

$$(xvty)^{u_0} = x^{u_0}v^{u_0}(u_0^{-1}tu_0)y^{u_0} = x^{u_0}v^{u_0}u_0^{-2}ty^{u_0}$$

and the uniqueness of the expression implies $v = v^{u_0}u_0^{-2} = u_0^{-1}vu_0^{-1}$. Then $v^{-1}u_0\, v = u_0^{-1}$ so v inverts u_0 as promised. (This step was lifted from Suzuki [8].)

(M) $N_\tau/(K_\tau \cap N_\tau) \simeq SL(2, 4)$ *or* $U(3, 4)$.

For each prime p dividing $|U| = q - 1$, we will write u_p for an element of order p in U, and let U_p be an S_p subgroup of U. The element v in K which inverts $u_0 = u_p$ in step (L) can be assumed to be a 2-element by raising v to an appropriate odd power. We will write v_p for v to indicate that this element depends on u_p.

Now since U is cyclic, $\langle u_p\rangle$ is characteristic in U which is normal in $L_\tau \cap K$ by (I)(iii). Since $S \leq C(\tau) \cap K \leq L_\tau \cap K$, it follows that S normalizes $\langle u_p\rangle$, for each choice of p, as well as normalizing U. Clearly S is a 2-Sylow subgroup of $N_K(\langle u_p\rangle)$, and v_p is a 2-element in $N_K(\langle u_p\rangle)$ which inverts u_p. Thus every element of U is inverted by an element in S. Conjugation by elements of S induce automorphisms of

$$\bar{N}_\tau = N_\tau/(N_\tau \cap K_\tau) \simeq SL(2, q),\ Sz(q) \text{ or } U(3, q),$$

which may invert any of the non-identity p-elements of its subgroup

$$\bar{U} = (U \times (K_\tau \cap N_\tau))/(K_\tau \cap N_\tau) \simeq Z_{q-1}.$$

Since these automorphisms correspond to field automorphisms of $GF(q)$ we see that $S/C_S(U)$ is cyclic. By step (K), S is generalized quaternion, and so $S/[S, S]$ has exponent 2. Thus $S/C_S(U) \simeq Z_2$, and the involution in this section must invert every element of prime order in U. It follows that this involution must invert every p-Sylow subgroup of U, and hence must invert U itself. On the other hand the involution in $S/C_S(U)$ must act now as a field automorphism of $GF(q)$ which inverts every non identity element of the multiplicative group $GF(q)^* = GF(q) - (0)$. It follows from this that $q = 4$. Thus

$$N_\tau/(N_\tau \cap K_\tau) \simeq SL(2, 4) \text{ or } U(3, 4).$$

(N) *A 2-Sylow subgroup of* $C_Q(\tau)$ *is a 2-Sylow subgroup of* Q.

Now $C_Q(\tau)$ is a 2-Sylow subgroup of either $SL(2, 4)$ or $U(3, 4)$. Thus $C_Q(\tau)$ has order 4 or 4^3. In either case, all involutions in $C_Q(\tau)$ belong to its center $T = Z(C_Q(\tau))$ which is a four-group.

Suppose $C_Q(\tau)$ is not a full 2-Sylow subgroup of Q. Then Q contains a τ-invariant 2-group S_0 containing $C_Q(\tau)$ as a subgroup of index 2, and $S_0 = \langle x, C_Q(\tau)\rangle$. Then $x^\tau = xc$ where $c \neq 1$, and $c \in C_Q(\tau)$. From $\tau^2 = 1$, it easily follows that c is an involution and therefore lies in $T^\#$. Now conjugation by x induces an automorphism on $T \times \langle\tau\rangle$, which is elementary of order 8. Since $x^2 \in C_Q(\tau)$, x^2 centralizes $T \times \langle\tau\rangle$ and so this automorphism has order 2. Consequently x centralizes $c = [x, \tau]$. This fact is critical in what follows.

Let t_0 be any involution in $T^\#$ and consider the class t_0^Q, which is τ-invariant. This class decomposes as

$$t_0^Q = (t_0^Q \cap T^\#) + \{t_1, t_1^\tau\} + \{t_2, t_2^\tau\} + \cdots + \{t_m, t_m^\tau\},$$

where $t_1, \cdots, t_m$ are representatives in t_0^Q of the τ-orbits of length 2. Setting $u_i = t_i t_i^\tau$, $i = 1, 2, \cdots, m$, we see that both t_i and τ invert u_i.

