LEBESGUE SPACES OF PARABOLIC POTENTIALS

BY

RICHARD J. BAGBY

Introduction

We define a class of spaces $\mathcal{L}_\alpha^p$ via Fourier transform techniques. These spaces have been studied previously by Sampson [11]. They arise in the study of the heat equation; they are the parabolic analogue of the spaces of Bessel potentials introduced by Aronszajn and Smith [1] and by Calderón [4]. The results obtained in this paper are analogous to results obtained by Strichartz [13] for Bessel potentials.

The first chapter contains the basic facts about $\mathcal{L}_\alpha^p$ spaces. In the second chapter we characterize some of these spaces in terms of an integral norm of a difference quotient. We develop an interpolation theory for these spaces in the third chapter. These results are of some interest in themselves; they are used in the fourth chapter to find sufficient conditions for the product of two functions to be in one of the spaces $\mathcal{L}_\alpha^p$.

Establishing the characterization of Chapter 2 requires a number of calculations. The appendix contains the worst of these.

This paper consists essentially of the author's doctoral dissertation at Rice University. I wish to thank my advisor Dr. B. Frank Jones for his help. Financial support was provided by the United States Air Force, N.A.S.A., and the Schlumberger Foundation.

1. Preliminaries

1.1 *Notation.* Let E^{n+1} denote Euclidean $(n + 1)$-space. Points in E^{n+1} will be denoted in the form (x, t), where $x \in E^n$. Unless explicitly stated otherwise, all function spaces are assumed to be spaces of functions defined on E^{n+1}.

The usual inner product in E^n will be denoted by $x \cdot y$. For $x \in E^n$, $|x| = (x \cdot x)^{1/2}$. Differential operators are expressed in the form

$$D_x^\alpha D_t^j = (\partial/\partial x_1)^{\alpha_1} \cdots (\partial/\partial x_n)^{\alpha_n}(\partial/\partial t)^j;$$

the order of the multi-index α is denoted by $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n$. The Laplace operator in E^n is denoted by Δ_x.

Let $\mathcal{S}$ denote the space of C^∞ functions ϕ satisfying

$$\sup_{(x,t)} |P(x, t) D_x^\alpha D_t^j \phi(x, t)| < \infty$$

for any polynomial P and any α, j. $\mathcal{S}$ is given the usual topology; see Schwartz [12]. The dual of $\mathcal{S}$ is denoted by $\mathcal{S}'$; its elements are called tempered distributions.

Received April 15, 1969.

The Fourier transform is defined on $\mathcal{S}$ by

$$\hat{\phi}(\xi, \tau) = (2\pi)^{-(n+1)/2} \iint e^{-ix\cdot\xi - it\tau}\phi(x, t)\, dxdt;$$

it is extended to $\mathcal{S}'$ in the usual manner. Where no confusion arises, the dual variables will also be denoted (x, t).

The letter C will be used to denote any positive constant whose exact value need not be known explicitly.

1.2 Definition. For arbitrary complex α, define $\mathcal{J}_\alpha : \mathcal{S}' \to \mathcal{S}'$ by

$$(\mathcal{J}_\alpha T)^\wedge = (1 + |x|^2 + it)^{-\alpha/2}\hat{T},$$

where

$$(1 + |x|^2 + it)^{-\alpha/2} = \exp\{-\tfrac{1}{2}\alpha[\ln|1 + |x|^2 + it| + i \arg(1 + |x|^2 + it)]\},$$

with $-\pi/2 < \arg(1 + |x|^2 + it) < \pi/2$.

Since $(1 + |x|^2 + it)^{-\alpha/2}$ is a C^∞ function each of whose derivatives are bounded by polynomials, $\mathcal{J}_\alpha$ defines a continuous operator from $\mathcal{S}'$ into itself. Note that $\mathcal{J}_{\alpha+\beta} = \mathcal{J}_\alpha \mathcal{J}_\beta$ and that formally $\mathcal{J}_\alpha = (1 - \Delta_x + D_t)^{-\alpha/2}$.

