REGULAR MODULES AND RINGS

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1. Introduction

This paper continues the study of purity and regularity started in [4] and [5]. As a biproduct we show that certain results which we obtained for one type of purity are in fact valid for many types.

Unless otherwise noted we use the notation and conventions of Mitchell [6] for categories and those of Bourbaki [2] for rings and modules. Subcategories are always full; rings are always associative and have 1, but are not necessarily commutative; and modules are always unital.

For any object A of a category $\mathbf{A}$, and for any set I, we denote by A^I the coproduct (if it exists) of I copies of A.

2. Homotopic purity

Let $\mathbf{A}$ be any abelian category and $\mathbf{E} = \mathbf{E}(\mathbf{A})$ be the additive category of short exact sequences over $\mathbf{A}$. We introduce the equivalence relation of homotopy into $\mathbf{E}$ by defining $(f_1, f_2, f_3) = f \sim 0$ by σ (and $f \sim g$ iff $f - g \sim 0$) iff, given the commutative diagram, in $\mathbf{E}$

$$\begin{array}{ccccccc} A : 0 \to A_1 & \xrightarrow{\alpha_1} & A_2 & \xrightarrow{\alpha_2} & A_3 \to 0 \\ f\downarrow \quad\quad f_1\downarrow & & f_2\downarrow & & \downarrow f_3 \\ B : 0 \to B_1 & \xrightarrow[\beta_1]{} & B_2 & \xrightarrow[\beta_2]{} & B_3 \to 0 \end{array}$$

any one of the following three equivalent conditions holds:

(1) there exists $\sigma_1 \in [A_2, B_1]$ with $\sigma_1 \alpha_1 = f_1$;

(2) there exist $\sigma_1 \in [A_2, B_1]$ and $\sigma_2 \in [A_3, B_3]$ with $f_2 = \beta_1 \sigma_1 + \sigma_2 \alpha_2$;

(3) there exists $\sigma_2 \in [A_3, B_2]$ with $\beta_2 \sigma_2 = f_3$.

This equivalence relation, defined on each $[A, B]_{\mathbf{E}}$ of $\mathbf{E}$ is compatible with the category structure of $\mathbf{E}$ and hence defines a quotient category $\mathbf{E}'$ and a binary relation θ on the objects of $\mathbf{E}$:

$$A \;\theta\; B \Leftrightarrow [A, B]' = [A, B]_{\mathbf{E}'} = 0.$$

Now θ in turn defines a polarity situation in the sense of Birkhoff [1], whose notation we use to define for any subclass $\mathbf{G}$ of $\mathbf{E}$ the subcategory

$$\mathbf{G}^* = (E \in \mathbf{E} \mid G \;\theta\; E \text{ for all } G \in \mathbf{G})$$

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The objects of $\mathbf{G}^*$ are called $\mathbf{G}$-pure exact sequences (generated by $\mathbf{G}$). When no confusion can arise we sometimes omit the $\mathbf{G}$. (Clearly $(\bigcup \mathbf{G}_i)^* = \bigcap \mathbf{G}_i^*$.)

We refer to [4] for the notions of pure projectivity, projective generation, existence of enough pure projectives, etc.

We write $P\gamma M$ to indicate that P is pure in M (i.e. $0 \to P \to M \to M/P \to 0$ is pure exact) for some (understood) purity.

3. Examples

(1) If $\mathbf{G}$ consists of all split exact sequences, then $\mathbf{G}^*$ consists of all short exact sequences. Dually:

(2) If $\mathbf{G}$ consists of all short exact sequences then $\mathbf{G}^*$ consists of all the split exact sequences.

These are the two extreme cases.

Let $(f_1, f_2) = f : \alpha \to \beta$ be a map in the category $\mathbf{A}^2$ of maps of $\mathbf{A}$, i.e. $f_2 \alpha = \beta f_1$. We call f *homotopic to* 0 *by* σ and write $f \sim_\sigma 0$ or just $f \sim 0$, iff there exists σ with $\sigma\alpha = f_1$. This introduces again an equivalence relation compatible with the category structure of $\mathbf{A}^2$.

