ON THE THEORY OF ENGEL ELEMENTS IN GROUPS

BY

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Let us define

$$[x, y] = x^{-1}y^{-1}xy \quad \text{and} \quad [x, (n+1)y] = [[x, ny], y].$$

An element x of the group G is called a *left* (*right*) *Engel element of* G if for every y in G there is an integer n such that $[y, nx] = 1$ ($[x, ny] = 1$). Denote the sets of left and right Engel elements of G by $L(G)$ and $R(G)$, respectively.

In this paper we shall be concerned with two problems. The first is to find large classes of groups in which the Engel elements for well-behaved subgroups (in a sense to be described in §1). We shall generalize some known results in §1–2. The second problem is, given a class of groups in which the Engel elements form subgroups, find alternate characterizations of these subgroups. In §1 we introduce special subgroups whose definitions are modeled on the Engel radicals introduced by Gruenberg [5] and use these to characterize the subgroups of Engel elements in some classes of groups. These special subgroups have element-wise definitions. Some elegant "global" descriptions of Engel elements have been obtained by Baer [2] and Gruenberg [6] for special classes of groups, namely: The right Engel elements coincide with the hypercenter and the left Engel elements coincide with the unique maximum normal hypercentral subgroup. There are some relatively uncomplicated groups in which the Engel elements cannot admit such a description. For instance, let G be the standard restricted wreath product of a cyclic group of order p, a prime, by an infinite elementary abelian p-group. Then G is metabelian and locally nilpotent, so $R(R) = L(G) = G$; yet G has trivial center. In §3 we introduce a generalization of hypercentrality which may be of independent interest. This generalization enables us to give a global description of the Engel elements analogous to those of Baer and Gruenberg for a class of groups containing the previous example. This description is developed in §4.

Notation

Let G be a group.

$\langle S\rangle$ = subgroup generated by set S.

$[A, B] = \langle [a, b] \mid a \in A, b \in B\rangle$.

Let σ be an automorphism of G and $x \in G$.

$[x, \sigma] = x^{-1}\cdot(x)\sigma$.

$A^B = \langle b^{-1}ab \mid b \in B, a \in A\rangle$.

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$Z(G)$ = center of G.
$\alpha(G)$ = hypercenter of G.
$\varphi(G)$ = Hirsch-Plotkin radical of G (maximum normal locally nilpotent subgroup).
J = infinite cyclic group.
J_m = cyclic group of order m.
A wr B = standard restricted wreath product of groups A and B.
The following classes of groups occur frequently.
$\mathfrak{A}$ = class of abelian groups.
$\mathfrak{M}$ = class of Noetherian groups.
$\mathfrak{Z}$ = class of groups with a central series (in the sense of P. Hall and B. Hartley [7, p. 3] or Kurosh [10, p. 171]).
Let $\mathfrak{X}$ be a group theoretic class, i.e., $1 \in \mathfrak{X}$ and groups isomorphic to $\mathfrak{X}$ groups are $\mathfrak{X}$ groups. Following P. Hall:
$L\mathfrak{X}$ = groups in which finitely generated subgroups are contained in $\mathfrak{X}$ groups.
$S\mathfrak{X}$ = subgroups of $\mathfrak{X}$ groups.
$Q\mathfrak{X}$ = homomorphic images of $\mathfrak{X}$ groups.
$\acute{E}\mathfrak{X}$ = groups having an ascending normal series with $\mathfrak{X}$ factors.
hyper-$\mathfrak{X}$ = groups having an ascending invariant series with $\mathfrak{X}$ factors.
If U is a set of the above operations, $\mathfrak{X}$ is U closed if $T\,\mathfrak{X} \subseteq \mathfrak{X}$, for $T \in U$.
H asc K = there is an ascending normal series of subgroups from H to K (If $H = \langle x \rangle$, we write x asc K.).

1. Engel classes and radicals

Gruenberg has shown in [5] that

$$\sigma_1(G) = \{x \mid x \text{ asc } G\}$$

and

$$\rho_1(G) = \{x \mid y \text{ asc } x^G \langle y \rangle \quad \text{for all} \quad y \in G\}$$

form characteristic subgroups of left and right Engel elements of the group G respectively. Evidently, x is ascendant in G iff x is ascendant in x^G, so we may define $\sigma_1(G)$ to be the set of elements of G which are ascendant in their normal closure. If H is a subgroup of G, we say that H is *weakly ascendant* in G if G has a local system $\{G_\gamma\}$ (see [10, p. 166]) of subgroups such that H asc G_γ, for all γ, and if $G_\gamma \subseteq G_\tau$, then G_γ asc G_τ. The notion of weak ascendance is due to Plotkin [11, p. 10]. We write H wasc G.

Now let us define

$$\sigma_2(G) = \{x \mid x \text{ wasc } x^G\},$$

$$\rho_2(G) = \{x \mid y \text{ wasc } x^G \langle y \rangle \quad \text{for all} \quad y \in G\}.$$

If x wasc x^G and $y \in G$, then $t = [y, x] \in x^G$. There is a subgroup K of X^G containing t such that x asc K. By induction on the length of the ascending series from $\langle x \rangle$ to K, one sees that there is an integer n such that $[t, nx] = 1$.

Thus, $x \in L(G)$. Similarly, $\rho_2(G) \subseteq R(G)$. The Engel radicals just defined are closely related to the Hirsch-Plotkin radical:

THEOREM 1.1. *Let G be a group; then*
(i) $\sigma_2(G) = \varphi(G)$ and
(ii) $\rho_2(G) = \varphi(G) \cap R(G)$.

Proof. If x wasc x^G, then $\varphi(\langle x \rangle) \subseteq \varphi(x^G)$ by Theorem 1 of [11]. Hence, $x \in \varphi(x^G)$ which is contained in $\varphi(G)$ by the same theorem.

Conversely, if $x \in \varphi(G)$ then x^G is a locally nilpotent group. Hence, the set of all finitely generated subgroups of x^G containing x forms the appropriate local system of x^G and x wasc x^G.

It is clear that $\rho_2(G) \subseteq \sigma_2(G)$, so that we obtain from the above remarks that

$$\rho_2(G) \subseteq \varphi(G) \cap R(G).$$

Conversely, if $x \in \varphi(G) \cap R(G)$ and $y \in G$, we let $x_1, \cdots, x_m$ be a finite number of conjugates of x by elements of G. Since any homomorphism maps Engel elements onto Engel elements, the x_i are right Engel elements of G. Let

$$M = \{[x_i, ny] \mid i = 1, \cdots, m \quad \text{and} \quad n \geq 0\}$$

(here $[x, Oy] = x$). Then M is a finite subset of $\varphi(G)$, so $\langle M \rangle$ is nilpotent. Since y acts as a left Engel element on M, y normalizes $\langle M \rangle$ by Lemma 2 of [5] and hence $\langle M, y \rangle$ is nilpotent by Proposition 1 of [5]. Let $B = \langle M, y \rangle$. Then the set of all B so obtained is a local system of nilpotent subgroups of $x^G\langle y \rangle$, each member of which contains y. Since every subgroup of a nilpotent group is subnormal (H is subnormal in K if there is a finite normal series from H to K), we conclude that y wasc $x^G\langle y \rangle$, so that $x \in \rho_2(G)$.

