# ON INTEGRABILITY AND SUMMABILITY IN VECTOR SPACES

BY

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**1.** In [10] we have studied the properties of a functionally defined integral for measures defined on $\sigma$-algebras and having their values in locally convex spaces. This paper extends those results to measures defined on $\delta$-rings, but is more concerned with the problem of determining which scalar valued functions are integrable (Definition 3.1).

The more general setting of $\delta$-rings is desirable because of the following considerations. Any sequence $(x_t) \subset E$ can be considered as an $E$ valued measure on the $\delta$-ring of finite subsets of the natural numbers, simply by defining $\mu(A)$ to be $\sum_{t \in A} x_t$. Setting $f(t) = 1$ for each $t$, it is apparent that the questions of the summability of $(x_t)$, and of the existence of solutions to the equations

$$(*) \qquad \langle x_A, x' \rangle = \int_A f(t) \langle \mu(dt), x' \rangle, \quad x' \in E'$$

for all $A \subset N$ are closely related. The chief topic of this paper is the relation between the behavior of summable sequences in a given space $E$, and the existence of solutions to $(*)$ for arbitrary $E$-valued measures and scalar-valued functions.

Sections 2 and 3 contain the basic facts about vector measures and their integrals needed in the latter sections. In Section 4 the relationship between integrability with respect to a given measure and its variations is considered, and it is shown that these two notions coincide in an $(F)$-space if and only if the space is nuclear. In Section 5 we show that in a $(B)$-space $E$ not containing $c_0$, a function is $\mu$-integrable if it is $\langle \mu, x' \rangle$-integrable for each $x' \in E'$. Integrals for measures into spaces of linear operators are the subject of §6. These results are applied to show that for a measure $\mu$ into a $(B)$-space $E$, a function is $\mu$-integrable if its integral in $E''$ is a norm measure. In the last section we make comparisons with some of the other integrals and related ideas in the literature.

**2.** Throughout this paper $\tau$ is a $\delta$-ring of subsets of a set $S$, i.e., $\tau$ is a collection of subsets of $S$ closed under relative complement, finite union and countable intersection. $C(\tau)$ is the $\sigma$-algebra of sets locally in $\tau$ and $E$ is a locally convex, Hausdorff linear topological space. The scalar field of $E$ may be either the real or complex numbers and is denoted by $\mathbb{C}$. The term zero neighborhood in $E$ means a closed, convex and circled set containing zero in its interior. The gauge of a zero neighborhood $U$ is the function $p_U(x) =$

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$\sup_{x' \in U^0} | \langle x, x' \rangle |$. As much as possible we will use the notation and terminology of Schaefer [13].

A collection $(x_\alpha) \subset E$ is summable if the net $(\sum_{\alpha \in A} x_\alpha)$, formed by sums over finite sets of indices and directed by inclusion, is convergent in the given topology on $E$. The collection $(x_\alpha)$ is absolutely summable if it is summable and $\sum_\alpha p_U(x_\alpha) < \infty$ for each zero neighborhood $U$. Finally, a sequence $(x_n)$ is subseries convergent if $\lim_n \sum_{k \leq n} \chi_A(k) x_k$ exists for each set $A$ of natural numbers. The basic relations between these three notations may be found in Day [5].

A measure with values in $E$ is an additive set function defined on $\tau$ with the property that $\sum \mu(A_n)$ is convergent to $\mu(\bigcup_n A_n)$ for each pairwise disjoint sequence $(A_n) \subset \tau$ with $\bigcup_n A_n \in \tau$. Notice that $(\mu(A_n))$ is both summable and subseries convergent to $\mu(\bigcup_n A_n)$. For $\mu : \tau \to E$ a measure, $\langle \mu, x' \rangle$ and $x'\mu$ are used to denote the composition of $\mu$ with a continuous linear functional on $E$, and in fact this composition is a scalar measure on $\tau$. Conversely, Grothendieck has shown [9, p. 166] that an additive set function $\mu : \tau \to E$ is a measure if $\langle \mu, x' \rangle$ is a scalar measure for each $x' \in E'$.

DEFINITION 2.1. Let $U$ be a zero neighborhood in $E$. The $U$-variation of $\mu$ is the set function on $C(\tau)$ defined by

$$\bigvee_U(\mu, A) = \sup \textstyle\sum_{i \leq n} p_U(\mu(A_i)),$$

where the supremum is taken over all finite pairwise disjoint collections $(A_i)_{i \leq n} \subset \tau$ of subsets of $A$.

The proof given by Dinculeanu [6, p. 35] shows that, for $\mu$ a measure, $\bigvee_U(\mu, \cdot)$ is an extended real-valued measure on $C(\tau)$. In case $E$ is normed (in particular if $E = \mathcal{C}$) only the variation over the closed unit ball is considered and the subscript is omitted. Measures of finite variation (i.e., those satisfying $\bigvee_U(\mu, A) < \infty$ for each zero neighborhood $U$ and $A \in \tau$) can be characterized by a strengthening of countable additivity.