Now suppose some u_j has odd order, $1 \leq j \leq m$. Then $\langle t_j, t_j^\tau\rangle$ is a τ-invariant dihedral group containing an odd number of members of t_0^Q. Thus τ centralizes one of these involutions, and this involution, then, is an element c_j in $T^\#$. Thus c_j inverts u_j and so τc_j centralizes u_j. Since U transitively permutes the three elements of $T^\#$, we can find an element u in U such that $c^u = c_j$. Then, since U is centralized by τ, we see that $x_1 = x^u = u^{-1}xu$ also normalizes $C_Q(\tau)$, that

$$[x_1, \tau] = [x^u, \tau] = [x, \tau]^u = c^u = c_j \quad \text{and} \quad [x_1, c_j] = [x, c]^u = 1^u = 1.$$

Since $\tau c_j = \tau^{x_1}$ centralizes u_j, τ centralizes $x_1 u_j x_i^{-1}$. Then

$$x_1 u_j x_1^{-1} \in C_Q(\tau)$$

which contains c_j as a central element. Thus, since c_j commutes with $x_1 u_j x_1^{-1}$ as well as x_1, we see that c_j also commutes with u_j. This contradicts the fact c_j inverts u_j (since u_j has odd order by assumption, and is non-trivial because $t_j \neq t_j^\tau$).

Thus we must assume that u_i has even order for $i = 1, \cdots, m$. Since u_i is always a non-identity element, some power of u_i is an involution z_i fixed by τ. Then $z_i \in T^\#$. In addition, t_i and t_i^τ both commute with z_i. Thus we see that every element t_0^Q commutes with at least one element of $T^\#$. Since this conclusion holds for each involution t_0 chosen in $T^\#$ we see from Proposition 2, that $T \cap O_2(Q) > \{1\}$. But this contradicts step (A).

Thus we must have that $C_Q(\tau)$ is a full 2-Sylow subgroup of Q.

(O) *$C_Q(\tau)$ is not a full 2-Sylow subgroup of Q.*

We prove this by showing that the assumption that it is a full S_2-subgroup of Q leads to an impossible situation concerning the fusion of involutions in a 2-Sylow subgroup of G.

Assume, as in (N), that $C_Q(\tau)$ is a full 2-Sylow subgroup of Q and as before set $T = Z(C_Q(\tau))$, an elementary group of order 4 containing all of the involutions in $C_Q(\tau)$. Let S be a 2-Sylow subgroup of K lying in $C(\tau)$. Then S normalizes $C_Q(\tau)$ and it is easy to see that the semidirect product $S^* = SC_Q(\tau)$ is a full 2-Sylow subgroup of G.

Suppose w is an involution in S^*. Then w lies in $\langle\tau\rangle \times C_Q(\tau)$, since $S^*/C_Q(\tau) \simeq S$ is generalized quaternion. Then clearly $w \in \langle\tau\rangle \times T$. Thus $\langle\tau\rangle \times T = \Omega_1(S^*)$. Now S induces an automorphism of order 2 on T fixing the involution z_1, say, in T. Then clearly

$$\langle\tau\rangle \times \langle z_1\rangle \simeq z_2 \times z_2$$

comprises the center of S^*. By the Burnside theorem on fusion, all elements of this group which are conjugate in G are conjugate in $N_G(S^*)$. But since $S^* \leq G_\alpha$ and $C_Q(\tau)$ is semiregular on $\Omega - (\alpha)$, α is the unique letter in Ω left fixed by S^*. Thus $N_G(S^*) \leq G_\alpha$ and so $N_G(S^*)$ normalizes Q and hence normalizes $Q \cap Z(S^*) = \langle z_1\rangle$. Thus z_1 is not fused to τ or τz_1 in G, and so, conjugating by U, we see that τ is not fused to any element of T in G.

If τ were not fused to any further involutions in S^*, then by the Z^*-theorem of Glauberman [5], $G = C_G(\tau)O_{2'}(G)$. But $O_{2'}(G) = 1$ by the Feit Thompson theorem [4] and step (E). Then $G = C_G(\tau)$. But this is absurd since $\tau \neq 1$, τ fixes α and β, and G is transitive on Ω.

Thus τ must be fused to some further involution in $\Omega_1(S^*) = \langle\tau\rangle \times T$, but is not fused to involutions in T. Thus τ is fused in G to an element τz_1 lying in the coset τT. From the action of U on $\Omega_1(S^*)$, it follows that τ is conjugate to τz_1. Since both of these elements lie in $Z(S^*)$, it follows that τ is conjugate to τz_1. Since both of these elements lie in $Z(S^*)$, the theorem of Burnside tells us that an element in $N_G(S^*)$ induces, by conjugation, an automorphism of $Z(S^*)$ which transposes τ and τz_1, but fixes z_1. Such an automorphism clearly has order 2 and this statement contradicts the fact that S^* has odd index in $N(S^*)$ (since S^* is a 2-Sylow subgroup of G).

This contradiction proves the step, and in fact proves

(P) *The theorem holds.*

This follows at once from the incompatibility of steps (N) and (O).

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