COROLLARY 1.1 *The subsets $\sigma_2(G)$ and $\rho_2(G)$ are characteristic subgroups of G.*

Proof. By Theorem 1.1, $\varphi(G) = \sigma_2(G)$. The non-empty set $\rho_2(G)$ is is clearly inverse closed and a characteristic subset of G. It therefore suffices to prove that if x and z are elements of $\rho_2(G)$, so is xz.

By Lemma 14 of [5], xz is a right Engel element of G. It is clear from the definition that x and z belong to $\sigma_2(G) = \varphi(G)$. Hence, $xz \in \varphi(G)$. By Theorem 1.1, $xz \in \rho_2(G)$.

Remark 1.1. It is possible to have $\rho_2(G) < \sigma_2(G)$, as an examination of J_p wr J reveals (here, $\rho_2(G) = 1$ and $\sigma_2(G)$ is the base group).

We now define the classes of groups

$$\mathfrak{L}_i = \{G \mid L(G) = \sigma_i(G)\} \quad \text{and} \quad \mathfrak{R}_i = \{G \mid R(G) = \rho_i(G)\}, \qquad i = 1, 2.$$

(These are the classes of groups in which left or right Engel elements are "well behaved" in reasonable senses).

Remark 1.2. From Theorem 1.1 it follows that $G \in \mathfrak{L}_2$ iff $\langle L(G)\rangle$ is locally nilpotent. Analogous statements hold for the other Engel classes.

PROPOSITION 1.1. *The following are equivalent:*
(1) $G \in \mathfrak{R}_2$.
(2) $\langle R(G)\rangle \in \mathfrak{L}_2$.
(3) $\langle R(G)\rangle$ *is locally nilpotent.*

Proof. Since $\rho_2(G) \subseteq \varphi(G)$, the fact that (1) implies (2) and (2) implies (3) is clear. Now if $\langle R(G)\rangle$ is locally nilpotent, then $\langle R(G)\rangle \subseteq \varphi(G)$. By Theorem 1.1 (ii), $R(G) \subseteq \rho_2(G)$. Consequently, we have equality and $G \in \mathfrak{R}_2$.

PROPOSITION 1.2. *The following are equivalent:*
(1) $G \in \mathfrak{R}_1$.
(2) $\langle R(G)\rangle$ *is* $\acute{E}\mathfrak{A}$.
(3) $\langle R(G)\rangle \in \mathfrak{L}_1$.

Proof. Baer has shown [3, Satz 3.3] that the locally nilpotent $\acute{E}\mathfrak{A}$ groups are precisely the groups generated by ascendant elements. Consequently, if $G \in \mathfrak{R}_1$ then $R(G)$ is a locally nilpotent $\acute{E}\mathfrak{A}$ group. If $\langle R(G)\rangle = K$ is $\acute{E}\mathfrak{A}$, then every left Engel element of K is ascendant in K by Proposition 3 of [5]. But Heineken has shown that the inverse of a right Engel element is a left Engel element [13, p. 210]. So K is generated by ascendant elements and is locally nilpotent; thus $K \in \mathfrak{L}_1$. Finally, if $K \in \mathfrak{L}_1$ then again K is generated by ascendant elements, and consequently K is locally nilpotent. Therefore, $K \subseteq \varphi(G)$ and $R(G) = K$ by Lemma 14 of [5]. Now if $x \in R(G)$ and $y \in G$, then $x^G\langle y\rangle$ is an extension of an $\acute{E}\mathfrak{A}$ group by a cyclic group and therefore $\acute{E}\mathfrak{A}$. But y is a left Engel element of $x^G\langle y\rangle$, so y asc $x^G\langle y\rangle$ by Lemma 14 of [5]. Hence, $G \in \mathfrak{R}_1$.

COROLLARY 1.2. *The group G is an $\mathfrak{L}_1$ group iff $\langle L(G)\rangle$ is an $\acute{E}\mathfrak{A}$ group.*

Relations between the Engel classes are clarified by the following result:

THEOREM 1.2. *The following are proper class inclusions:*
(i) $\mathfrak{L}_1 \subseteq \mathfrak{L}_2$
(ii) $\mathfrak{R}_1 \subseteq \mathfrak{R}_2$
(iii) $\mathfrak{L}_1 \subseteq \mathfrak{R}_1$
(iv) $\mathfrak{L}_2 \subseteq \mathfrak{R}_2$.

Proof. The inclusions (i) and (ii) follow from the inclusions $\sigma_1(G) \subseteq \sigma_2(G)$ and $\rho_1(G) \subseteq \rho_2(G)$. That they are proper follows from the fact that there exist locally nilpotent groups which are not $\acute{E}\mathfrak{A}$ (see [8]). If $G \in \mathfrak{L}_1$, then the right Engel elements of G are contained in $\sigma_1(G)$, since inverses of right Engel elements are left Engel elements and $L(G) = \sigma_1(G)$. By Proposition 1.2, $G \in \mathfrak{R}_1$. A similar application of Proposition 1.1 shows that $\mathfrak{L}_2 \subseteq \mathfrak{R}_2$. It is known that there exist groups in which every element is an Engel element and which are locally nilpotent (see [4, p. 274, footnote]). Let G be such a group. From the definition of wreath products, it is easy to see that $R(G \text{ wr } J) = 1$, so that G wr J belongs to $\mathfrak{R}_1$. However, every element of the base group is a left Engel element of G wr J and the base group has subgroups isomorphic to G, so is not locally nilpotent.

Remark 1.3. Inclusions (iii) and (iv) of the previous theorem are particularly interesting. They say, in effect, that whenever the left Engel elements are well behaved, so are the right Engel elements. Consequently, the classes $\mathfrak{L}_1$ and $\mathfrak{L}_2$ are of special interest.