THEOREM 2.2. *A measure $\mu : \tau \to E$ is of finite variation if and only if $(\mu(A_n))_n$ is absolutely summable to $\mu(\bigcup_n A_n)$ for each pairwise disjoint sequence $(A_n) \subset \tau$ with $\bigcup_n A_n \in \tau$.*

*Proof.* The necessity of the last condition is obvious. Conversely, suppose $\bigvee_U(\mu, A_0) = \infty$ for some $A_0 \in \tau$ and zero neighborhood $U$. For some partition $\mathcal{P}_1$ of $A_0$, $1 < \sum_{A \in \mathcal{P}_1} p_U(\mu(A))$. Since $\bigvee_U(\mu, \cdot)$ is additive, $\bigvee_U(\mu, A_1) = \infty$ for some $A_1 \in \mathcal{P}_1$. Continuing in this manner there is a decreasing sequence $(A_n)_{n \geq 0}$ in $\tau$ and a sequence $(\mathcal{P}_n)_{n \geq 1}$ of partitions such that $\mathcal{P}_n$ is a partition of $A_{n-1}$, $A_n \in \mathcal{P}_n$ and

$$n - \textstyle\sum_{i \leq n-1} \sum_{A \in \mathcal{P}_i \setminus \{A_i\}} p_U(\mu(A)) < \sum_{A \in \mathcal{P}_n} p_U(\mu(A)), \quad n > 1.$$

The sequence $\bigcup_n (\mathcal{P}_n \setminus \{A_n\})$ is pairwise disjoint and its union is in $\tau$, but $\sum_n \sum_{A \in \mathcal{P}_n \setminus \{A_n\}} p_U(\mu(A)) = \infty$.

DEFINITION 2.3. Let $U$ be a zero neighborhood in $E$. The $U$-semi-variation of $\mu$ is the set function on $C(\tau)$ defined by

$$\|\mu\|_U(A) = \sup_{x' \in U^0} \bigvee (x'\mu, A).$$

Only the semi-variation over the closed unit ball is considered when $E$ is normed. For $\mu$ a measure, each semi-variation assumes finite values on $\tau$ and is countably subadditive—further, the following is proved in [10].

THEOREM 2.4. *If $\mu$ is a measure and $(A_n)_{n \geq 0}$ is a sequence in $\tau$ satisfying $A_n \subset A_0$ and $\lim_n A_n = \emptyset$, then $\lim_n \|\mu\|_U(A_n) = 0$ for each zero neighborhood $U$.*

**3.** Throughout this section $\mu$ is a fixed $E$-valued measure on $\tau$. We suppose that the real and imaginary parts of all functions considered below are measurable with respect to the $\sigma$-algebra $C(\tau)$.

DEFINITION 3.1. A function $f : S \to \mathbb{C}$ is $\mu$-integrable if (1) $f$ is $\langle \mu, x' \rangle$-integrable for each $x' \in E'$ and (2) the functional equation

$$\left\langle \int_A f(t)\mu\,(dt), x' \right\rangle = \int_A f(t) \langle \mu\,(dt), x' \rangle$$

has a solution for each $A \in C(\tau)$.

The integral defined in equation (2) is clearly linear, gives the correct vector for $\tau$-simple functions and has the property that continuous linear operators can be brought inside the integral.

THEOREM 3.2. *If $f$ is $\mu$-integrable, then the indefinite integral*

$$\lambda(A) = \int_A f(t)\mu\,(dt)$$

*is a measure in $C(\tau)$ and*

$$\|\lambda\|_U(A) = \sup_{x' \in U^0} \int_A |f(t)| \bigvee (x'\mu, dt)$$

*for each zero neighborhood $U$ and $A \in C(\tau)$.*

*Proof.* This follows immediately from Grothendieck's result and the definition of the semi-variation.

THEOREM 3.3. *Let $(f_n)$ be a sequence of $\mu$-integrable functions which converges pointwise to $f$, and $g$ be a $\mu$-integrable function such that $|f_n| \leq |g|$ for each $n$.*

(1) *If $E$ is semi-complete, then $f$ is $\mu$-integrable.*

(2) *If $f$ is $\mu$-integrable, then*

$$\int_A f(t)\mu\,(dt) = \lim_n \int_A f_n(t)\mu\,(dt)$$

*uniformly with respect to $A \in C(\tau)$.*

To reduce to the case of $\mu$ defined on a $\sigma$-algebra we need

LEMMA 3.4. *If $g$ is $\mu$-integrable, $U$ is a zero neighborhood in $E$ and $\varepsilon > 0$, then there is an $A \in \tau$ such that*

$$\sup_{x' \in U^0} \int_{S \setminus A} |f(t)| \vee (x'\mu, dt) < \varepsilon.$$

*Proof of lemma.* Let $\lambda$ be the indefinite integral of $g$. If the lemma is false, there is an increasing sequence $(A_n) \subset \tau$ and a sequence $(x'_n) \subset U^0$ satisfying

$$\vee (x'_n \lambda, A_n) > \varepsilon \quad \text{and} \quad \vee (x'_n \lambda, A_{n+1}) < \varepsilon/2$$

for each $n$. Let $A = \bigcup_n A_n$. The sequence $(A \setminus A_n)$ is decreasing in $C(\tau)$ and has empty intersection, but

$$\| \lambda \|_U (A \setminus A_{n+1}) \geq \vee (x'_n \lambda, A) - \vee (x'_n \lambda, A_{n+1}) > \varepsilon/2,$$

contradicting Theorem 2.4.

