MORSE FUNCTIONS ON GRASSMANNIANS

BY
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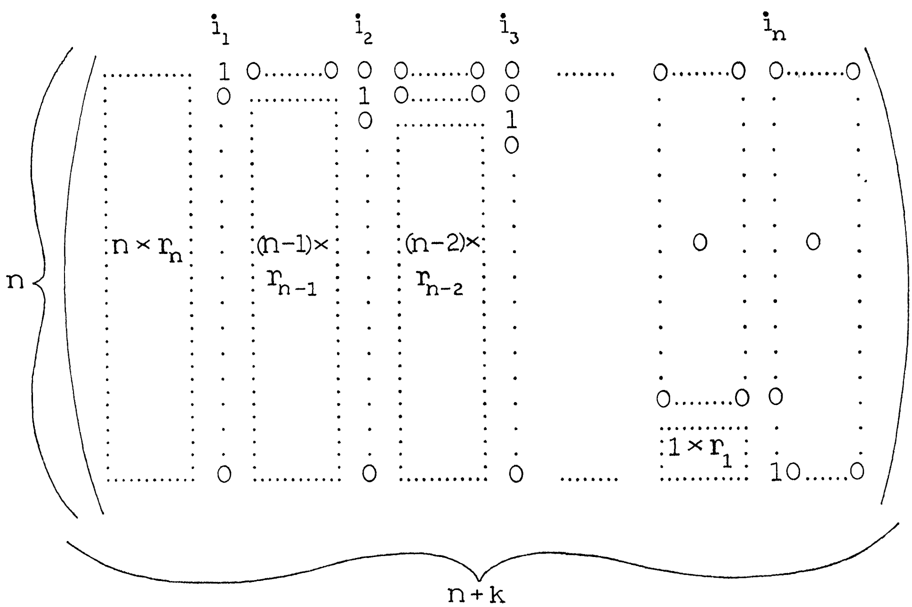
It is occassionally useful to have an explicit Morse function on a manifold. In this note we describe a quite explicit "nice" Morse function on the Grassmann manifolds $G(n, k)$ of n planes in $(n + k)$-space. We work on real Grassmanns for definitiveness; however with obvious adaptions, the procedure works for complex or quaternionic Grassmanns. In the case of



$G(1, k)$ (projective space), the resulting functions are of the type defined in [3, p. 26]. In the case of $G(2, k)$, they have the form of sectional curvature calculated at a point. It was consideration of this special case as developed in [2] that led to these results. We refer the reader to [3] for a general discussion of Morse theory. It is no doubt the case that several of the methods and/or results of this note are known; however we have followed the path of least resistance, which is to reprove them and to put them into a cohesive form rather than search the literature.

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The statement

We consider a particular well known model of $G(n, k)$. Let $M(n, k)$ be the set of all $n \times (n + k)$ matrices of rank n. $GL(n) = M(n, 0)$. If $X, Y \in M(n, k)$ define $X \sim Y$ if $X = \tau Y$ for some $\tau \in GL(n)$. This is the equivalence relation of row equivalence. Then $G(n, k) = M(n, k)/\sim$. The correspondence is as follows: fix a basis in $(n + k)$ space and then the row vectors of X determine an n-plane. It is a known result, and is in any event a straightforward computation in linear algebra, that a set of representatives of $G(n, k)$ are the reduced echelon matrices; those of the form shown in the figure where the elements in the "boxes" of size $\mu \times r_\mu$ are arbitrary. The i_μ are column numbers. If the i_μ are fixed and the elements in these boxes vary over all values, the resulting set in $G(n, k)$ is a cell. This decomposition into cells is the Schubert decomposition. It is minimal. Let $I = (i_1, \cdots, i_n)$ with $1 \leq i_1 < \cdots < i_n \leq n + k$. Denote this cell $\Delta(I)$. Its dimension is

$$d(I) = \sum_{\mu=1}^{n} \mu r_\mu = \sum_{\mu=1}^{n} (n - \mu + 1)(i_\mu - i_{\mu-1} - 1). \tag{2}$$