2. The classes

Plotkin has shown in [11, Lemma 5] that $L\mathfrak{L}_2 = \mathfrak{L}_2$. Since there are locally nilpotent non-$\acute{E}\mathfrak{A}$ groups, $L\mathfrak{L}_1 \neq \mathfrak{L}_1$. The closure properties of the classes $\mathfrak{L}_i$ are not encouraging in this respect:

PROPOSITION 2.1. *The classes $\mathfrak{L}_1$ and $\mathfrak{L}_2$ are not subgroup or quotient closed.*

Proof. As in the proof of Theorem 1.2 let G be a non-locally nilpotent group whose elements are all Engel elements. Let H be the free product of G and J. It follows readily from the properties of free products that if x is a non-trivial element of H, then there is an element $y \in H$ such that

$$[x, ny] \neq 1 \neq [y, nx]$$

for all positive integers n. Therefore, $L(G) = R(G) = 1$. So H is an $\mathfrak{L}_1$ group. However, H has a subgroup, G, which is not an $\mathfrak{L}_2$ group. Also, $H/J^H \cong G$, so H has a factor group which does not belong to $\mathfrak{L}_2$. Hence, $\mathfrak{L}_i$ is neither subgroup or quotient closed, $i = 1, 2$.

Let us define $\mathfrak{E}_i$ as the union of all classes $\mathfrak{X}$ which contain all cyclic groups, are contained in $\mathfrak{L}_i$ and are S, Q and $\acute{E}$ closed ($i = 1, 2$). The $\mathfrak{E}_i$ are large subclasses of $\mathfrak{L}_i$. As an illustration, Plotkin has shown in [12] that $\acute{E}LM$ groups (i.e., the LM-radical groups) belong to $\mathfrak{E}_2$. By Corollary 1.2, the $\acute{E}\mathfrak{A}$ groups belong to $\mathfrak{E}_1$. We shall see in Corollary 2.1 that $\acute{E}\mathfrak{M}$ groups belong to $\mathfrak{E}_1$.

THEOREM 2.1. *$\mathfrak{E}_i$ is the largest $\{S, Q, \acute{E}\}$ closed subclass of $\mathfrak{L}_i$ containing the cyclic groups ($i = 1, 2$).*

Proof. The classes $\acute{E}\mathfrak{E}_i$ evidently contain the cyclic groups and are $\acute{E}$ closed. Furthermore, any factor (homomorphic image of a subgroup) of an $\acute{E}\mathfrak{E}_i$ group has an ascending series with factors (quotient group of successive terms) which are isomorphic to factors of $\mathfrak{E}_i$ groups. Consequently, $\acute{E}\mathfrak{E}_i$ are S and Q closed classes. Thus, to complete the proof of the Theorem it is sufficient to prove that $\acute{E}\mathfrak{E}_i \subseteq \mathfrak{L}_i$. Let $G \in \acute{E}\,\mathfrak{E}_i$; then G has a series

$$1 = G_0 \subseteq G_1 \subseteq .... \subseteq G_\lambda = G$$

such that $G_{\mu+1}/G_\mu \in \mathfrak{E}_i$, $O \leq \mu < \lambda$. Suppose $1 \neq G_1 \in \mathfrak{X}$, where $\mathfrak{X}$ is one of the classes defining $\mathfrak{E}_i$. By Satz 2.1 of [3, p. 49], G has a characteristic $\mathfrak{X}$ subgroup C containing G_1. Let $x_1, \ldots, x_n$ be left Engel elements of G. If the group generated by these elements, say B, is not nilpotent, then there is a subgroup M normal in B such that B/M is not nilpotent but every proper homomorphic image is by Lemma 4 of [1, p. 410]. Hence, there is an $\acute{E}\mathfrak{E}_i$ group K which is generate by a finite number of left Engel elements, non-nilpotent, and every proper homomorphic image of K is nilpotent. By the preceding discussion, K has a non-trivial normal $\mathfrak{X}$ subgroup C, where $\mathfrak{X}$ is one of the classes defining $\mathfrak{E}_i$. If $C = K$, K is nilpotent since $\mathfrak{X} \subseteq \mathfrak{L}_i$. Hence, $C \neq K$ and K/C is nilpotent. If x is one of the Engel element generators of K, then $\langle xC \rangle$ is subnormal in K/C, so that $C\langle x \rangle$ is subnormal in K. But $C\langle x \rangle$ is an $\mathfrak{X}$ group, since $\mathfrak{X}$ contains the cyclic groups and is $\acute{E}$ closed. Since $\mathfrak{X} \subseteq \mathfrak{L}_i$ and x is a left Engel element of $C\langle x \rangle$, $x \in \varphi(C\langle x \rangle)$, which is subnormal in G. By Theorem 1 of [11], $\varphi(C\langle x \rangle) \subseteq \varphi(K)$, so that K is nilpotent, a contradiction. Hence, any $\acute{E}\mathfrak{E}_i$ group is an $\mathfrak{L}_2$ group.

It remains to prove that if G is $\acute{E}\mathfrak{E}_1$ and $x \in L(G)$, then x asc G. By repeating the argument mentioned above, transfinitely if necessary, we see that G has an ascending characteristic series

$$1 = G_0 \subseteq G_1 \subseteq \cdots \subseteq G_\lambda = G,$$

where $G_{\mu+1}/G_\mu$ is an $\mathfrak{X}$ group ($\mathfrak{X}$ is as above). Let $x \in L(G)$. For $\mu < \lambda$, we have $G_\mu \triangleleft G_{\mu+1}\langle x \rangle$. Furthermore, $G_{\mu+1}\langle x \rangle/G_\mu$ is an $\mathfrak{X}$ group, since $\mathfrak{X}$ is $\acute{E}$ closed and contains the cyclic groups. Thus,

$$xG_\mu \text{ asc } G_{\mu+1}\langle x \rangle/G_\mu \quad \text{and} \quad G_\mu\langle x \rangle \text{ asc } G_{\mu+1}\langle x \rangle.$$

Since $G_0\langle x \rangle = \langle x \rangle$, x asc G and the theorem follows.

COROLLARY 2.1 *If every non-trivial homomorphic image of the group G has a non-trivial ascendant subgroup which is noetherian or $\acute{E}\mathfrak{A}$, then $G \in \mathfrak{L}_1$.*

Proof. Let G be such a group. Then G has an ascending invariant series whose factors are groups generated by their ascendant $\acute{E}\mathfrak{A}$ or their ascendant noetherian groups. Hence, it is sufficient to prove that $\acute{E}\mathfrak{A} \subseteq \mathfrak{E}_1$; for then $G \in \acute{E}\,\mathfrak{E}_1$, since $\acute{E}\mathfrak{A} \subseteq \mathfrak{E}_1$. Let $K \in \acute{E}\mathfrak{M}$; then K has an ascending normal series with noetherian factors. Since $\acute{E}L\mathfrak{M} \subseteq \mathfrak{E}_2$, K is an $\mathfrak{E}_2$ group; so if $x \in L(K)$, then x^G is a normal locally nilpotent $\acute{E}\mathfrak{M}$ group. But noetherian locally nilpotent groups are nilpotent. Hence, an $\acute{E}\mathfrak{M}$ series for x^G can be refined to an ascending series with abelian factors. So x^G is an $\acute{E}\mathfrak{A}$ group and x asc $x^G \triangleleft G$. Therefore, $\acute{E}\mathfrak{M} \subseteq \mathfrak{E}_1$ and the result follows.