*Proof of theorem.* By the proof of Theorem 2.2 in [10], it is sufficient to show that the sequence $(\int_A f_n(t)\mu(dt))$ is Cauchy uniformly with respect to $A \in C(\tau)$. Let $U$ be a zero neighborhood in $E$ and $A$ be the element of $\tau$ guaranteed by Lemma 3.4 with $\varepsilon = \frac{1}{2}$. Then

$$p_U\left(\int_{A \cap B} f_n(t)\mu\,(dt) - \int_B f_n(t)\mu\,(dt)\right) < \frac{1}{2}$$

for all $n$ and $B \in C(\tau)$. Define $\vee : C(\tau) \to E$ by $\vee(B) = \mu(A \cap B)$. Since $C(\tau)$ is a $\sigma$-algebra, the proof in [10] shows that, for some $n_0$, the inequality

$$p_U\left(\int_B f_n(t) \vee (dt) - \int_B f_m(t) \vee (dt)\right) < \frac{1}{2}$$

holds for all $B \in C(\tau)$ and $n, m \geq n_0$. Combining these two inequalities yields the Cauchy condition.

The lemma also gives the following approximation condition for integrable functions.

THEOREM 3.5. *If $g$ is $\mu$-integrable, $U$ is a zero neighborhood in $E$ and $\varepsilon > 0$, then there is a $\tau$-simple function $f$ satisfying*

$$\sup_{x' \in U^0} \int |f(t) - g(t)| \vee (x'\mu, dt) < \varepsilon.$$

*Proof.* Choose $A \in \tau$ to satisfy

$$\sup_{x' \in U^0} \int_{S \setminus A} |g(t)| \vee (x'\mu, dt) < \epsilon/2.$$

Since $g\chi_A$ is $C(\tau)$-measurable and concentrated on an element of $\tau$, there is a

$\tau$-simple function vanishing off $A$ and satisfying

$$\sup_{s\in A} |f(s) - g(s)| < \varepsilon/2[\|\mu\|_U(A) + 1]^{-1}.$$

The function $f$ clearly works.

With the aid of Lemma 3.4 a version of the Orlicz-Pettis Theorem (see [11], and [9, Corollary 2, p. 141]) can be recovered from the integration theory developed above. Since the general method of proof will be used several times later we adopt some notation for dealing with discrete $E$-valued measures. For a set $S$, $\tau_S$ is the $\delta$-ring of all finite subsets of $S$. Every subset of $S$ is locally in $\tau_S$. A collection $(x_\alpha)_{\alpha\in S}$ defines a measure $\mu$ from $\tau_S$ into $E$—specifically, let $\mu(\emptyset) = 0$ and $\mu(A) = \sum_{\alpha\in A} x_\alpha$ if $A \neq \emptyset$. We refer to the measure defined in this way as the measure induced by $(x_\alpha)_{\alpha\in S}$. The function on $S$ whose only value is 1 is denoted by $1(\cdot)$. $N$ always denotes the set of positive integers.

THEOREM 3.6. *If $(x_\alpha)_{\alpha\in S} \subset E$ and $(\chi_A(\alpha)x_\alpha)$ is summable in $\sigma(E, E')$ for each $A \subset S$, then each subcollection $(\chi_A(\alpha)x_\alpha)$ is summable in $\tau(E, E')$.)*

*Proof.* It is sufficient to show that $(x_\alpha)$ is summable. Let $\mu$ be the measure induced by $(x_\alpha)$. For $A \subset S$ and $x' \in E'$,

$$\langle \textstyle\sum_\alpha \chi_A(\alpha)x_\alpha, x'\rangle = \int_A 1(\alpha)\langle\mu(d\alpha), x'\rangle,$$

so $1(\cdot)$ is $\mu$-integrable in $\tau(E, E')$. Let $U$ be a $\tau(E, E')$ zero neighborhood. By Lemma 3.4, there is an $A_0 \in \tau_S$ such that

$$\sup_{x'\in U^0} \int_{S\setminus A_0} 1(\alpha)v(x'\mu, d\alpha) < 1.$$

For $A \in \tau_S$ and $A \supset A_0$,

$$p_U\left(\int_S 1(\alpha)u(d\alpha) - \textstyle\sum_{\alpha\in A} x_\alpha\right) \leq \sup_{x'\in U^0} \int_{S\setminus A} 1(\alpha)v(x'\mu, da) < 1.$$

**4.** In this section we investigate the relation between $\mu$-integrability and integrability with respect to the variations $\bigvee_U(\mu, \cdot)$. It is not necessary to assume that $\mu$ is of finite variations.

THEOREM 4.1. *If $E$ is quasi-complete and $f$ is $\bigvee_U(\mu, \cdot)$-integrable for each zero neighborhood $U$, then $f$ is $\mu$-integrable.*

*Proof.* Notice first that $f$ must be $\langle\mu, x'\rangle$-integrable for each $x' \in E'$, since $\{x \in E : |\langle x, x'\rangle| \leq 1\}$ is a zero neighborhood. Consider the collection $\beta$ of all zero neighborhoods in $E$ to be directed by inclusion, and for each $U \in \beta$ choose a $\tau$-simple function $f_U$ such that $|f_U| \leq 4|f|$ and

$$\int |f(t) - f_U(t)| \bigvee_U(\mu, dt) < \frac{1}{2}.$$

Let $A \in C(\tau)$. The net $(\int_A f_U(t)\mu(dt))$ is bounded because

$$p_V\left(\int_A f_U(t)\mu\,(dt)\right) \leq \int 4|f(t)\,|\bigvee_V\,(\mu, dt)$$

for each $U$, $V \in \beta$. Since we will need a similar argument later, we will prove that the net $(\int_A f_U(t)\mu(dt))$ converges to the integral of $f$ over $A$ assuming only that the net is bounded and satisfies

$$\sup_{x' \in U^0} \int |f(t) - f_U(t)| \bigvee (x'\mu, dt) < \frac{1}{2}.$$

This condition holds for the functions considered above since $\bigvee(x'\mu, \cdot) \leq \bigvee_U(\mu, \cdot)$ for $x' \in U^0$.