1.3 Definition. For $1 \leqq p \leqq \infty$, $\mathcal{L}^p_\alpha$ is the Banach space of tempered distributions T such that $\mathcal{J}_{-\alpha} T \in L^p$, with the norm $\| T \|_{p,\alpha} = \| \mathcal{J}_{-\alpha} T \|_p$. Clearly $\mathcal{L}^p_\alpha = \mathcal{J}_\alpha(L^p)$ and $\mathcal{L}^p_{\alpha+\beta} = \mathcal{J}_\beta(\mathcal{L}^p_\alpha)$.

1.4. Definition. A locally integrable function $m(x, t)$ is said to be a *multiplier* (on Fourier transforms of functions) of type (p, q) if for every $\phi \in S$, $m\phi \in S'$ and the operator $T : \mathcal{S} \to \mathcal{S}'$ defined by $(T\phi)^\wedge = m\hat{\phi}$ satisfies $T\phi \in L^q$ with $\| T\phi \|_q \leq C\| \phi \|_p$, C independent of $\phi \in S$. The space of all multipliers of type (p, q) is denoted M^q_p; these spaces are treated in Hörmander [7].

Due to the form of the operator $\mathcal{J}_\alpha$, the following theorem will be extremely useful. It is a special case of a theorem proved in Fabes and Riviére [5].

1.5 Theorem. *Let $m \in L^\infty$ and suppose*

$$\sup_{(x,t)\neq(0,0)} (|x|^2 + |t|)^{|\beta|+k} | D^\beta_x D^k_t m(x, t) | \leq C_0,$$

whenever $|\beta| + 2k \leq N$, where $N > (n + 2)/2$. Then $m \in M^p_p$ for $1 < p < \infty$ and the norm of the associated operator is bounded by $C_0 C_p$, where C_p depends only on n and p.

Applying (1.5) to the function $(1 + |x|^2 + it)^{-\alpha/2}$, we see that $\mathcal{J}_\alpha : L^p \to L^p$ continuously if Re $(\alpha) \geq 0$ and $1 < p < \infty$; the operator norm of $\mathcal{J}_\alpha$ is bounded by $C_p e^{(\pi/2)\operatorname{Im}\alpha} | p_n(\alpha) |$ where P_n is a polynomial depending only on n. As a consequence, $\mathcal{L}^p_\alpha = \mathcal{L}^p_{\operatorname{Re}(\alpha)}$ for $1 < p < \infty$. Since our new results are valid only in the case $1 < p < \infty$, we will restrict our attention to the case of real α.

1.6 LEMMA. *If $\alpha > 0$, then the function $\mathcal{G}_\alpha$ defined by*

$$\begin{aligned} \mathcal{G}_\alpha(x, t) &= (4\pi)^{-n/2}\Gamma(\alpha/2)^{-1}t^{(\alpha-n)/2-1} \exp\{-t - |x|^2/4t\}, \quad t > 0 \\ &= 0, \quad t \leqq 0 \end{aligned}$$

satisfies:

(i) $\mathcal{G}_\alpha \in L^1$.

(ii) $\hat{\mathcal{G}}_\alpha(x, t) = (1 + |x|^2 + it)^{-\alpha/2}$.

(iii) *For* $0 < \alpha < n + 2$, $\mathcal{G}_\alpha \in L^r$ *if* $1 \leq r < (n + 2)/(n + 2 - \alpha)$ *and* $E(\eta) \equiv |\{(x, t) : \mathcal{G}_\alpha(x, t) > \eta\}| \leq c_{\alpha,n}\,\eta^{-(n+2)/(n+2-\alpha)}$ *for* $\eta > 0$.

(iv) $\mathcal{G}_\alpha \in L^\infty$ *if* $\alpha \geq n + 2$.