For $\mathfrak{E}_2$ groups Theorem 2.1 can be extended to the following result:

THEOREM 2.2. *Hyper-$L\mathfrak{E}_2$ groups are $\mathfrak{L}_2$ groups.*

Proof. Deny the assertion. Since the class of hyper-$L\mathfrak{E}_2$ groups is S and Q closed, we can repeat the reduction process of Theorem 2.1 and obtain a non-nilpotent hyper-$L\mathfrak{E}_2$ group G which is generated by a finite number of left Engel elements and is such that every proper homomorphic image is nilpotent. Then G has a non-trivial normal $L\mathfrak{E}_2$ subgroup K. If $K = G$, then $G \epsilon L\mathfrak{K}_2 = \mathfrak{K}_2$ (by Theorem 1 of [11]), a contradiction. So $K \neq G$ and G/K is nilpotent. So if $x \epsilon L(G)$, then $K\langle x\rangle$ is subnormal in G. Now if $x_1, \cdots, x_n$ belong to K, let

$$M = \{[x_i, mx] \mid i = 1, \cdots, n \quad \text{and} \quad m \geq 0\}.$$

Since $\mathfrak{E}_2$ is subgroup closed, $\langle M\rangle$ is an $\mathfrak{E}_2$ group normalized by the left Engel element x (as in the proof of Theorem 1.1, apply Lemma 2 and Proposition 1 of [5]). But the extension $\langle M\rangle \langle x\rangle$ is an $\mathfrak{E}_2$ group, since $\mathfrak{E}_2$ contains the cyclic groups. The collection of all groups so obtained forms a local system of $\mathfrak{E}_2$ groups for $\langle M, x\rangle$. Hence, $\langle M, x\rangle$ is an $\mathfrak{K}_2$ group and

$$x \epsilon \varphi(\langle K, x\rangle) \subseteq \varphi(G).$$

Since x was an arbitrary element of $L(G)$, G is nilpotent, a contradiction.

COROLLARY 2.2. *If every non-trivial homomorphic image of the group G has a normal $L\acute{E}L\mathfrak{M}$ subgroup, then $G \epsilon \bar{K}_2$.*

Remark 2.1. We have not been able to decide whether or not $L\mathfrak{E}_2 = \mathfrak{E}_2$. If this were the case, one would have $\acute{E}L\mathfrak{E}_2 \subseteq \mathfrak{E}_2$, a significant extension of Theorem 2.2.

3. The weak hypercenter

Recall that a normal subgroup K of G is called *hypercentral* (*finitely hypercentral*) *in* G if K has an ascending (finite) series which is central in G, i.e., if (L, M) is a jump in the series, then $[G, M] \subseteq L$. Equivalently, $K \subseteq \alpha(G)$. If G is hypercentral (finitely hypercentral) in itself, we simply say G is a hypercentral (nilpotent) group.

More generally, we say that K is *Z hypercentral in G* if K has a normal series (in the sense of P. Hall [7, p. 2]) which is central in G, i.e., if (L, M) is a jump in the series, than $[G, M] \subseteq L$. It follows that $\mathfrak{Z}$ is the class of groups which are Z hypercentral in themselves. A disadvantage of Z hypercentrality is that Z hypercentral groups need not map onto Z hypercentral groups under homomorphisms. More suitable for our purposes is the following concept of hypercentrality: The normal subgroup K of G is *weakly hypercentral in* G if for any G-invariant subgroup M of K, K/M is Z hypercentral in themselves is the largest quotient closed subclass of $\mathfrak{Z}$ (the Z groups in Kurosh's terminology). Such groups are called weakly hypercentral groups.

Remark 3.1. The concepts of hypercentrality given above are distinct. Non-finitely hypercentral groups which are hypercentral are well known. Every free group is Z hypercentral, since the descending central series has trivial intersection. However, non-abelian simple groups are homomorphic images of free groups, so free groups are not weakly hypercentral.

In fact, while it is true that subgroups of finitely hypercentral, hypercentral and Z hypercentral groups have the corresponding property, this is not so for weakly hypercentral groups. Kargapolov has constructed in [8] a weakly hypercentral group with a non-abelian free subgroup. But non-abelian free groups are not weakly hypercentral. The situation is somewhat different with normal subgroups.

LEMMA 3.1. *If K is weakly hypcercentral in G and $M \lhd G$, then $M \cap K$ is weakly hypercentral in G.*

Proof. If L is a G-invariant subgroup of $M \cap K$, then K/L is Z hypercentral in G/L and omit repeats. Since $M \cap K$ is normal in G, the series so obtained is a Z hypercentral series for $M \cap K$ in G. Since L was arbitrary, $M \cap K$ is weakly hypercentral in G.

LEMMA 3.2. *If β maps G onto H and K is weakly hypercentral in G, then $K\beta$ is weakly hypercentral in H.*

Proof. Let $L = \ker \beta$ so that without loss of generality, $H = G/L$ and β is the natural map. The map β induces a homomorphism from $G/(K \cap L)$ to G/L whose restriction to $K/(K \cap L)$, π, is an isomorphism. Clearly, $K\beta = KL/L$, which is the image of π. If M/L is any G/L-invariant subroup of KL/L, then $M \cap K$ is a G-invariant subgroup of K. Since K is weakly hypercentral in G, there is an invariant (in G) series running from $M \cap K$ to K whose factors are centralized by elements of G. Since π is an isomorphism, π maps such a series onto a series in G/L running from M/L to KL/L whose factors are centralized by elements of G/L. It follows that $K\beta/(M/L)$ is Z hypercentral in $(G/L)/(M/L)$; since M was arbitrary, $K\beta$ is weakly hypercentral in G/L.