For $U$, $V \subset W$,

$$\begin{aligned} p_W\left(\int_A f_U(t)\mu(dt)\right) - \int_A f_V(t)\mu(dt)\Big) &\leq \sup_{x' \in W^0} \int |\, f_U(t) - f(t)\,| \bigvee (x'\mu, dt) \\ &\quad + \sup_{x' \in W^0} \int |\, f_V(t) - f(t)\,| \bigvee (x'\mu, dt) \\ &\leq \sup_{x' \in U^0} \int |\, f_U(t) - f(t)\,| \bigvee (x'\mu, dt) \\ &\quad + \sup_{x' \in V^0} \int |\, f_V(t) - f(t)\,| \bigvee (x'\mu\, dt) \\ &< 1. \end{aligned}$$

Thus the net is Cauchy in $E$ and is convergent to some element $x_A$ in $E$.

Let $x' \in E'$, $\varepsilon > 0$ and choose $U \in \beta$ so that $|\langle \cdot\,, x'\rangle|$ is at most $\varepsilon$ on $U$. Then $\varepsilon^{-1}x \in U^0$ and

$$\begin{aligned} &\left|\langle x_A, x'\rangle - \int_A f(t)\,\langle \mu(dt), x'\rangle\right| \\ &\quad\leq \varepsilon\left|\left\langle x_A - \int_A f_U(t)\mu(dt), \varepsilon^{-1}x'\right\rangle\right| + \varepsilon \int |\, f(t) - f_U(t)\,| \bigvee (\varepsilon^{-1}x', dt) \\ &\quad< 3\varepsilon/2. \end{aligned}$$

The converse of the preceding theorem need not be true, but we can give some description of the functions for which it holds.

THEOREM 4.2. *Let $f$ be a $\mu$-integrable function with $\lambda$ its indefinite integral. The function is $\bigvee_U(\mu, \cdot)$-integrable for each zero neighborhood $U$ if and only if $\lambda$ is of finite variation, in which case*

$$\bigvee_U\,(\lambda, A) = \int_A |\, f(t)\,| \bigvee_U\,(\mu, dt).$$

*Proof.* For $A \in C(\tau)$ and $U$ any zero neighborhood,

$$p_U(\lambda(A)) \leq \int_A |f(t)| \bigvee_U (\mu, dt).$$

Thus $\lambda$ is of finite variation and

$$\bigvee_U (\lambda, A) \leq \int_A |f(t)| \bigvee_U (\mu, dt)$$

whenever $f$ is $\bigvee_U(\mu)$ integrable.

Conversely, suppose that $\lambda$ is of finite variation. To show that $|f|$ is $\bigvee_U(\mu, \quad)$-integrable it is sufficient to show that

$$|a| \bigvee_U(\mu, A) \leq \bigvee_U(\lambda, A)$$

whenever $|a\chi_A| \leq |f|$, since this inequality implies that

$$\int_A g(t) \bigvee_U (\mu, dt) \leq \bigvee_U (\lambda, A)$$

for every measurable set $A$ and $\tau$-simple function $g$ dominated by $|f|$. This will also prove the integral representation for the variation of $\lambda$. Let $B \in \tau$ be any subset of $A$. By the Hahn-Banach theorem,

$$p_U(\mu(B)) = |\langle \mu(B), x' \rangle|$$

for some $x' \in U^0$ and so

$$p_U(\mu(B)) \leq |a|^{-1} \int_B |f(t)| \bigvee (x'\mu, dt)$$

$$= |a|^{-1} \bigvee (x'\lambda, B) \leq |a|^{-1} \bigvee_U (\lambda, B).$$

By the definition of variation, $\bigvee_U(\mu, A) \leq |a|^{-1} \bigvee_U(\lambda, A)$ and the theorem is proven.

COROLLARY 4.3. *For a space $E$ the following are equivalent.*

(1) *Every summable sequence $(x_n)$ in $E$ is absolutely summable.*

(2) *If $\mu$ is any $E$-valued measure defined on a $\delta$-ring, then $\mu$ is of finite variation.*

(3) *If $\mu$ is any $E$-valued measure defined on a $\delta$-ring, then a $\mu$-integrable function is $\bigvee_U(\mu, \cdot)$-integrable for every zero neighborhood $U$.*

*Proof.* The implications (1) implies (2) and (2) implies (3) follow from Theorems 2.2 and 4.2, respectively. Assume (3), let $(x_n)$ be summable and $\mu$ the measure on $\tau_N$ induced by the sequence. As in the proof of Theorem 3.6, the function $1(\cdot)$ is $\mu$-integrable on $N$ and by (3) is $\bigvee_U(\mu, \cdot)$-integrable for each zero neighborhood $U$. Then

$$\sum_n p_U(x_n) = \int_N 1(n) \bigvee_U (\mu, dn) < \infty,$$

so $(x_n)$ is absolutely summable.