Proof. (i) is immediate. (ii) is given in Jones [8]. For the last part of (iii), note that $\mathcal{G}_\alpha(x, t) \leq ct^{(\alpha-n)/2-1}e^{-|x|^2/4t}$ for $t > 0$. Consequently

$$\mathcal{G}_\alpha(\lambda x, \lambda^2 t) \leq c\lambda^{\alpha-n-2}t^{(\alpha-n)/2-1}e^{-|x|^2/4t} \quad \text{for } \lambda, t > 0.$$

Then

$$\begin{aligned} E(\eta) &= \lambda^{-n-2}|\{(x, t) : \mathcal{G}_\alpha(\lambda x, \lambda^2 t) > \eta\}| \\ &\leqq \lambda^{-n-2}|\{(x, t) : t > 0,\ ct^{(\alpha-n)/2-1}e^{-|x|^2/4t} > \eta\lambda^{n+2-\alpha}\}|. \end{aligned}$$

Setting $\lambda = \eta^{1/(n+2-\alpha)}$,

$$\begin{aligned} E(\eta) &\leq \eta^{-(n+2)/(n+2-\alpha)}|\{(x, t) : t > 0,\ ct^{(\alpha-n)/2-1}e^{-|x|^2/4t} > 1\}| \\ &= c\eta^{-(n+2)/(n+2-\alpha)}. \end{aligned}$$

The first part of (iii) follows by a direct calculation; it also follows from the estimate for $E(\eta)$ and the fact that $\mathcal{G}_\alpha \in L^1$.

(iv) is obvious.

1.7 THEOREM. *Let α, β be real.*

(i) $\mathcal{L}^p_\alpha \subset \mathcal{L}^p_\beta$ *if* $\alpha > \beta$; *in particular*, $\mathcal{L}^p_\alpha \subset L^p$ *if* $\alpha > 0$.

(ii) *For* $1 \leqq p < q \leqq \infty$, $\mathcal{L}^p_\alpha \subset \mathcal{L}^q_\beta$ *if* $1/p < 1/q + (\alpha - \beta)/(n + 2)$.

(iii) *If* $1 < p < q < \infty$, *then* $\mathcal{L}^p_\alpha \subset \mathcal{L}^q_\beta$ *also if* $1/p = 1/q + (\alpha - \beta)/n$.

Proof. Let $f \in \mathcal{L}^p_\alpha$. Then $f = \mathcal{J}_\alpha \phi$, with $\phi \in L^p$. For $\beta < \alpha$,

$$f = \mathcal{J}_\beta \mathcal{J}_{\alpha-\beta}\phi = \mathcal{J}_\beta(\mathcal{G}_{\alpha-\beta} * \phi).$$

By part (i) of (1.6), $\mathcal{G}_{\alpha-\beta} \in L^1$ and hence $\mathcal{G}_{\alpha-\beta} * \phi \in L^p$. Consequently $f \in \mathcal{L}^p_\beta$. If $1/p < 1/q + (\alpha - \beta)/(n + 2)$ then by (1.6), $\mathcal{G}_{\alpha-\beta} \in L^r$ where $1/p + 1/r = 1/q + 1$. Thus by Young's theorem, $\mathcal{G}_{\alpha-\beta} * \phi \in L^q$ and hence $f \in \mathcal{L}^q_\beta$. In the case $1 < p < q < \infty$ and $1/p = 1/q + (\alpha - \beta)/(n + 2)$, this is a simple variant of the standard fractional integration theorem as proved in Zygmund [16] and extended by O'Neil [10].

1.8 THEOREM. *If α is real and $1 < p < \infty$, then $\mathcal{L}^p_\alpha$ is reflexive and its dual is $\mathcal{L}^{p'}_{-\alpha}$, where $1/p + 1/p' = 1$. The pairing between $\mathcal{L}^p_\alpha$ and $\mathcal{L}^{p'}_{-\alpha}$ is defined by*

$$[\phi, \psi] = \iint \phi(x, t)\psi(-x, -t)\, dxdt \quad \textit{for } \phi, \psi \in \mathcal{S}.$$