(We let $i_0 = 0$.) Let $c(I)$ denote the center of the cell, that point where all values in all boxes are zero. Let

$$Q = Q(n, k) = \{I = (i_1, \cdots, i_n) \mid 1 \leq i_1 < \cdots < i_n \leq n + k\}.$$

$Q(n, k)$ has $C(n, k) = (n + k)!/n!k!$ elements. We order Q as follows: if $I = (i_1, \cdots, i_n)$, $J = (j_1, \cdots, j_n)$, define $I \leq J$ if $i_\mu \leq j_\mu$ for $\mu = 1, \cdots, n$. From (2), we get

1. Lemma. *For $I, J \in Q(n, k)$,*

$$d(I) - d(J) = \sum_{\mu=1}^{n} (i_\mu - j_\mu).$$

Hence $I > J$ implies $d(I) > d(J)$.

For $X = (x_{\alpha\beta}) \in M(n, k)$, let $\det_I(X) = \det_I(x_{\alpha\beta})$ be the $n \times n$ determinant of X obtained by choosing the $i_1, \cdots, i_n$ rows (we do not care about the sign).

2. Theorem. *Let $\tilde{f}_{n,k} : M(n, k) \to \mathbf{R}$ be*

$$\tilde{f}_{n,k}(X) = \frac{\sum_{I \in Q} d(I) \det_I^2(X)}{\sum_{I \in Q} \det_I^2(X)}. \tag{3}$$

Then $\tilde{f}_{n,k}$ induces a C^∞ function $f = f_{n,k} : G(n, k) \to \mathbf{R}$ with the following properties.

(a) $f(c(I)) = d(I)$.

(b) *The $c(I)$ are precisely the critical points of f; they are non-degenerate.*

(c) $\mathrm{Index}_{c(I)} f = d(I)$.

(d) *If $G(n, k)$ is considered the subset of $G(n, k + 1)$ with last column zero,* $f_{n,k+1} \mid G(n, k) = f_{n,k}$.

(e) *The m-skeleton of the Schubert decomposition is a subset of* $V_a = f_{n,k}^{-1}$ $(-\infty, a]$ *if* $a \geq m$.

The general construction

Let $q^2(n, k)$ be the set of all pairs (I, J) where I, J are *unordered* n-tuples of the first $(n + k)$ positive integers. Let R: $q^2(n, k) \to \mathbf{R}$ $((I, J) \to R_{I,J})$ satisfy

$$R_{I,J} = R_{J,I} \tag{4a}$$

$$R_{s(I),J} = (-1)^{|s|} R_{I,J} \quad \text{for } s \in S_n . \tag{4b}$$

Here S_n is the symmetric group and $|s| = \text{sign } s$.

$$s(i_1, \cdots, i_n) = (i_{s(1)}, \cdots, i_{s(n)}).$$

Let $X = (x_{\alpha\beta})$ and $Y = (y_{\alpha\beta})$ be members of $M(n, k)$. Let $I = (i_1, \cdots i_n)$, $J = (j_1, \cdots, j_n)$. Define

$$g_R(X, Y) = \sum\nolimits_{(I,J)\epsilon q} R_{I,J}\, x_{1,i_1} \cdots x_{n,i_n} y_{1,j_1} \cdots y_{n,j_n} \tag{5}$$

3. Lemma. *If* $X' = \sigma X$, $Y' = \tau Y$ *for* σ, $\tau \in GL(n)$,

$$g_R(X', Y') = (\det \sigma)(\det \tau) g_R(X, Y). \tag{6}$$

Proof. Fix Y. Let the rows of X be vectors in $(n + k)$-space. X thus determines an n-plane. Consider $g(X) = g_R(X, Y)$ as an n-linear function in this n-plane. By (4b) it is alternating. Thus if $X' = \sigma X$, $g(X') = (\det \sigma)$ $g(X)$. The full result follows from (4a).