Condition (1) of the preceeding corollary holds in a Banach space if and only if the space is finite dimensional. Using the Dvoretzky-Rogers Theorem [7] and the construction given in the proof that (3) implies (1), a counter-example to the converse of Theorem 4.1 can be constructed in any infinite-dimensional Banach space—moreover, the measure can be taken to be of finite variation on its domain. In Fréchet spaces (1) holds only in nuclear spaces (see [13, Corollary 2, p. 184]).

**5.** The integrability of a function $f$ with respect to each of the measures $\langle \mu, x' \rangle$, $x' \in E'$, is not generally sufficient for the integrability of $f$. For example, if $(e_n)$ is the sequence of unit vectors in $c_0$ and $\mu$ the measure on $\tau_N$ induced by the sequence, then $1(\cdot)$ is $\langle \mu, x' \rangle$-integrable for each $x' \in l^1$ but is clearly not integrable with respect to $\mu$. In a certain sense $c_0$ is the smallest $(B)$-space in which "weak" integrability does not imply integrability. Bessaga and Pelczynski [3] have shown that, for $E$ a $(B)$-space, Condition (1) of the next theorem is equivalent to the assertion that $E$ has no subspace isomorphic to $c_0$.

THEOREM 5.1. *The following are equivalent in a quasi-complete space $E$.*

(1) *If $(x_n)$ is any sequence in $E$ satisfying*

$$\textstyle\sum_n |\langle x_n, x' \rangle| < \infty$$

*for each $x' \in E'$, then $(x_n)$ is summable*

(2) *If $\mu$ is any $E$-valued measure defined on a $\delta$-ring and $f$ is $\langle \mu, x' \rangle$-integrable for each $x' \in E'$, then $f$ is integrable.*

*Proof.* Assume (2), let $(x_n)$ be a sequence as in (1) and let $\mu$ be the measure on $\tau_N$ determined by the sequence. Clearly $1(\cdot)$ is $\langle \mu, x' \rangle$-integrable for each $x' \in E'$, so $1(\cdot)$ must be $\mu$-integrable. As we have previously noted, this implies that $(x_n)$ is summable.

In order to prove that (1) implies (2) we will first show that $f\chi_A$ is integrable for each $A \in \tau$. We may reduce to the case in which $f$ is non-negative. Fix $A \in \tau$ and for each $n$ let $A_n$ be the points in $A$ satisfying $|g(s)| \leq n$, where $g = f\chi_A$. Since $g$ is $C(\tau)$-measurable and concentrated on an element of $\tau$, we may choose inductively a sequence $(f_n)$ of non-negative $\tau$-simple functions satisfying $f_n$ vanishes off $A_n$, $f_1 \leq g \leq f_1 + 1$ on $A_1$, and

$$f_n \leq g - \textstyle\sum_{i<n} f_i \leq f_n + n^{-1} \quad \text{on } A_n \text{ for } n > 1.$$

The sequence $(\sum_{i\leq n} f_i)$ converges monotonically to $g$ and for each $x' \in E'$ and $B \in C(\tau)$,

$$\sum_n \left| \left\langle \int_B f_n(t)\mu(dt), x' \right\rangle \right| \leq \int g(t) \vee (x'\mu, dt).$$

By (1), the sequence $(\int_B f_n(t)\mu(dt))$ is summable for every $B \in c(\tau)$. By dominated conveyence,

$$\left\langle \sum_n \int_B f_n(t)\mu(dt), x' \right\rangle = \int_B g(t)\langle \mu(dt), x' \rangle$$

for each $x' \in E'$.

We next claim that for each zero neighborhood $U$ there is a set $A \in \tau$ such that

$$(*) \qquad \sup_{x' \in U^0} \int_{S \setminus A} | f(t) | \vee (x'\mu, dt) < 1.$$

If this is not true for a certain zero neighborhood $U$, we may choose an increasing sequence $(A_n) \subset \tau$ and a sequence $(x_n') \subset U^0$ so that

$$\int_{S \setminus A_n} | f(t) | \vee (x_n' \mu, dt) < \frac{1}{2} \quad \text{and} \quad \int_{S \setminus A_n} | f(t) | \vee (x_{n+1}' \mu, dt) > 1$$

for each $n$. Then also

$$\int_{A_{n+1} \setminus A_n} | f(t) | \vee (x_{n+1}' \mu, dt) > \frac{1}{2}$$

for each $n$ and hence there is another sequence $(B_n) \subset \tau$ with $B_n \subset A_{n+1} \setminus A_n$ and

$$\left| \int_{B_n} f(t) \langle \mu(dt), x_{n+1}' \rangle \right| > \frac{1}{8}.$$

Let $x_n$ be the $\mu$-integral of $f\chi_{B_n}$. Since $(B_n)$ is a pairwise disjoint collection,

$$\sum_n | \langle x_n, x' \rangle | \leq \int | f(t) | \vee (x'\mu, dt)$$

for every $x' \in E'$. By (1), this sequence is summable and so $\lim p_U(x_n) = 0$. This is a contradiction since

$$p_U(x_n) \geq | \langle x_n, x_{n+1}' \rangle | \geq \tfrac{1}{8}$$

for all $n$.