Proof. By Parseval's formula,

$$[\phi, \psi] = \iint \hat{\phi}(\xi, \tau)\hat{\psi}(\xi, \tau)\, d\xi\, d\tau = \iint (\mathcal{J}_{-\alpha}\phi)^{\wedge}(\xi, \tau)(\mathcal{J}_{\alpha}\psi)^{\wedge}(\xi, \tau)\, d\xi\, d\tau$$

$$= \iint \mathcal{J}_{-\alpha}\phi(x, t)\mathcal{J}_{\alpha}\psi(-x, -t)\, dx\, dt.$$

Hence

$$|[\phi, \psi]| \leq \| \mathcal{J}_{-\alpha}\phi\|_p \| \mathcal{J}_{\alpha}\psi\|_{p'} = \|\phi\|_{p,\alpha}\|\psi\|_{p',-\alpha}.$$

Since $\mathcal{S}$ is dense in every $\mathcal{L}^p_\alpha$ space with $p < \infty$, $[\cdot\,,\,\cdot]$ has a unique extension to a continuous bilinear form on $\mathcal{L}^p_\alpha \times \mathcal{L}^{p'}_{-\alpha}$.

Conversely, if F is in the dual of $\mathcal{L}^p_\alpha$, then $F \circ \mathcal{J}_\alpha$ is in the dual of L^p and hence can be identified with a function $g \in L^{p'}$. But then $\mathcal{J}_{-\alpha}\, g \in \mathcal{L}^{p'}_{-\alpha}$ and $\mathcal{J}_{-\alpha}\, g$ can be identified with F.

1.9 THEOREM. *Let $1 < p < \infty$, $\alpha > 0$, k a positive integer such that $2k \leq \alpha$. Then*

$$\|f\|_{p,\alpha} \approx \sum_{|\gamma|+2j\leq 2k} \| D_x^\gamma D_t^j f\|_{p,\alpha-2k}.$$

Proof. Since $\mathcal{J}_\beta$ is an isometry of $\mathcal{L}^p_\alpha$ onto $\mathcal{L}^p_{\alpha+\beta}$ and $\mathcal{J}_\beta$ commutes with differentiation, it suffices to consider the case $\alpha = 2k$.

We have $\mathcal{J}_{-2k}f = (1 - \Delta_x + D_t)^{2k}f$, so clearly

$$\|f\|_{p,2k} = \|\mathcal{J}_{-2k}f\|_p = \|(1 - \Delta_x + D_t)^{2k}f\|_p \leq c\sum_{|\gamma|+2j\leq 2k} \|D_x^\gamma D_t^j f\|_p.$$

For the reverse inequality, let $f = \mathcal{J}_{2k}\, g$, $g \in L^p$. Then $D_x^\gamma D_t^j f = D_x^\gamma D_t^j \mathcal{J}_{2k}\, g$. Thus

$$(D_x^\gamma D_t^j f)^{\wedge} = \frac{i^{|\gamma|+j}x^\gamma t^j}{(1 + |x|^2 + it)^k}\hat{g}.$$

Applying (1.5), $x^\gamma t^j/(1 + |x|^2 + it)^k \in M_p^p$ if $|\gamma| + 2j \leqq 2k$; hence

$$\|D_x^\gamma D_t^j f\|_p \leqq c\|g\|_p = c\|f\|_{p,\alpha}.$$

Using (1.9) it is often possible to reduce questions about $\mathcal{L}^p_\alpha$ spaces to the case $0 \leqq \alpha < 2$.

We now introduce a function H_α which is similar to $\mathcal{G}_\alpha$. H_α will have homogeneity properties which are useful in characterizing $\mathcal{L}^p_\alpha$ spaces.

(1.10), (1.11), and (1.12) below are due to Sampson [11].