We can now prove the following basic fact about weakly hypercentral subgroups:

THEOREM 3.1. *Every group G has a unique maximum subgroup which is weakly hypercentral in G.*

Proof. From the definition, subgroups of G which are weakly hypercentral in G are normal in G. Well order the set of such subgroups to obtain the collection $\{G_\gamma \mid 0 \leq \gamma \leq \lambda\}$ (set $G_0 = 1$). Let H be the subgroup of G generated by this collection of normal subgroups and let asterisks denote images of elements of G under a homomorphism. Define

$$K_\gamma = \langle G_\alpha^* \mid \alpha \leq \gamma \rangle, \qquad 0 \leq \gamma \leq \lambda.$$

Then if $\alpha < \beta$, $K_\alpha \subseteq K_\beta$, $K_0 = 1$ and $K_\lambda = H^*$. For $\gamma < \lambda$, we have

$$K_{\gamma+1}/K_\gamma \cong G_{\gamma+1}^*/(G_{\gamma+1}^* \cap K_\gamma) = T,$$

which is Z hypercentral in $G^*/(G_{\gamma+1}^* \cap K_\gamma)$. The map from this last group onto G^*/K_γ maps a Z hypercentral series of T in $G^*/(G_{\gamma+1}^* \cap K_\gamma)$ onto a

Z hypercentral series of $K_{\gamma+1}/K_\gamma$ in G^*/K_γ. Hence, we may refine the ascending invariant series $\{K_\gamma\}$ to a Z hypercentral series of H^* in G^*. Since the homomorphism was arbitrary, H is weakly hypercentral in G. The group H obviously contains every such subgroup, which proves the theorem.

Remark 3.2. We shall call the unique maximum subgroup of G which is weakly hypercentral in G the *weak hypercenter* of G and denote it by $\bar{\alpha}(G)$. Clearly, $\alpha(G) \subseteq \bar{\alpha}(G)$ and Kargapolav's example in [8] shows that inclusion may be proper. It is also shown in [8] that every group G has a unique maximum normal weakly hypercentral subgroup, which we denote by $\bar{\varphi}(G)$. Since homomorphic images of locally nilpotent groups are locally nilpotent and the class $\mathfrak{Z}$ satisfies the local theorem (see [10, p. 218]), it follows that $\bar{\varphi}(G) \subseteq \bar{\varphi}(G)$.

The property of being Z hypercentral in G is a "local" property in the following sense:

THEOREM 3.2. *If H is a subgroup of G and G has a local system $\{G_\beta\}$ of subgroups such that $H \cap G_\beta$ is Z hypercentral in G_β, for each β, then H is Z hypercentral in G.*

Proof. The proof is almost the same as Kurosh's proof of the local theorem for SN groups in [10, p. 183]. We will therefore assume the reader has this text at hand and indicate the necessary modifications. Let $\mathfrak{S}_\beta$ be a Z hypercentral series of $H \cap G_\beta$ in G_β. For elements $h \in H$ and $g \in G$ and every G_β containing both of these elements let $(C^\beta_{h,g}, \bar{C}^\beta_{h,g})$ be the jump such that $C^\beta_{h,g}$ is the largest member of the system $\mathfrak{S}_\beta$ avoiding the non-trivial elements of $\{h, g\}$ and $C^\beta_{h,g}$ contains $H \{h, g\}$. Then $[g, h] \in C^\beta_{h,g}$. Now construct the local systems linked with pairs and sets of pairs of the form (h, g), $h \in H$, $g \in G$, as Kurosh does, and form the system $\{H(a, b)\}$ of subgroups of H. The completion of this system yields a Z hypercentral series of $[H, G]$ in G. Add H to this series and the theorem follows.

We now find some conditions under which weak hypercentrality implies hypercentrality.

PROPOSITION 3.1. *Let K be a Noetherian solvable subgroup of G which is weakly hypercentral in G. Then K is hypercentral in G.*

Proof. Deny the conclusion. Then there is a group G with non-hypercentral Noetherian solvable subgroup K which is weakly hypercentral in G. By passing to the factor group $G/\alpha(G) \cap K)$ and replacing K by $K/\alpha(G) \cap K)$, we may assume $\alpha(G) \cap K = 1$. Since K is solvable, K has a non-trivial characteristic abelian subgroup which is weakly hypercentral in G by Lemma 3.1. Replace K by this subgroup and we may assume that K is abelian and $K \cap \alpha(G) = 1$. The set of normal subgroups H of G contained in K such that K/H is not hypercentral in G/H is non-empty, since 1 is one of them. This collection has a maximal element L, since K is Noetherian. By passing to the factor group G/L we may assume that K is non-hypercentral in G, but if M is a proper G-invariant subgroup of K, then K/M is hypercentral in G/M. Hence, $K \cap Z(G) = 1$. For if not, then $K/K \cap Z(G))$ is hypercentral in $G/K \cap Z(G))$, which implies that K is hypercentral in G, a contradiction. If K is not torsion-free, then the periodic subgroup P of G is non-trivial and characteristic in K, so normal in G. Therefore, P is Z hypercentral in G. But K is Noetherian, so P is finite and hence $K \cap Z(G) \neq 1$, contradiction. Consequently K is torsion-free. If p is a prime, then K^p is a proper characteristic subgroup of K, so K/K^p is hypercentral in G/K^p. Also, K/K^p has order p^n, where n is the rank of K. Consequently we have

$$[K, nG] \subseteq \bigcap \{K^p \mid p \text{ is prime}\} = 1.$$

It follows that K is hypercentral in G, a contradiction, and the proof is complete.

COROLLARY 3.1. *Every Noetherian solvable weakly hypercentral group is hypercentral.*

COROLLARY 3.2. *If K is weakly hypercentral in G and K has an ascending G-invariant series whose factors are Noetherian solvable, then K is hypercentral in G.*

Proof. Deny the conclusion. Then $K \cap \alpha(G) < K$. By passing to the factor group of G by $K \cap \alpha(G)$, we may assume that $K \cap \alpha(G) = 1$, since the image of K will be weakly hypercentral in the image of G. However, K has a non-trivial Noetherian solvable subgroup M which is normal in G. By Lemma 3.1, M is weakly hypercentral in G. Hence $M \subseteq \alpha(G)$ by Proposition 3.1, a contradiction.

The automorphism β of G is *algebraic* if for every $x \in G$, the group generated by $S = \{[x, n\beta] \mid n \geq 0\}$ can be finitely generated. It is easy to verify that the group generated by S is the same as the group generated by $\{(x)\beta^n \mid n \geq 0\}$. Consequently, if for every $x \in G$ there is a non-zero integer m such that $(x)\beta^m = x$, then β is algebraic.

THEOREM 3.2. *Let G be a group; the following are equivalent:*

(A) *G is a hypercentral group.*

(B) (i) *G is a weakly hypercentral group.*

(ii) *Non-trivial homomorphic images of G have non-trivial normal Noetherian solvable subgroups.*

(C) (i) *G is a weakly hypercentral group.*

(ii) *Non-trivial homomorphic images of G have non-trivial finitely generated normal $\acute{E}\mathfrak{A}$ subgroups.*

(iii) *Inner automorphisms of G are algebraic.*

Proof. If G is hypercentral, the upper central series of G can be refined to an ascending invariant series with cyclic factors. Since any homomorphic image of G also has this property, G satisfies (B) (ii). That G satisfies

(B) (i) is clear. Conversely, if G satisfies (B), it follows from Corollary 3.2 that G is a hypercentral group.