4. Corollary. *If* $\bar{g}_R(X) = g_R(X, X)$, *then*

$$\bar{g}_R(\sigma X) = (\det \sigma)^2 \bar{g}_R(X). \tag{7}$$

Call R diagonal if

$$\begin{aligned} R_{I,J} &= (-1)^{|s|} R_{I,I} && \text{if } J = s(I) \text{ for some } s \in S_n \\ &= 0 && \text{if } J \neq s(I) \text{ for any } s \in S_n . \end{aligned} \tag{8}$$

In this event, denote $R_{I,I} = \lambda_I$. If, in addition, all $\lambda_I = 1$, call R trivial and denote g_R by g_0. In these cases we can put g_R in two other convenient forms. Suppose $(n + k)$-space has an inner product $<\ ,\ >$ and the fixed basis is orthonormal. Let $X_1, \cdots, X_n$ be the row vectors of X and likewise for Y.

5. Lemma. *If R is diagonal,*

$$g_R(X, Y) = \sum\nolimits_{I\epsilon Q(n,k)} \lambda_I \det_I(X) \det_I(Y). \tag{9}$$

If R is trivial,

$$g_0(X, Y) = g_R(X, Y) = \sum\nolimits_{s\epsilon S_n} (-1)^{|s|} \langle X_1, Y_{\sigma(1)}\rangle \cdots \langle X_n, Y_{\sigma(n)}\rangle. \tag{10}$$

Proof. (10) comes from expanding the right hand side and comparing with (9). (9) comes from expanding the right hand side and comparing with (5).

6. COROLLARY. *If R is diagonal,*

(11) $$\bar{g}_R(X) = g_R(X, X) = \textstyle\sum_{I \epsilon Q(n,k)} \lambda_I \det_I^2(X).$$

In particular $\bar{g}_0(X) = g_0(X, X) > 0$ *for all* $X \epsilon M(n, k)$.

The following is now obvious from corollaries 4 and 6.

7. PROPOSITION. *For any R satisfying conditions* (4), $f_R = \bar{g}_R/\bar{g}_0$ *is* (*induces*) *a well-defined* C^∞ *function on* $G(n, k)$.

8. *Remarks.* (a) Let $O(n, k) \subset M(n, k)$ be the set of matrices whose row vectors are orthonormal. Then $O(n, 0) = O(n)$, the orthogonal group. If, for $X, Y \epsilon O(n, k)$, we define $X \sim Y$ if $X = \sigma Y$ for some $\sigma \epsilon O(n)$, $G(n, k) = O(n, k)/\sim$. Furthermore $\bar{g}_0 \mid O(n, k) \equiv 1$. Hence $f_R = \bar{g}_R \mid O(n, k)$ and no division is necessary.

(b) If for $n = 2$, $R_{I,J}$ is denoted R_{ijkl}, $X_1 = (x_{11}, \cdots, X_{1,n+k})$, $X_2 = (x_{21}, \cdots, x_{2,n+k})$ are denoted respectively $X = (x_1, \cdots, x_{n+k})$, $Y = (y_1, \cdots, s_{n+k})$,

$$f_R = \frac{\sum R_{ijkl}\, x_i\, y_j\, x_k\, x_l}{\langle X, X\rangle\langle Y, Y\rangle - \langle X, Y\rangle^2}$$

which is the form of sectional curvature in local coordinates [2].