1.10 PROPOSITION. *Let*

$$H_\alpha(x, t) = t^{(\alpha-n)/2-1}\exp\{-|x|^2/4t\}, \quad t > 0$$
$$= 0, \qquad t \leq 0.$$

Then for $\alpha > 0$, $H_\alpha \in \mathcal{S}'$. If $0 < \alpha < n + 2$, $\hat{H}_\alpha$ is a function and

$$\hat{H}_\alpha(x, t) = c(\alpha, n)(|x|^2 + it)^{-\alpha/2}.$$

1.11 LEMMA. *For $\alpha > 0$, there exist bounded measures μ, μ_1, μ_2 such that*

$$(|x|^2 + it)^{\alpha/2} = (1 + |x|^2 + it)^{\alpha/2}\hat{\mu}$$

and

$$(1 + |x|^2 + it)^{\alpha/2} = \hat{\mu}_1 + (|x|^2 + it)^{\alpha/2}\hat{\mu}_2 .$$

1.12 THEOREM. *Let $\alpha > 0$. Let $f \in L^p$. Then $f \in \mathcal{L}^p_\alpha$ iff there exists $g \in L^p$ such that $(|x|^2 + it)^{\alpha/2}\hat{f} = \hat{g}$, in which case $\| f \|_{p,\alpha} \approx \| f \|_p + \| g \|_{p'}$.*

If $0 < \alpha < n + 2$, then $H_\alpha \in L^1 + L^\infty$. Hence if the function g above is in $L^1 \cap L^\infty$, we have $f = c(\alpha, n)^{-1} H_\alpha * g$.

2. A characterization of $\mathcal{G}^p_\alpha$

Let

$$\Omega_r = \{ (y, s) \in E^{n+1} : | y | < r, -r^2 < s < r^2 \}.$$

Let

$$\Omega_r^+ = \{ (y, s) \in \Omega_r : s > 0 \}.$$

For brevity Ω_1 and Ω_1^+ will be denoted by Ω and Ω^+.

2.1 DEFINITION. For $f \in L^1_{\text{loc}}$, let

$$S_\alpha f(x, t) = \left(\int_0^\infty \left[\iint_{\Omega^+} | f(x - ry, t - r^2 s) - f(x, t) | \, dy \, ds \right]^2 r^{-1-2\alpha} \, dr \right)^{1/2}$$

2.2 THEOREM. *For $0 < \alpha < 1$ and $1 < p < \infty$, $f \in \mathcal{L}^p_\alpha$ iff $f \in L^p$ and $S_\alpha f \in L^p$; in which case $\| f \|_{p,\alpha} \approx \| f \|_p + \| S_\alpha f \|_p$.*

In the case $p = 2$, the inequality $\| S_\alpha f \|_2 + \| f \|_2 \leq C \| f \|_{2,\alpha}$ is proved using Fourier transform techniques. According to (1.12), $f \in \mathcal{L}^p_\alpha$ iff $f \in L^2$ and $\hat{f} = \hat{H}_\alpha \hat{\Phi}$ for some $\Phi \in L^2$; moreover, $\| f \|_{2,\alpha} \approx \| f \|_2 + \| \Phi \|_2$.
Applying Schwarz's inequality and then Fubini's theorem,

$$\begin{aligned}
S_\alpha f(x, t)^2 &= \int_0^\infty \left(\iint_{\Omega^+} | f(x - ry, t - r^2 s) - f(x, t) | \, dy \, ds \right)^2 r^{-1-2\alpha} \, dr \\
&\leq C \int_0^\infty \left(\iint_{|y| + \sqrt{s} \leq 2} | f(x - ry, t - r^2 s) \right. \\
&\qquad \left. - f(x, t) |^2 \, dy \, ds \right) r^{-1-2\alpha} \, dr \\
&= C \int_0^\infty \left(\iint_{|y| + \sqrt{s} \leq 2} | f(x - y, t - s) - f(x, t) |^2 \, dy \, ds \right) r^{-n-3-2\alpha} \, dr \\
&= C \iint_{s > 0} | f(x - y, t - s) - f(x, t) |^2 \, dy \, ds \int_{\frac{1}{2}(|y| + \sqrt{s})}^\infty r^{-n-3-2\alpha} \, dr \\
&= C \iint_{s > 0} | f(x - y, t - s) - f(x, t) |^2 (|y| + \sqrt{s})^{-n-2-2\alpha} \, dy \, ds.
\end{aligned}$$