Any map α factors as $\alpha = me$ with m monic and e epic. Hence $f : \alpha \to \beta$ is homotopic to 0 iff there exists σ with $\sigma m = f$. These two notions of homotopy are clearly compatible. If we let $c = \operatorname{cok} m$ and associate to α the short exact sequence

$$\xrightarrow{m} \quad \xrightarrow{c}$$

then each class $\mathbf{C}$ of maps of $\mathbf{A}$ defines in this way a class of associated short exact sequences $\mathbf{G} = \mathbf{G}(\mathbf{C})$ which in turn defines $\mathbf{G}$-purity.

For the following examples let $\mathbf{A}$ be the category of unitary left A-modules over an arbitrary associative ring A with 1. Proofs follow from more general results given later.

(3) We have shown in [4] that the class $\mathbf{C} = \bigcup[A^I, A^J]$, with the union taken over all finite sets I and J, defines the tensor product purity of Cohn [3] i.e. $P\gamma M$ iff $N \otimes P \to N \otimes M$ is monic for all right A-modules N.

(4) If $\mathbf{C} = \bigcup[A, A^J]$ with the union taken over all finite sets J then $P\gamma M$ iff $KM \cap P = KP$ for all (or equivalently all finitely generated) right ideals K of A.

(5) If $\mathbf{C} = \bigcup[A^I, A]$ with the union taken over all finite sets I, then $P\gamma M$ iff whenever a finite system of equations $a_i x = p_i \in P$ is solvable in M it is solvable in P.

(6) If $\mathbf{C} = [A, A]$ then $P\gamma M$ iff $aM \cap P = aP$ for all $a \in A$.

4. (I, J) purity

Let P be a submodule of M in the category $\mathbf{A}$ of left A-modules. The purity defined by the set $[A^I, A^J]$ where I and J are arbitrary non-empty sets is called (I, J) *purity* and we write $P_I \gamma_J M$. Note that this purity is projectively generated, as defined in [4].

For arbitrary non-empty classes **I** and **J** the purity defined by the class $\bigcup[A^I, A^J]$ with the union taken over all (I, J) in $\mathbf{I} \times \mathbf{J}$ is called $(\mathbf{I}, \mathbf{J})$ purity and we write $P_{\mathbf{I}} \gamma_{\mathbf{J}} M$.

We call a module (J, I) *presented* iff there exists an exact sequence $A^J \to A^I \to N \to 0$.

THEOREM 4.1. *The following statements are equivalent.*

(1) $P_I \gamma_J M$.

(2) (*Relative solvability of equations*) *Every system of linear equations* $\sum a_{ij} x_j = p_i \in P$, *indexed by* I, *in a set of unknowns indexed by* J, *and having constant terms in* P *is solvable in* P *whenever it is solvable in* M.

(3) $N \otimes P \to N \otimes M$ *is monic for all* (J, I) *presented right* A*-modules* N.

Proof. (1) $\Leftrightarrow$ (2). Such a system of equations with a solution in M is equivalent to the existence of the commutative diagram:

$$\begin{array}{ccc} A^I & \xrightarrow{\alpha} & A^J \\ f : f_1 \downarrow & & \downarrow f_2 \\ P & \xrightarrow[\beta]{} & M. \end{array}$$

(1) $\Rightarrow$ (2). Since $P_I \gamma_J M$ there exists $\sigma \in [A^J, P]$ with $\sigma\alpha = f_1$ and the images of the base elements of A^J under σ is the desired solution in P.

(2) $\Rightarrow$ (1). Conversely if we have a solution in P we can define σ by sending base elements to elements of the solution, to make $f \sim 0$ by σ.