9. LEMMA. $c(I)$ *is a critical point of* f_R *if and only if* $R_{I,J} = 0$ *if J differs from I in precisely one element.*

Proof. (See [2, Prop. 5.1].) We use the following local coordinate system around $c(I)$ suggested by (1). Let $I = (i_1, \cdots, i_n) \epsilon Q(n, k)$. Fix

$$x_{j,i_r} = 1, \quad j = r \qquad (r = 1, \cdots, n)$$
$$= 0, \quad j \neq r.$$

Then $\{x_{j,k} \mid k \neq i_r \text{ for } r = 1, \cdots, n\}$ are local coordinates around $c(I)$. Note that $f_R(c(I)) = R_{I,I}$. For $\varepsilon \neq$ any i_r, let $I(l, \varepsilon)$ be the sequence $(\varepsilon_1, \cdots, \varepsilon_n)$ with

$$\varepsilon_j = i_j, \quad j \neq l$$
$$= \varepsilon, \quad j = l.$$

We need to prove that $c(I)$ is critical if and only if all $R_{I,I(k,\epsilon)} = 0$.

In this coordinate system

$$g_0 f_R = g_R = R_{I,I} + 2\Sigma R_{I,I(l,\epsilon)}\, x_{l,\epsilon} + \text{quadratic and higher order terms.}$$

Here the sum ranges over all relevant l, ε. Hence

(12) $$\frac{\partial g_0}{\partial x_{l,\epsilon}} f_R + g_0 \frac{\partial f_R}{\partial x_{l,\epsilon}} = 2R_{I,I(l,\epsilon)} + \text{linear and higher order terms.}$$

Since $g_0 = 1 +$ quadratic and higher order terms, (12) becomes at $c(I)$ where all $x_{l,\epsilon} = 0$,

$$\left.\frac{\partial f_R}{\partial x_{l,\epsilon}}\right|_{c(I)} = 2R_{I,I(k,\epsilon)}.$$

The result is immediate.

The special case

We return to the f described in theorem 2. It is clearly, by Corollary 6, the case of f_R when R is diagonal with $\lambda_I = d(I)$. Thus by Prop. 7, it is a well-defined C^∞ function. At $c(I)$, $\det_I = 1$ is the only non-zero determinant; hence (a) is established. (d) is trivial. By Lemma 9, the $c(I)$ are critical. To prove they are nondegenerate and to find their indices we proceed as follows. Let $d(I) = \lambda_I$. Since $f(c(I)) = \lambda_I$, we consider $f(X) - \lambda_I$.

$$(13) \qquad f(X) - \lambda_I = \frac{\sum_{J \epsilon Q} (\lambda_J - \lambda_I) \det_J^2 (X)}{\bar{g}_0}$$

If $X \epsilon \Delta(I)$, $\det_J(X) = 0$ unless $J \leq I$. (In terms of the coordinate system used in the proof of lemma 9, this amounts to holding $x_{j,k} = 0$ if $j \leq r$, $k > i_r$.) Invoking Lemma 1, we see that

$$(14) \qquad f(X) - \lambda_I < 0 \quad \text{if } X \epsilon \Delta(I) - c(I).$$

Thus the dimension of the negative-definite subspace of the Hessian at $c(I)$ is $\geq d(I)$. On the other hand, if we let the complementary variables vary; that is hold $x_{j,k} = 0$ if $j > r$, $k < i_r$, we get that $\det_J(X) = 0$ unless $J \geq I$. Thus the dimension of the positive-definite subspace of the Hessian at $c(I)$ is $\geq \dim G(n, k) - d(I) = nk - d(I)$. Combining these two facts proves that $c(I)$ is non-degenerate and the index there is $d(I)$. (14) also proves part (e).

We are left with proving that the $c(I)$ are the only critical points. We do this by proving that the gradient $\nabla f \neq 0$ at $X \neq c(I)$.