Thus by Fubini's theorem and Parseval's equation,

$$\| S_\alpha f \|_2^2$$

$$\leq C \iint_{s>0} (|y| + \sqrt{s})^{-n-2-2\alpha}\, dy\, ds \iint |[f(\cdot - y, \cdot - s) - f]^\wedge(\xi, \tau)|^2 d\xi\, d\tau$$

Noting that

$$[f(\cdot - y, \cdot - s) - f]^\wedge(\xi, \tau) = \hat{\Phi}(\xi, \tau)[H_\alpha(\cdot - y, \cdot - s) - H_\alpha]^\wedge(\xi, \tau)$$
$$= \hat{\phi}(\xi, \tau)[e^{-iy\cdot\xi - is\tau} - 1](|\xi|^2 + i\tau)^{-\alpha/2}$$

and again changing the order of integration,

$$\| S_\alpha f \|_2^2 \leq C \iint |\hat{\phi}(\xi, \tau)|^2 \| \xi |^2$$

$$+ i\tau|^{-\alpha}\, d\xi\, d\tau \iint_{s>0} |e^{-iy\cdot\xi - is\tau} - 1|^2 (|y|^2 + \sqrt{s})^{-n-2-2\alpha}\, dy\, ds$$

Substituting $y = (|\xi|^2 + i\tau)^{-1/2} y'$, $s = (|\xi|^2 + i\tau)^{-1} s'$ and using the mean value theorem to estimate the resulting integrand for y, s near 0, it is readily seen that

$$\iint_{s>0} |e^{-iy\cdot\xi - is\tau} - 1|^2 (|y|^2 + \sqrt{s})^{-n-2-2\alpha}\, dy\, ds \leq C \| \xi |^2 + i\tau|^\alpha$$

Thus

$$\| S_\alpha f \|_2^2 \leq C \iint |\hat{\phi}(\xi, \tau)|^2\, d\xi\, d\tau = C \| \hat{\phi} \|_2^2.$$

As in Strichartz [12, I.2.3], (2.2) is proved using results from the theory of convolution of operators on Banach space valued functions. These results are given below; for a thorough treatment of Banach space valued functions see Hille and Phillips [6].

Let X be a Banach space with norm $\| \cdot \|_X$. Let $M(X)$ denote the space of strongly measurable functions defined on E^{n+1} with values in X. $L^p(X)$ is the Banach space of functions in $M(X)$ such that the function $(x, t) \to \| f(x, t) \|_X$ is in L^p. $L^\infty_{\text{com}}(X)$ is the class of functions in $L^\infty(X)$ having compact support.

2.3 Theorem. *Let X, Y be Banach spaces. Let $A : L^\infty_{\text{com}}(X) \to M(Y)$ be given by*

$$A\phi(x, t) = \iint k(x - y, t - s)\phi(y, s)\, dy\, ds$$

where $k(x, t)$ is a bounded operator from X into Y for a.e. (x, t). *Suppose that*

1°. $\| A\phi \|_{L^2(Y)} \leq C_0 \| \phi \|_{L^2(X)}$ *for* $\phi \in L^\infty_{\text{com}}(X)$

2°. $\iint_{C\Omega_{2r}} \| k(x - z, t - u) - k(x, t) \|_{\mathcal{L}(X,Y)}\, dx\, dt \leq C_1$ *for all* $(z, y) \in \Omega_r$,

where C_1 is independent of r.

Then $\| A\phi \|_{L^p(Y)} \leq C_p \| \phi \|_{L^p(X)}$ *for* $1 < p < \infty$, *all* $\phi \in L^\infty_{\text{com}}(X)$.

Theorem (2.3) appears in Lewis [9) in a slightly more general form. Theorem (2.4) below is a modification of Theorem 4 of Benedek, Calderón and Panzone [2]. It may be proved along the same lines using (1.5) in place of the multiplier theorem of Hormander.