The equivalence of the conditions (B) and (C) follow from the facts that an $\acute{E}\mathfrak{A}$ group whose inner automorphisms are algebraic is locally Noetherian [3, Folgerung 4.11] and hypercentral groups are locally nilpotent.

Remark 3.3. The hypotheses of Proposition 3.1, its corollaries and Theorem 3.2 could be weakened if it could be shown that Noetherian weakly hypercentral groups are nilpotent, a possibility we have not been able to confirm or deny.

4. The global descriptions

Conditions under which $R(G)$ and $L(G)$ may be described as finitely hypercentral (in G) and nilpotent subgroups of G are rather special. Baer has shown, for instance, that if $G \in \mathfrak{L}_2$ it is sufficient for $\varphi(G)$ to satisfy the maximum condition on abelian subgroups (see [2, p. 257]).

By applying a hypercentrality criterion of Gruenberg, one easily obtains sufficient conditions for hypercentral descriptions of Engel elements of $\mathfrak{L}_2$ groups. Recall that the abelian group A has finite rank n if every finitely generated subgroup is contained in one generated by at most n elements.

PROPOSITION 4.1. *If $\varphi(G)$ has an ascending G-invariant series whose factors are abelian of finite rank, then $\rho_1(G) = \alpha(G)$ and $\sigma_1(G) = \varphi(G)$.*

Proof. To show that $\rho_1(G) = \alpha(G)$ it is sufficient to prove that

$$\rho_1(G) \cap Z(G) > 1,$$

since $\alpha(G) \subseteq \rho_1(G)$ and images in any factor group of ascendant elements are themselves ascendant elements. But $\rho_1(G) \subseteq \varphi(G)$ so that $\rho_1(G)$ has an abelian normal subgroup of finite rank and consisting of right Engel elements. By Proposition 1.1 (ii) of [6],

$$\rho_1(G) \cap Z(G) > 1$$

and the result follows.

The equality $\sigma_1(G) = \varphi(G)$ follows from the fact that $\varphi(G)$ is a locally nilpotent $\acute{E}\mathfrak{A}$ group, so that $\varphi(G) \subseteq \sigma_1(\varphi(G)) = \sigma_1(G)$.

Remark 4.1. If G satisfies the hypotheses of Proposition 4.1 and $G \in \mathfrak{L}_2$, then $L(G) = \varphi(G)$, which is the maximum hypercentral normal subgroup of G, since

$$\alpha(\varphi(G)) = \rho_1(\varphi(G)) = \varphi(G)$$

by Proposition 4.1. It also follows from Proposition 4.1 and Theorem 1.2 (iii) that $R(G) = \alpha(G)$, the hypercenter of G. In the remainder of this section we develop $\bar{\alpha}$ and $\bar{\varphi}$ analogues to these hypercentral descriptions of Engel elements.

LEMMA 4.1. *If K is a G-invariant subgroup of the normal subgroup H of*

G, G/H is finite or $G/H = \langle gH \rangle$, where g acts as a left Engel element on K, $\{K_\gamma\}$ is an H-invariant series of K and

$$L_\gamma = \bigcap \{K_\gamma^g \mid g \in G\},$$

then the system $\mathfrak{L} = \{L_\gamma\}$ yields, after omission of repeats, a G-invariant series for K.

Proof. Let S be a non-empty subset of the index set Γ. Since

$$\bigcap \{K_\delta^g \mid \delta \in S \text{ and } g \in G\} = \bigcap \{\bigcap \{K_\delta^g \mid \delta \in S\} \mid g \in G\},$$

intersection of terms of the system $\mathfrak{L}$ belong to the system. Let

$$K_\mu = \bigcup \{K_\delta \mid \delta \in S\}.$$

Clearly, $\bigcup \{L_\delta \mid \delta \in S\} \subseteq L_\mu$. If μ has a largest predecessor, the converse equality follows. Suppose μ has no largest predecessor and let $x \in L_\mu$.

Case 1. G/H is finite. Let $\{g_1, \cdots, g_n\} = T$ be a complete set of coset representatives of H in G. Since the K_γ are invariant in H, $L_\gamma = \{K_\gamma^g \mid g \in T\}$. So there are elements $x_i \in K$ such that

$$x = x_i^{g_i}, \qquad i = 1, \cdots, n.$$

Hence, there is a $\delta < \mu$ such that $x_i \in K_\delta$, $i = 1, \cdots, n$. It follows that

$$x \in \bigcap \{K_\delta^g \mid g \in T\} = L_\delta .$$

Case 2. $G/H = \langle gH \rangle$ and g acts as a left Engel element on K. Then the set $\{[x, ng] \mid n \geq 0\}$ is finite and contained in L_μ (for L_μ is clearly normalized by G). Hence, there is a $\delta < \mu$ such that this finite set is contained in K_δ. By Lemma 4 of [5], $x \in L_\delta$.

In either case, it follows that unions of members of $\mathfrak{L}$ belong to $\mathfrak{L}$. Clearly 1 and K belong to $\mathfrak{L}$.

Now suppose that (L_λ, L_μ) is a jump in $\mathfrak{L}$. Since $\{K_\gamma\}$ is complete, there is a largest $\sigma \in \Gamma$ such that $L_\sigma = L_\lambda$ and a smallest $\rho \in \Gamma$ such that $L_\rho = L_\mu$. It follows that (K_δ, K_ρ) is a jump in $\{K_\gamma\}$. Hence, $K_\sigma \lhd K_\rho$. By definition of the members of $\mathfrak{L}$ we conclude that $L_\sigma \lhd L_\rho$. Hence, $\mathfrak{L}$ is a normal series of K (after omission of repeats).

LEMMA 4.2. *If K is a normal (in G) subgroup of right Engel elements of G which is Z hypercentral in the normal subgroup H of G, and G/H is cyclic or finite, then K is Z hypercentral in G.*

Proof. Let $\{K_\gamma\}$ be a Z hypercentral series of K in H. Form the series $\{L_\gamma\}$ as in Lemma 4.1. Since $K \subseteq R(G)$, elements of G act as left Engel elements on K. Consequently, $\{L_\gamma\}$ is a normal series of K. Clearly, each L_γ is normalized by G. Let (L_λ, L_μ) be a jump of the series. One sees from the proof of Lemma 4.1 that we may assume (K_λ, K_μ) is a jump in $\{K_\gamma\}$. Therefore, if $g \in G$, $[H, K_\mu^g] \subseteq K_\lambda^g$, since $\{K_\gamma\}$ is a Z hypercentral series in H. It follows from the definition of L_γ that $[H, L_\mu] \subseteq L_\lambda$. Therefore, $\{L_\gamma\}$ is a Z hypercentral series of K in H.