The gradient

We first set some notation. Let $X = (x_{ij}) \epsilon M(n, k) \subset E^{n(n+k)}$. The coordinates in $E^{n(n+k)}$ are $\{x_{ij}\}$. Let $e_{ij} = \partial/\partial x_{ij}$; a typical tangent vector is $\sum y_{ij} e_{ij}$. Let $N(n, k)$ be the set of all $n \times (n + k)$ matrices; then $Y = (y_{ij}) \epsilon N(n, k)$ and thus the tangent bundle of $M(n, k)$ is

$$\{(X, Y) \mid X \epsilon M(n, k), Y \epsilon N(n, k)\}.$$

Let $O(n, k)$ be the Stiefel manifold of orthonormal n frames in $(n + k)$-space. (See Remark 8(a).) Let π_Λ be the projection from $\Lambda = \Lambda(n, k)$ to $G(n, k)$ for $\Lambda = M$ or O. Let Y^T denote the transpose of the matrix Y. 0_n denotes the 0 $n \times n$ matrix. If $\Lambda = \Lambda(n, k)$ is a manifold, T^Λ is its tangent bundle.

10. PROPOSITION.

$$T^M = \{(X, Y) \mid X \in M(n, k),\ Y \in N(n, k)\}$$

$$T^0 = \{(X, Y) \mid X \in O(n, k),\ Y \in N(n, k),\ XY^T \text{ is antisymmetric}\}$$

$$\pi_\Lambda^* T^G = \{(X, Y) \mid X \in \Lambda(n, k),\ Y \in N(n, k),\ XY^T = 0_n\}$$

$T^G = \pi_\Lambda^* T^G$ factored out by the equivalence $(X, Y) \sim (\sigma X, \sigma Y)$ for $\sigma \in \Gamma$. Here $\Lambda = M$ or O whence Γ is respectively GL or O.

Proof. We remark that the conditions can easily be verified to be dimensionally correct. T^M was determined above. We consider T^0. Let E be an antisymmetric $(n + k) \times (n + k)$ matrix, $X \in O(n, k)$. $\exp E =$ the exponent of $E \in O(n, k)$.

$$Y = \lim_{t \to 0} \frac{X(\exp tE) - X}{t}$$

is a tangent vector to $O(n, k)$ at X and furthermore, all tangent vectors are of this form. We have

$$YX^T = \lim_{t \to 0} \frac{X(\exp tE)X^T - XX^T}{t} = XEX^T.$$

Thus $XY^T = XE^TX^T = -XEX^T = -(XY^T)^T$ and T^0 is as stated.

In considering T^G, we let $\Lambda = M$; the case $\Lambda = O$ is similar. For $X \in M(n, k)$, $\pi_M^{-1}\pi_M(X)$ is a submanifold of $G(n, k)$; let T_1 be the subbundle of T^G which is tangent at each point to this submanifold. Let S be the bundle claimed to be T^G.

We will show there is a split exact sequence of bundles

$$0 \to T_1 \to T^M \to \pi_M^* S \to 0$$

which, along with its splitting, is respected by the action of $GL(n)$. This will establish the result.

Let $E \in GL(n)$. At $X \in GL(n)$, all vectors in the fiber of T_1 are of the form

$$\lim_{t \to 0} \frac{(\exp tE)X - X}{t} = EX.$$

Thus $T_1 = \{(X, X_1) \in M(n, k) \times N(n, k) \mid$ each row of X_1 is linearly dependent on the rows of $X\}$. Hence $\pi_M^* S$ is the orthogonal complement of T_1 and the split sequence is valid. For $\sigma \in GL(n)$, define $\tilde{\sigma}: T^M \to T^M$ by $\tilde{\sigma}(X, Y) = (\sigma X, \sigma Y)$. $\tilde{\sigma}$ induces maps on T_1 and $\pi_M^* S$ which behave as claimed. Thus the result is proved.

We now want to consider gradients. We use the notation $\nabla_\Lambda f$ which is a vector field in T^Λ. All functions are at least C^1.