Finally, let $B \in C(\tau)$ and for each $A \in \tau$ let $x_A$ be the integral of $f\chi_{A \cap B}$. The collection $(x_A)_{A \in \tau}$ is directed by inclusion of the indices. The net is bounded (it is obviously weakly bounded) and by (*) is Cauchy. An easy calculation shows that the limit $x_B$ must satisfy

$$\langle x_B, x' \rangle = \int_B f(t) \langle \mu(dt), x' \rangle.$$

In particular the previous theorem enables us to describe the classical $(B)$-spaces in which "weak" integrability always implies integrability. Each reflexive Banach space and each $L^1$-space has this property, since these two classes of $(B)$-space are weakly sequentially complete and hence cannot contain $c_0$. No separable dual $(B)$-space can have a subspace isomorphic to $c_0$ [3, Corollary 10, p. 161], and neither can a space with a separable bidual. No infinite-dimensional $C(K)$-space ($K$ compact Hausdorff) has this property, since each such space has a subspace isomorphic to $c_0$.

We remark that for (2) of the preceeding theorem to hold it is sufficient that $E''$ have no subspace isomorphic to $c_0$. There appears to be no known example of a $(B)$-space not containing $c_0$ whose bidual does.

**6.** In this section $E$ and $F$ denote locally convex, Hausdorff spaces and $G$ is a collection of bounded subsets of $E$, which covers $E$ and is directed by containment. The space of continuous linear operators from $E$ to $F$ is denoted by $\mathfrak{L}(E, F)$—the subcsripts of $s$, $c$, $b$ and $G$ refer to $\mathfrak{L}(E, F)$ as a topological vector space under the topologies of simple, pre-compact, bounded and $G$ convergence, respectively.

The dual of $\mathfrak{L}_s(E, F)$ can be identified (algebraically) with $E \otimes F'$, where $\langle u, x \otimes y' \rangle = \langle u(x), y' \rangle$ for each basic tensor $x \otimes y'$ and $u \in \mathfrak{L}(E, F)$ [13, Corollary 4, p. 139]. Thus a scalar-valued function $f$ is integrable for a measure $\mu : \tau \to \mathfrak{L}_s(E, F)$ if and only if

$$\langle \lambda(A)x, y' \rangle = \int_A f(t) \langle \mu(dt)x, y' \rangle$$

holds for some measure $\lambda$ from $C(\tau)$ into $\mathfrak{L}_s(E, F)$. We will first give an apparently weaker condition which is equivalent to integrability in $\mathfrak{L}_s(E, F)$, and next a condition sufficient to push integrability in $\mathfrak{L}(E, F)$ to integrability in $\mathfrak{L}_G(E, F)$.

THEOREM 6.1. *Suppose $E$ is barreled and $\mu : \tau \to \mathfrak{L}_s(E, F)$ is a measure. A function $f$ is $\mu$-integrable if and only if $f$ is $\mu(\cdot)x$-integrable for each $x \in E$.*

*Proof.* The necessity of the last condition is obvious. For $x \in E$ let

$$\lambda_x(A) = \int_A f(t)\mu(dt)x, \quad A \in C(\tau).$$

To show that $f$ is $\mu$-integrable we must show that the linear maps $x \to \lambda_x(A)$ are continuous. Let $U$ be a zero neighborhood in $F$ and

$$B = \{x \in E : \| \lambda_x \|_U(S) \leq 1\}.$$

$B$ is circled, convex and radial (since the range of the $F$-valued measure $\lambda_x$ is bounded). To show that $B$ is a barrel we need only show that $B$ is closed.

Let $x_0$ be in the closure of $B$, $y' \in U^0$ and $\varepsilon > 0$. There is a pairwise disjoint collection $(A_i)_{i \leq n} \subset \tau$ and scalars $(a_i)_{i \leq n}$ such that

$$\left| \textstyle\sum_{i \leq n} a_i \chi_{A_i} \right| \leq |f|$$

and

$$\int | f(t) | \vee ((x_0 \otimes y')\mu, dt) - \varepsilon/2 < \sum_{i \leq n} | a_i | \, | \langle \mu(A_i)x_0, y' \rangle |.$$

Choose a zero neighborhood $V$ in $E$ such that

$$\mu(A_i)[V] \subset \varepsilon/2n\,[\max_{i\leq n}|a_i| + 1]^{-1}U$$

for each $i \leq n$, and then an element $x$ of $B$ such that $x - x_0 \in V$. Then

$$\int |f(t)| \vee ((x_0 \otimes y')\mu, dt) - \varepsilon \leq \sum_{i\leq n} |a_i|\,|\langle\mu(A_i)x, x'\rangle|$$
$$\leq \|\lambda_x\|_U(S) \leq 1.$$

Since $y'$ and $\varepsilon > 0$ were arbitrary,

$$\|\lambda_{x_0}\|_U(S) = \sup_{y'\in U^0} \int |f(t)| \vee ((x_0 \otimes y')\mu, dt) \leq 1.$$

Since $E$ is barreled, zero is interior to $B$, and for any $x \in B$ and $A \in C(\tau)$

$$p_U(\lambda_x(A)) \leq \|\lambda_x\|_U(S) \leq 1.$$

THEOREM 6.2. *Suppose $E$ is barreled, $F$ is quasi-complete and*

$$\mu : \tau \to \mathfrak{L}_G(E, F)$$

*is a measure. A function $f : S \to \mathbb{C}$ is $\mu$-integrable in $\mathfrak{L}_G(E, F)$ if and only if (1) $f$ is $\mu$-integrable in $\mathfrak{L}_s(E, F)$ and (2) the indefinite integral of $f$ is a measure in $\mathfrak{L}_G(E, F)$.*

*Proof.* Again, the necessity of the last two conditions is obvious. Suppose $f$ satisfies (1) and (2) and let $\lambda : C(\tau) \to \mathfrak{L}_G(E, F)$ be a measure satisfying

$$\langle\lambda(A), x \otimes y'\rangle = \int_A f(t)\langle\mu(dt), x \otimes y'\rangle.$$

We will first show that $f$ is $\langle\mu, \varphi\rangle$-integrable for each $\varphi \in \mathfrak{L}_G(E, F)'$.