Again, let (L_λ, L_μ) be a jump. Since $[H, L_\mu] \subseteq L_\lambda$, the group of automorphisms A of L_μ/L_λ induced by elements of G is a homomorphic image of G/H, so is finite or cyclic. Let T_g denote the automorphism induced by g.

Case 1. $A = \langle T_g \rangle$, for some $g \in G$. Define $A_0 = L$ and, inductively,

$$A_{n+1} = \{x \in L_\mu \mid [x, g] \in A_n\}.$$

Since L_μ/L_λ is abelian, the chain

$$A_0 \subseteq A_1 \subseteq \cdots \subseteq A_n \subseteq \cdots$$

is an increasing chain of subgroups. Since g acts as a left Engel element on K, L_μ is the union of all the A_i.

Case 2. $A = \{T_{g_1}, \cdots, T_{g_m}\}$. Then let $A_0 = L$ and, inductively,

$$A_{n+1} = \{x \in L_\mu \mid [x, g_i] \in A_n,\ i = 1, \cdots, m\}.$$

Then if $x \in A_{n+1}$, and $g \in G$, $[x, g] \in A_n$. As in Case 1, we obtain an ascending chain of subgroups. If $x \in L$, there is a positive integer r such that $[x, rg_i] = 1$, $i = 1, \cdots, m$, since $x \in R(G)$. Therefore, $x \in A_r$, so L_μ is the union of all the A_i.

In either case we may insert the A_i into the series $\{L_\gamma\}$ at the appropriate jump. The new series obtained by doing this at every jump is a Z hypercentral series of K in G, since $[G, A_{n+1}] \subseteq A_n$. This proves the lemma.

We shall denote by $\mathfrak{N}$ the class of all groups posessing an ascending normal series with factors which are cyclic or finite. Note that $\mathfrak{N} \subseteq \acute{E}\mathfrak{M}$.

PROPOSITION 4.2. *If G is a hyper-$L\,\mathfrak{N}$ group, then $R(G) \subseteq \bar{\alpha}(G)$.*

Proof. Assume the ascending invariant series

$$1 = G_0 \subseteq G_1 \subseteq \cdots \subseteq G_\rho = G$$

has $L\mathfrak{N}$ factors. Then G is an $\mathfrak{S}_2$ group by Corollary 2.2, so that $R(G)$ is a locally nilpotent subgroup of G by Theorem 1.2 (iv). We proceed by induction to prove that $R(G) \cap G_\beta$ is Z hypercentral in G_β.

If $\beta = 1$ and H is a finitely generated subgroup of G_1, then $R(G) \cap H$ is a normal locally nilpotent subgroup of right Engel elements of H. Consequently, $H/(H \cap R(G))$ has an ascending normal series with cyclic or finite factors, say

$$\{K_\gamma/(R(G) \cap H) \mid 0 \leq \gamma \leq \alpha\}.$$

Induct on the oridinal α to show that $R(G) \cap H$ is Z hypercentral in H. The case $\alpha = 0$ is clear. If $\alpha > 0$ and the result is true for series of smaller length, there are two cases to consider. If α is a limt ordinal, then $R(G) \cap H$ is Z hypercentral in K_δ, $\delta < \alpha$, and $H = \bigcup \{K_\delta \mid \delta < \alpha\}$. By Theorem 3.2,

$R(G) \cap H$ is Z hypercentral in H. If $\alpha = \delta + 1$, then $R(G) \cap H$ is Z hypercentral in K_δ, K_α/K_δ is finite or cyclic and generated by elements which act as left Engel elements on $R(G) \cap H$. By Lemma 4.2, $R(G) \cap H$ is Z hypercentral in $K_\alpha = H$, completing the induction. The set of all finitely generated subgroups of G_1 form a local system for G_1, so that $R(G) \cap G_1$ is Z hypercentral in G_1 by Theorem 3.2.

Inductively, suppose $R(G) \cap G_\delta$ is Z hypercentral in G_δ, for all $\delta < \beta$, where $\beta \leq \rho$. If β is a limit ordinal, then G_β is the union of all G_δ, $\delta < \beta$, and $R(G) \cap G_\delta$ is Z hypercentral in G_δ; therefore, $R(G) \cap G_\beta$ is Z hypercentral in G_β by Theorem 3.2.

If β is not a limit ordinal, say $\beta = \delta + 1$, then $R(G) \cap G_\delta$ has a series $\{M_\gamma\}$ making $R(G) \cap G_\delta$ Z hypercentral in G_δ. Let H be generated by G_δ and a finite number of elements of G_β. Then H/G_δ is an $\acute{E}\mathfrak{N}$ group. We can repeat the induction argument used for $\beta = 1$ to obtain that $R(G) \cap G_\delta$ is Z hypercentral in H. Such H form a local system for G_β. By Theorem 3.2, $R(G) \cap G_\delta$ is Z hypercentral in G_β. We have also shown in the case $\beta = 1$ that the right Engel elements of any $L\mathfrak{N}$ group form a subgroup which is Z hypercentral in the group. Since any homomorphism maps right Engel elements onto right Engel elements, it follows that $R(G_\beta/G_\delta)$ is Z hypercentral in G_β/G_δ.

Let $T = R(G) \cap G_\beta$. Then TG_δ/G_δ is a normal subgroup of G_β/G_δ consisting of right Engel elements. Therefore, TG_δ/G_δ is Z hypercentral in G_β/G_δ. Let π be the usual isomorphism from TG_δ/G_δ onto $T/(T \cap G_\delta)$. Consider these groups to be operator groups with conjugation by elements of G as operators. For $t \in T$ and $G \in G_\beta$, we have

$$(tT)^g\pi = t^g(T \cap G_\delta) = ((tT)\pi)^g,$$

so that π is an operator isomorphism. If $\{L_\gamma\}$ is a Z hypercentral series for TG_δ/G_δ in G_β/G_δ, then $\{L_\gamma\pi\}$ is a series of G-admissible normal subgroups of $T/(T \cap G_\delta)$. Since π is an operator isomorphism, the series $\{L_\gamma\pi\}$ is a Z hypercentral series for $T/(T \cap G_\delta)$ in $G_\beta/(T \cap G_\delta)$.

If we take inverse images under the natural map of T onto $T/(T \cap G_\delta)$ of this series and add to this system the terms of a Z hypercentral series for $T \cap G_\delta$ in G_β, we obtain a Z hypercentral series for T in G_β; this proves that $R(G) \cap G_\beta$ is Z hypercentral in G_β and completes the induction. Therefore, $R(G) \cap G_\rho = R(G)$ is Z hypercentral in $G_\rho = G$.