Suppose F is a function on $O(n, k)$. If F is invariant under the left action of $O(n)$, F induces a function f on $G(n, k)$. $\nabla_G f$ can be determined as follows: let $\theta: T^0 \to \pi_0^* T^G$ be the orthogonal projection. Note that, extending the notation of the last proof, $\tilde{\sigma}\theta = \theta\tilde{\sigma}$. Then $\pi\tilde{\sigma}\nabla_G f = \theta\nabla_0 F$.

11. PROPOSITION. *Let* $X = (x_{ij}) \in O(n, k)$. *Let* $\nabla_0 F(X) = (X, Y) \in T^0$ *where* $Y = (y_{ij}(X))$. *Then at* $\pi_0(X) \in G(n, k)$, $\nabla_G f$ *is the equivalence class of* (X, Z) *where* $Z = (z_{ij}(X))$ *with*

$$z_{ij} = y_{ij} - x_{ij} \sum_{k,l} x_{ik}\, y_{lk}\,. \tag{15}$$

Proof. Let the rows of X be $X_1, \cdots, X_n$, those of Y, $Y_1, \cdots, Y_n$. $X_1, \cdots, X_n$ are orthonormal. If $\theta' \colon T^0 \to T_1$ is the projection orthogonal to θ, $\theta = 1 - \theta'$. Let θ' be the projection to the space spanned by X_l. Then $\theta' = \sum_{l=1}^{n} \theta'_l$. Since $\theta'_l(\Sigma_k Y_k) = \Sigma_k \langle X_l, Y_k\rangle X_l$,

$$\theta'_l\Big(\sum_{k,j} y_{kj}\, e_{kj}\Big) = \sum_{h,k,j} x_{lj}\, y_{kj}\, x_{lh}\, e_{lh}\,.$$

Thus $\theta'\big(\sum_{k,j} y_{kj}\, e_{kj}\big) = \sum_{h,k,j,l} x_{lj}\, y_{kj}\, x_{lh}\, e_{lh}$. The result now is trivial.

Now let $G\colon M(n, k) \to R$ be a function, let $i\colon O(n, k) \to M(n, k)$ be the inclusion and $\Psi\colon i^*T^M \to T^0$ the orthogonal projection. It is a special case of a standard result that

$$\nabla_0(G \mid O(n, k)) = \Psi(i^*\nabla_M G).$$

Suppose, as in our case, that $G = g/g_0$ with $g_0 \mid O(n, k) \equiv 1$. Since

$$\Psi(i^*\nabla_M g_0) = 0,$$

$$\nabla_0(G \mid O(n, k)) = g_0^{-2}\Psi i^*(g_0 \nabla_M g - g\nabla_M g_0) = \Sigma(i^*\nabla_M g). \tag{16}$$

We have $g = \sum_I \lambda_I \det_I{}^2(X)$. Let $\min_I (ij)(X)$ denote the determinant of the $(i, j)^{\text{th}}$ cofactor in the matrix of which $\det_I (X)$ is the determinant. If $j \notin I$, let $\min_I (ij)(X) = 0$. Then $\nabla_M f(X) = (X, Y)$ where $Y = (y_{ij})$ and

$$y_{ij} = \sum_I 2\lambda_I \det_I (X) \min_I (ij)(X).$$

Using (15) and (16) we have, for $X \in O(n, k)$, $\pi_0^*\nabla_G f(\pi_0 X) = (X, Y)$ where $Y = (y_{ij})$ with

$$\begin{aligned} y_{ij} = {} & \sum_I 2\lambda_I \det_I (X) \min_I (ij)(X) \\ & - \sum_{l,k,J} 2\lambda_J\, x_{ij}\, x_{ik} \det_J (X) \min_J (lk)(X). \end{aligned} \tag{17}$$

Since, for any i,

$$\sum_k x_{ik} \min_J (lk)(X) = \delta_{il} \det_J (X), \tag{18}$$

it is the case that

$$\begin{aligned} y_{ij} &= 2\sum_I \lambda_I \det_I (X) \min_I (ij)(X) - \sum_J 2\lambda_J\, x_{ij} \det_J{}^2(X) \\ &= 2\sum_I \lambda_I \det_I (X) \min_I (ij)(X) - 2x_{ij} f(X). \end{aligned} \tag{19}$$