Let

$$\mathfrak{A} = \left\{\int g(t)\mu(dt) : g \quad \text{is } \tau\text{-simple and} \quad |g| \leq |f|\right\}.$$

$\mathfrak{A}$ is simply bounded, since

$$\left|\left\langle\int g(t)\mu(dt)x, y'\right\rangle\right| \leq \int |f(t)| \vee ((x \otimes y')\mu, dt) < \infty$$

for each $\tau$-simple function $g$ with $|g| \leq |f|$, and each $x \in E$, $y' \in F'$. Since $E$ is barreled, $\mathfrak{A}$ is equicontinuous and hence bounded in $\mathfrak{L}_G(E, F)$. Let $\varphi \in \mathfrak{L}_G(E, F)'$ and $M$ be an upper bound for $|\varphi(\cdot)|$ on $\mathfrak{A}$. Let $g$ be a $\tau$-simple function such that $|g| \leq |f|$. Then $|g\chi_A| \leq |f|$ for each $A \in C(\tau)$ and

$$\left|\int_A g(t)\langle\mu(dt), \varphi\rangle\right| = \left|\left\langle\int g\chi_A(t)\mu(dt), \varphi\right\rangle\right| \leq M.$$

Thus $\int |g(t)| \vee (\varphi\mu, dt)$, the variation of the indefinite integral of $g$, is at most $4M$. Since $g$ was arbitrary, $f$ is $\langle\mu, \varphi\rangle$-integrable.

Let $M \in G$, $U$ be a zero neighborhood in $F$ and let

$$W = \{w \in \mathfrak{L}(E, F) : w[M] \subset U\}.$$

For each such zero neighborhood $W$ we will show that there is a $\tau$-simple function $f_W$ satisfying $|f_W| \leq 4\,|f|$ and

$$(*) \qquad \sup_{\varphi \in W^0} \int | f(t) - f_W(t) | \vee (\varphi\mu, dt) < \frac{1}{2}$$

For each set $A$ locally in $\tau$,

$$\begin{aligned} \| \lambda \|_W(A) &\leq 4 \sup_{B \subset A} p_W(\mu(B)) \\ &\leq 4 \sup_{B \subset A} \sup_{x \otimes y' \in M \otimes U^0} |\langle \mu(B), x \otimes y' \rangle| \\ &\leq 4 \sup_{M \otimes U^0} \vee ((x \otimes y')\mu, A). \end{aligned}$$

Using this inequality and the fact that $\lambda$ is a measure in $\mathfrak{L}_G(E, F)$, the proof of Lemma 3.4 shows that $\| \lambda \|_W(S \setminus A) < \frac{1}{8}$ for some $A \in \tau$. Now choose a set $B \in \tau$ such that $B \subset A$, $f$ is bounded on $B$ and $\| \lambda \|_W(A \setminus B) < \frac{1}{8}$. Finally, let $f_W$ be any $\tau$-simple function vanishing off $B$ and satisfying $|f_W| \leq 4\,|f|$ and

$$\sup_{s \in B} |f(s) - f_W(s)| < \tfrac{1}{8}[\| \mu \|_W(B) + 1]^{-1}.$$

Let $A \in C(\tau)$. The net $(\int_A f_W(t)\mu(dt))$ is bounded since

$$p_U\left(\int_A f_W(t)\mu(dt)\right) \leq 4 \| \lambda \|_U(S) < \infty$$

for all $U$ and $W$. The space $\mathfrak{L}_G(E, F)$ is quasi-complete since $E$ is barreled and $F$ is quasi-complete. Repeating the argument used in Theorem 4.1, $f$ must be $\mu$-integrable in $\mathfrak{L}_G(E, F)$.

COROLLARY 6.3. *Suppose $E$ is barreled, $F$ quasi-complete and $\mu : \tau \to \mathfrak{L}_c(E, F)$ is a measure. A function $f : S \to \mathbb{C}$ is $\mu$-integrable in $\mathfrak{L}_c(E, F)$ if and only if $f$ is $\mu(\ \ )x$-integrable for each $x \in E$.*

*Proof.* By Theorem 6.1, $f$ is $\mu$-integrable in $\mathfrak{L}_s(E, F)$. We need only show that its indefinite integral $\lambda$ is a measure in $\mathfrak{L}_c(E, F)$. If $(A_n) \subset C(\tau)$ is a decreasing sequence with empty intersection, then the sequence $(\lambda(A_n))$ converges to zero in $\mathfrak{L}_c(E, F)$ since it converges pointwise and $E$ is barreled.

COROLLARY 6.4. *Let $E$ be a barreled space and $\mu : \tau \to E'_\sigma$ be a measure. A function $f$ is $\mu$-integrable in $E'_\sigma$ if and only if $f$ is $\langle x, \mu \rangle$-integrable for every $x \in E$. If in addition $\mu$ is a measure in $E'_\beta$, then $f$ is integrable in $E'_\beta$ if and only if the indefinite integral of $f$ is a measure in $E'_\beta$.*

*Proof.* Identity $E'_\sigma$ and $E'_\beta$ with $\mathfrak{L}_s(E, \mathbb{C})$ and $\mathfrak{L}_b(E, \mathbb{C})$, respectively, and apply Theorems 6.1 and 6.2 in turn.