Finally, suppose K is a G-invariant subgroup of $R(G)$. Then G/K is also a hyper-$L\mathfrak{N}$ group and $R(G)/K$ is a normal subgroup of right Engel elements of G/K. Since $R(G/K)$ is Z hypercentral in G/K, so is $R(G)/K$. It follows that $R(G)$ is weakly hypercentral in G, so $R(G) \subseteq \bar{\alpha}(G)$ by Theorem 3.1.

THEOREM 4.1. *Suppose the group G has an ascending invariant series*

$$1 = G_0 \subseteq G_1 \subseteq \cdots \subseteq G_\sigma = G$$

such that if $\delta < \sigma$, *then*

(1) $G_{\delta+1}/G_\delta$ *is Noetherian abelian, or*

(2) $G_{\delta+1}/G_\delta$ *is locally finite and the automorphisms induced by* G *are algebraic; then* $L(G) = \bar{\varphi}(G)$ *and* $R(G) = \bar{\alpha}(G)$.

Proof. By Proposition 4.2, $R(G) \subseteq \bar{\alpha}(G)$. Suppose this inclusion is proper. Then there is a first term G_δ containing an element x in $\bar{\alpha}(G)$ but not $R(G)$. Clearly δ is not a limit ordinal, so there is an ordinal β such that $\delta = \beta + 1$. Let asterisks denote images of G in G/G_β.

Let $K = x^G \subseteq G_{\beta+1}$. Then K^* is weakly hypercentral in G^* by Lemmas 3.1 and 3.2.

Case 1. $G_{\beta+1}/G_\beta$ satisfies condition (1) of the theorem. Since K^* is Noetherian abelian, K^* is hypercentral in G^* by Proposition 3.1. Hence, if $y \in G$, there is a positive integer n such that $z = [x, ny] \in G_\beta$. But $\bar{\alpha}(G)$ is normal in G, so that $z \in \bar{\alpha}(G) \cap G_\beta$. By our choice of β, $z \in R(G)$, so there is a positive interger m such that

$$1 = [z, my] = [x, (n + m)y].$$

Therefore, $x \in R(G)$, a contradiction.

Case 2. $G_{\beta+1}/G_\beta$ satisfies condition (2) of the theorem. Then if $M = \{[x, ny] \mid n \geq 0\}$, where y is any element of G, $\langle M^* \rangle$ is finitely generated and therefore finite. Since K^* is weakly hypercentral in G^*, K^* has a Z hypercentral series in G^*. By intersecting $\langle M^* \rangle$ with the series for K^*, we obtain a finite series for $\langle M^* \rangle$, say

$$1 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_t = \langle M^* \rangle.$$

Furthermore, if $1 \leq i < t$, there is a jump (K_δ, K_γ) in the Z hypercentral series for K^* such that $K_\delta \cap \langle M^* \rangle = M_i$ and $K_\gamma \cap \langle M^* \rangle = M_{i+1}$. Since $[G^*, K_\gamma] \subseteq K_\delta$, we obtain that

$$[\langle y^*, M^* \rangle, M_{i+1}] \subseteq K_\gamma \cap \langle M^* \rangle = M_i .$$

Therefore, the group $\langle M^*, y^* \rangle$ is nilpotent and there is an integer n such that $[x, ny] \in G_\beta$. Now repeat the argument at the end of Case 1 to obtain a contradiction. We conclude that $R(G) = \bar{\alpha}(G)$.

Since G is an $\mathfrak{L}_2$ group (Corollary 2.2), we have $L(G) = \varphi(G) \subseteq \bar{\varphi}(G)$. Consider the ascending invariant series $\{G_\gamma \cap \bar{\varphi}(G)\}$ of $\varphi(G)$; for $\gamma < \sigma$, the group

$$(G_{\gamma+1} \cap \bar{\varphi}(G))/(G_\gamma \cap \bar{\varphi}(G))$$

is isomorphic to a subgroup of $G_{\gamma+1}/G_\gamma$. Furthermore, if $G_{\gamma+1}/G_\gamma$ is locally finite, an automorphism of

$$(G_{\gamma+1} \cap \bar{\varphi}(G))/G_\gamma \cap \bar{\varphi}(G))$$

induced by an element of $\bar{\varphi}(G)$ is algebraic, since the corresponding automorphism of $G_{\gamma+1}/G_\gamma$ is algebraic. Therefore, $\bar{\varphi}(G)$ satisfies the hypotheses of Theorem 4.1. By our previous arguments, $R(\bar{\varphi}(G)) = \bar{\alpha}(\bar{\varphi}(G)) = \bar{\varphi}(G)$. Therefore, $\bar{\varphi}(G) \subseteq L(G)$, which proves that $\bar{\varphi}(G) = L(G)$.

THEOREM 4.2. *If G is a hyper-$L\mathfrak{N}$ group and $\varphi(G)$ has a G-invariant ascending series whose factors satisfy conditions* (1) *or* (2) *of Theorem* 4.1, *then* $L(G) = \varphi(G) \cap \varphi(G)$ *and* $R(G) = \varphi(G) \cap \bar{\alpha}(G)$.

Proof. Since $\varphi(G)$ is weakly hypercentral,

$$\varphi(G) = \varphi(G) \cap \bar{\varphi}(G) = L(G)$$

($G \in \mathfrak{L}_2$ by Corollary 2.2). Also, $R(G) \subseteq \varphi(G)$ by Theorem 1.2 (iv). By Proposition 4.2, $R(G) \subseteq \bar{\alpha}(G)$. The proof of the converse inclusion

$$\bar{\alpha}(G) \cap \varphi(G) \subseteq R(G)$$

is carried out in exactly the same way as the inclusion $\bar{\alpha}(G) \subseteq R(G)$ in Theorem 4.1, except that we replace $\bar{\alpha}(G)$ by $\bar{\alpha}(G) \cap \varphi(G)$. This completes the proof.

Remark 4.2. Theorems 4.1 and 4.2 are partial converses to Proposition 4.2. It seems likely improvements to these theorems can be made. In this connection, we have not been able to find an $\acute{E}\mathfrak{A}$ weakly hypercentral group that is not locally nilpotent.

Added in proof. It has come to the author's attention that the term "weakly hypercentral" has already been used by R. Baer in his article *Das Hyperzentrum einer Gruppe III*, Math. Zeitschrift, vol. 59(1953), pp. 299–338, where a stronger version of Proposition 3.1 is proved. A more suitable term for the hypercentrality discussed in this paper is "$\bar{Z}$-hypercentrality". For more on this notion see the author's article *On $\bar{Z}$-hypercentral normal subgroups*, Arch. Math., vol. 21(1970), pp. 344–348.

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