(1) $\Leftrightarrow$ (3). This equivalence follows by chasing the following commutative diagram, with exact rows and columns:

$$\begin{array}{ccccc} 0 \to & A^J \otimes P & \to & A^J \otimes M \\ & \downarrow & & \downarrow \\ 0 \to & A^I \otimes P & \to & A^I \otimes M \\ & \downarrow & & \downarrow \\ & N \otimes P & \to & N \otimes M. \end{array}$$

COROLLARY. *There is an obvious analogue for* $(\mathbf{I}, \mathbf{J})$ *purity, which we leave for the reader's formulation.*

Remark. It is now clear that we can define (I, J) purity with respect to an object A of an arbitrary abelian category **A**. If A is a progenerator then we have a projectively generated purity, with enough projectives, etc.

5. Regularity

We return now the general situation where **A** is an arbitrary abelian category.

THEOREM 5.1. *Let $E \subseteq F \subseteq G$ be objects of* **A**. *Then for any purity we have*

(1) $E\gamma F$ *and* $F\gamma G \Rightarrow E\gamma G$,

(2) $E\gamma G \Rightarrow E\gamma F$.

For projectively generated purity we have also

(3) $F\gamma G \Rightarrow F/E\gamma G/E$,

(4) $E\gamma G$ *and* $F/E\gamma G/E \Rightarrow F\gamma G$.

Proof. We consider the commutative diagrams naturally induced by the given inclusions, and apply Theorem 4.3 of [4]. For example to show part (4) we use the commutative diagram, with exact rows:

$$\begin{array}{ccccccccc} 0 & \to & E & \to & G & \to & G/E & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & F & \to & G & \to & G/F & \to & 0. \end{array}$$

Henceforth, unless otherwise specified, we restrict ourselves to projectively generated purities.

COROLLARY. *Under the natural correspondence between subobjects of G/E and subobjects of G containing E, pure subobjects correspond to pure subobjects.*

PROPOSITION 5.2. *If P and Q are subobjects of an object M then*

$$(P \cap Q)\gamma M \text{ and } (P \cup Q)\gamma M \quad \Rightarrow \quad \text{both } P\gamma M \text{ and } Q\gamma M.$$

Proof. Observe that the proof of Proposition 8.3 of [4] carries through in this case.

An object R of **A** will be called *regular* iff every short exact sequence with middle term R is pure exact.

THEOREM 5.3. *If $0 \to R \to S \to T \to 0$ is a short exact sequence in* **A** *then S is a regular object iff both R and T are regular and $R\gamma S$.*

$\Rightarrow$. For any subobject V of S we have $(V \cup R)/R\gamma T$ and hence $(V \cup R)\gamma S$ since $R\gamma S$. Also $(V \cap R)\gamma S$ since $(V \cap R)\gamma S$ since $(V \cap R)\gamma R$. Hence $V\gamma S$.

$\Leftarrow$. Given in Theorem 6.2 of [4].

6. Finitary purity

In this section we return to the category of left A-modules. If both I and J are finite sets, or if the classes **I** and **J** are classes of finite sets then we call the resulting purity finitary. Most (but not all) of our earlier examples were finitary. In this section we shall consider only finitary purity.

THEOREM 6.1. *For any finitary purity γ, if $P_k\ \gamma M_k$ for all $k \in K$, a directed set, then* dir lim $P_k\ \gamma$ dir lim M_k.

Proof. Any map $f : \alpha \to \beta$ where $\alpha \in [A^I, A^J]$ and $\beta \in [P, M]$ can be factored through some $\beta_k \in [P_k, M_k]$ for suitable $k \in K$, since both I and J are finite. The existence of the homotopy is then immediate.

COROLLARY. *Finitary purity is an inductive property.*

THEOREM 6.2. *If* $R = \sum_{k \in K} R_k$, *K any index set, then R is regular iff each* R_k *is regular.*

Proof. The proof given in [4] for Theorem 8.5 goes through in this more general situation. The crucial direct limit argument holds since we have finitary purity.

Finally we observe that for any finitary purity we can define a regular socle, which is a torsion socle, as in [4].

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