2.4 THEOREM. *Let H be a Hilbert space, and for each $p \in (1, \infty)$ let $B : L^p \to L^p(H)$ continuously. For $\phi \in L^\infty_{\text{com}}$, suppose $B\phi$ is given by*

$$(B\phi)^\wedge(x, t) = \hat{\phi}(x, t)h(x, t),$$

where h is an H-valued function such that

1°. *h is bounded in $E^{n+1} \sim (0, 0)$, and*

2°. *the family of functions $\{h(\rho x, \rho^2 t) : 0 < \rho < \infty\}$ is uniformly equicontinuous in $1/2 \leq (|x|^2 + |t|)^{1/2} \leq 2$.*

Suppose that $\| B\phi \|_{L^2(H)} \geq C \| \phi \|_2$, all $\phi \in L^2$. Then also

$$\| B\phi \|_{L^p(H)} \geq C_p(B) \| \phi \|_p \quad \textit{for all } \phi \in L^p,\ 1 < p < \infty.$$

In the original version of (2.4), h is an operator-valued function. Although it is not noted in the statement of the theorem, the proof requires that the family of operators $\{h^* h\}$ commute. In our case, $\{h^* h\}$ is a family of complex numbers, so the question of commutativity does not arise.

As a first step in proving (2.2); we have the following:

2.5 LEMMA. *Let $1 < p < \infty$, $\phi \in L^\infty_{\text{com}}$. Let $f = H_\alpha * \phi$. Then*

$$\| S_\alpha f \|_p \leq C_{p,\alpha} \| \phi \|_p \quad \textit{for } 0 < \alpha < 1.$$

Proof. We use (2.3) with $X = \mathbf{C}$ and Y the Banach space of functions $g(r, y, s)$ defined on $(0, \infty) \times \Omega^+$ such that

$$\| g \|_Y = \left(\int_0^\infty \left[\iint_{\Omega^+} | g(r, y, s) | \, dy \, ds \right]^2 r^{-1-2\alpha} \, dr \right)^{1|2} < \infty.$$

Define $p_{r,y,s}(x, t) = H_\alpha(x - ry, t - r^2 s) - H_\alpha(x, t)$. We will show that $p_{r,y,s}(x, t) \in Y$ for all (x, t) and that the operator $k(x, t) : \mathbf{C} \to Y$ defined by $k(x, t)\lambda = \lambda p_{r,y,s}(x, t)$ satisfies the hypotheses of (2.3). Since the operator A of (2.3) is convolution with $k(x, t)$, we have

$$A\phi(x, t) = [H_\alpha(\cdot - ry, \cdot - r^2 s) - H_\alpha] * \phi(x, t) = f(x - ry, t - r^2 s) - f(x, t).$$

Thus

$$\| A\phi(x, t) \|_Y = \left(\int_0^\infty \left[\iint_{\Omega^+} | f(x - ry, t - r^2 s) - f(x, t) | \, dy \, ds \right]^2 r^{-1-2\alpha} \, dr \right)^{1/2}$$
$$= S_\alpha f(x, t).$$

Hence the conclusion of (2.3) is precisely

$$\| S_\alpha f \|_p \leq C_{p,\alpha} \| \phi \|_p .$$

As a first step, we show

$$\int_0^\infty \left[\iint_{\Omega^+} |p_{r,y,s}(x, t)| \, dy \, ds\right]^2 r^{-1-2\alpha} \, dr < \infty$$

and hence $p_{r,y,s}(x, t) \in Y$. We have

$$\begin{aligned} p_{r,y,s}(x, t) &= (t - r^2s)^{(\alpha-n-2)/2} \exp\{-|x - ry|^2/4(t - r^2s)\} - t^{(\alpha-n-2)/2} \exp\{-|x|^2/4t\} && \text{for } 0 \leqq r^2s < t \\ &= -t^{(\alpha-n-2)/2} \exp\{-|x|^2/4t\} && \text{for } 0 < t \leq r^2s \\ &= 0 && \text{for } t \leqq 0 \end{aligned}$$