This corollary and the next point out one distinction between the Pettis integral for vector functions and our Pettis type integral for vector measures. In [12, p. 143], Phillips has given an example of a non-Pettis integrable function into a $(B)$-space $E$ whose (Dunford-Gelfand) integral in $E''$ is a measure in the norm topology. The corresponding situation cannot happen for vector measures.

COROLLARY 6.5. *Let $E$ be a $(B)$-space and $\varphi$ the natural map of $E$ into $E''$. A scalar function $f$ is integrable for an $E$-valued measure $\mu$ if and only if the indefinite integral in $E''_\sigma$ of $f$ with respect to $\varphi \circ \mu$ is a measure in norm.*

*Proof.* The necessity of the last condition is obvious. Conversely suppose that $\lambda : C(\tau) \to E''$ is a measure (in norm) satisfying

$$\langle x', \lambda(A)\rangle = \int_A f(t)\langle x', \varphi \circ \mu(dt)\rangle.$$

By Corollary 6.4, $f$ is $\varphi \circ \mu$-integrable in $E''_\beta$. If $y \in E'''_\beta$ vanishes on $\varphi[E]$, then

$$\langle \lambda(A), y\rangle = \int_A f(t)\langle \varphi \circ \mu(dt), y\rangle = 0$$

for all $A \in C(\tau)$. This certainly implies that $\lambda(A)$ is always in $\varphi[E]$ and

$$\langle \varphi^{-1}\lambda(A), x'\rangle = \int_A f(t)\langle \mu(dt), x'\rangle$$

is clearly satisfied.

**7.** In this section we will relate the abstract integral defined in Section 1 with some of the others already in the literature. In all the cases considered below the given measure $\mu$ has values in a $(B)$-space.

In [6], N. Dinculeanu has developed a bilinear integral for measures of finite variation which are defined on $\delta$-rings. For the special case of scalar valved functions his definition [6, p. 120] of an integrable function forces the function to be integrable with respect to the variation of the given measure. The results of §4 indicates that our theory properly contains that of [6] for scalar functions.

Another bilinear integral is the $*$-integral of R. G. Bartle [1]. Here the measure is defined on a $\sigma$-field and for scalar functions the theory reduces to that of Bartle, Dunford and Schwartz (compare Theorem 9 of [1] and Definition 2.5 of [2]). It can be shown [10, Theorem 2.4] that our integral and that in the paper of Bartle, Dunford and Schwartz coincide in their setting, i.e. for Banach-valued measures defined on $\sigma$-fields.

The countably additive integral of Gould [8] is over a quasi-measure space $(S, \Sigma_1, \Sigma_0, \mu)$, where $\Sigma_0$, the domain of $\mu$, is a $\delta$-ring and $\Sigma_1 \supset \Sigma_0$ is some $\sigma$-algebra of sets locally in $\Sigma_0$. Additionally, the measure space is assumed to be semistandard (Definition 5.7 of [8]), a condition we do not require. No problems arise in §2 above if we consider such a $\sigma$-algebra $\Sigma_1$ instead of $C(\Sigma_0)$, once we restrict our attention to $\Sigma_1$-measurable functions. Whenever the concepts of $\Sigma_1$-measurability and total measurability for $(S, \Sigma_1, \Sigma_0, \mu)$ coincide, our integral and his (Definition 6.1 of [8]) agree. This is true in particular for complete measures defined on $\sigma$-algebras.

The paper on Garrett Birkhoff [4] is apparently the first to relate unconditional convergence and integrability (for vector functions and scalar measures) in Banach spaces. However, the methods of Birkhoff are not easily adaptable to our setting. Consider the measure $\mu$ on the Lebesgue sets of $[0, 1)$ which sends a set to its characteristic function in $L^1[0, 1)$. If the roles of vectors and scalars are interchanged, the analog of Theorem 11 of [4] fails for the function $f = \chi_{[1/3,2/3)}$ and the partitions $\{[0, \frac{1}{3}), [\frac{1}{3}, 1)\}$ and $\}[0, \frac{2}{3}), [\frac{2}{3}, 1)\}$ of $[0, 1)$, and Theorem 12 also fails if $f$ is to be integrable. Adapting Theorem 13, p. 367 of [4] as a definition yields an integral which obviously can be approximated in norm by an unconditionally convergent series. A much strong series approximation is obtained by our methods for complete Banach valued measures defined on $\sigma$-algebras. In this case, recursive applications of Theorem 3.5 above give the following: a function $f$ is $\mu$-integrable if and only if there are sequences $(a_n)_{n\geq 1} \subset \mathcal{C}$, $(A_n)_{n\geq 0} \subset \tau$ such that $A_0$ has $\mu$-measure zero,

$$f(s) = \sum_{n\geq 1} a_n \chi_{A_n}(s) \quad \text{for } s \notin A_0$$

and $\sum_{n\geq 1} | a_n | \bigvee (x'\mu, A_n)$ is uniformly summable for $\| x' \| \leq 1$. The integral of $f$ over $A$ is then the unconditionally convergent series

$$\sum_{n\geq 1} a_n \, \mu(A \cap A_n).$$

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