If all $\lambda_I = 1$, $f \equiv 1$ and all $y_{ij} = 0$. Thus we have proved

12. LEMMA. *If* $X = (x_{ij}) \in O(n, k)$,

$$\sum_I \det_I (X) \min_I (ij)(X) = y_{ij} \tag{20}$$

If we substitute the expression for x_{ij} from (20) into (19), we get finally

$$y_{ij} = 2\sum_I (\lambda_I - f(X)) \det_I (X) \min_I (ij)(X). \tag{21}$$

The embedding

We have to prove that for $\pi_0 (X) \neq c(I)$, some y_{ij} in (21) is not zero. Rather than do this directly, we embed $G = G(n, k)$ as a nonsingular variety in a projective space P and extend f to a functions on all of P. The critical point set on P will tell us about that on G.

Consider a $(C(n, k) - 1)$-dimensional projective space P with homogeneous coordinates $(\xi_I)_{I \epsilon Q(n,k)}$. We assume $\sum_I \xi_I^2 = 1$. The map $e\colon G \to P$ given by $\xi_I = \det_I (X)$ is known to embed G as a submanifold and non-singular algebraic variety [1, chap. VII]. The $\det_I$ are up to sign the Plücker coordinates. Let

$$\Xi = (\xi_I) \;\epsilon\; O(1, C(n, k)), \qquad H = (\eta_I) \;\epsilon\; N(1, C(n, k)).$$

Then $\pi_0^* T^P = \{(\Xi, H) \mid \sum_I \xi_I \eta_I = 0\}$ as in Proposition 10.

If $X \epsilon O(n, k)$ and $(X, Y) \;\epsilon\; \pi_0^* T^G$, the map $e_* : T^G \to T^P$ is given at X by

$$\eta_I = \sum_{ij} y_{ij} \min_I (ij)(X). \tag{22}$$

Note that then

$$\sum_I \xi_I \eta_I = \sum_{I,i,j} y_{ij} \det_I (X) \min_I (ij)(X) = \sum_{i,j} y_{ij}\, x_{ij} = 0$$

by Lemma 12 and Proposition 10. Thus $e_*(T^G) \subset T^P$.

The standard metric $\langle\ ,\ \rangle$ in Euclidean space induces one in $O(n, k)$. This metric is invariant under the left action of $O(n)$ and is respected by the splitting in Proposition 10. Hence it induces a metric in $G(n, k)$ which is the one we have used to define ∇_G. Thus if $Y = (y_{ij})$, $Y' = (y'_{ij})$ are ϵ $N(n, k)$, the fiber of T^G at X,

$$\langle Y, Y' \rangle_G = \sum_{ij} y_{ij}\, y'_{ij}.$$

13. Proposition. *e is an isometry into.*

Proof. Because of the homogeneity of the situation, we need prove it at only one point $\pi_0 X$, say $X = (x_{ij})$ where

$$\begin{aligned} x_{ij} &= 1, \quad 1 \le i, j \le n, \\ &= 0, \quad \text{otherwise} \end{aligned}$$

Then if $(X, Y) \;\epsilon\; \pi_0^* T^G$, $y_{ij} = 0$ for $j \le n$. For $1 \le i \le n$, $j > n$, let $[i, j]$ ϵ $Q(n, k)$ be the sequence $(1, 2, \cdots, \hat{\imath}, \cdots, n_{ij})$ where $\hat{}$ denotes omission. Then

$$\begin{aligned} \min_I (ij)(X) &= 1, \quad I = [i, j], \\ &= 0, \quad \text{otherwise.} \end{aligned}$$

Thus

$$\langle e_* Y, e_* Y' \rangle_P = \sum_I \big(\sum_{\alpha\beta} y_{\alpha\beta} \min_I (\alpha\beta)(X) \cdot \sum_{\gamma\delta} y'_{\gamma\delta} \min_I (\gamma\delta)(X)\big)$$
$$= \sum_{1 \leq i \leq n, j > n} y_{ij} y'_{ij} = \langle Y, Y' \rangle_G .$$

This proves the result.

We now identify G with its image in P as Riemannian manifolds and drop the symbol e.

Define $F\colon P \to R$ by

$$F(\Xi) = \sum_I \lambda_I \xi_I^2 . \tag{23}$$

Then $F \mid G = f$. Also $\nabla_P F(\Xi)$ is the class of (Σ, H) where

$$\eta_I = 2(\lambda_I - F(\Xi))\xi_I . \tag{24}$$

Let $\chi_I \in P$ be the point (ξ_J) where

$$\xi_J = 1, \quad J = I,$$
$$= 0, \quad J \neq I.$$

Set χ_I equivalent to χ_J if $\lambda_I = \lambda_J$. Then from (24), we have

14. PROPOSITION. $\nabla_P F = 0$ *precisely on the sub-projective spaces spanned by equivalent points* χ_I.

Call this set S_F.

The singular point set on G is determined by the following.

15. PROPOSITION. *If* $X \in G \subset P$, $\nabla_P F(X)$ *is already tangent to* G. *Thus*

$$\nabla_P F(X) = \nabla_G f(X).$$

Accordingly, $\nabla_G f = 0$ *precisely on* $S_F \cap G$.

Proof. The metric on P (and on $G \subset P$) identifies the tangent bundle T and its dual. Hence we may regard (21) and (24) as the components of df and dF respectively. Thus $e^* dF = df$ or equivalently $e_* \nabla_G f = \nabla_P F$. This is what is needed for the proof.

The final step in the proof of Theorem 2 can now be given. It is simply a case of noting that if $\lambda_I < \lambda_J$ whenever $I < J$, then

$$S_F \cap G = \{\chi_I = c(I) \mid I \in Q(n, k)\},$$

and invoking Proposition 1.

Comments

Although we have presented the results in this note as a proof of Theorem 2, that theorem is actually a special application of the methods used. Functions of the type in Proposition 7 occur naturally in several contexts and the methods give information about their critical point sets. In particular

Propositions 14 and 15 completely determine the critical point set if R is diagonal.

We can look at the more general case as follows: let $\Lambda^n E^{n+k}$ be the nth exterior product of E^{n+k}. If $\varepsilon_1, \cdots, \varepsilon_{n+k}$ is a standard basis for E^{n+k} and $\varepsilon^1, \cdots, \varepsilon^{n+k}$ is its dual basis, we can consider $R = \{R_{I,J}\}$ as the compotents of a symmetric bilinear form on $\Lambda^n E^{n+k}$ and write it

$$\sum_{(I,J)} R_{I,J}\, \varepsilon^I \otimes \varepsilon^J$$

where $\varepsilon^I = \varepsilon^{i_1} \wedge \cdots \wedge \varepsilon^{i_n}$ and the sum is over all $(I, J) \in (Q(n, k))^2$.

It can be diagonalized in $\Lambda^n E^{n+k}$ by a $C(n, k) \times C(n, k)$ orthogonal transformation Ω, although not necessarily by a power matrix (i.e. Ω need not be induced by an orthogonal transformation in E^{n+k}). Ω operates naturally on $P = P^{C(n,k)-1}$ and we form $e_\Omega : G \to P$ by $e_\Omega = \Omega^{-1} e$. e_Ω is thus a "non-standard" isometric embedding of G in P. It is easy to see that the function f_R of Proposition 7 is the restriction to $e_\Omega G$ of a function of type (23) on P. Unfortunately there is no guarantee of a result analogous to proposition 15.

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