If $t \leq 0$, then obviously $p_{r,y,s}(x, t) = 0 \in Y$. Let $t > 0$. For $r^2 < \frac{1}{2}t$, $p_{r,y,s}(x, t)$ is given by a C^∞ function and by the mean value theorem it is $O(r)$ uniformly for $(y, s) \in \Omega^+$. Hence

$$\int_0^{(\frac{1}{2}t)^{1/2}} \left[\iint_{\Omega^+} |p_{r,y,s}(x, t)| \, dy \, ds\right]^2 r^{-1-2\alpha} \, dr \leqq C_{x,t} \int_0^{(\frac{1}{2}t)^{1/2}} r^{1-2\alpha} \, dr \leqq C_{x,t}$$

since $0 < \alpha < 1$. Since $\int_{(\frac{1}{2}t)^{1/2}}^\infty r^{-1-2\alpha} \, dr < \infty$, to conclude that

$$\int_{(\frac{1}{2}t)^{1/2}}^\infty \left[\iint_{\Omega^+} |p_{r,y,s}(x, t)| \, dy \, ds\right]^2 r^{-1-2\alpha} \, dr < \infty$$

it suffices to show that

$$\iint_{(y,s) \in \Omega^+,\ t - r^2s > 0} (t - r^2s)^{(\alpha-n-2)/2} \exp\{-|x - ry|^2/4(t - r^2s)\} \, dy \, ds \leqq C_t$$

for $r^2 \geq \frac{1}{2}t$. Making the change of variables $x - ry = y'$, $t - r^2s = s'$, we see that this last integral is dominated by

$$r^{-n-2} \int_0^t ds \int s^{(\alpha-n-2)/2} e^{-|y|^2/4s} \, dy = cr^{-n-2} \int_0^t s^{(\alpha-2)/2} \, ds = c_t r^{-n-2}$$

since $0 < \alpha < 1$. Hence $p_{r,y,s}(x, t) \in Y$ for all (x, t).

We have previously shown $A : L^2 \to L^2(Y)$ continuously. It remains only to show

$$\iint_{C\Omega_{2a}} \| k(x - z, t - u) - k(x, t) \|_{\mathfrak{L}(C,Y)} \, dx \, dt \leqq C$$

for all $(z, u) \in \Omega_a$, c independent of $a > 0$. This amounts to bounding

$$\iint_{C\Omega_{2a}} dx \, dt \left(\int_0^\infty \left[\iint_{\Omega^+} |p_{r,y,s}(x - z, t - u) - p_{r,y,s}(x, t)| \, dx \, ds\right]^2 r^{-1-2\alpha} \, dr\right)^{1/2}$$

The computation is quite lengthy; it is given in the appendix.

2.6 LEMMA. *Let* $\phi \in L^\infty_{\text{com}}$, $f = H_\alpha * \phi$, *where* $0 < \alpha < 1$. *Then*

$$\| \phi \|_p \leq C_{p,\alpha} \| S_\alpha f \|_p \quad \textit{for } 1 < p < \infty .$$

Proof. Define

$$T_\alpha f(x, t) = \left(\int_0^\infty \left| \iint_{\Omega^+} [f(x - ry, t - r^2 s) - f(x, t)] \, dy \, ds \right|^2 r^{-1-2\alpha} \, dr \right)^{1/2} .$$

Clearly $0 \leqq T_\alpha f \leqq S_\alpha f$; we will use (2.4) to show $\| \phi \|_p \leqq C_{p,\alpha} \| T_\alpha f \|_p$.

Define

$$k_r(x, t) = \iint_{\Omega^+} p_{r,y,s}(x, t) \, dy \, ds,$$

where $p_{r,r,s}(x, t) = H_\alpha(x - ry, t - r^2 s) - H_\alpha(x, t)$ as before. Then $k_r \in L^1$ since

$$\iint | k_r(x, t) | \, dx \, dt \leq \iint dx \, dt \iint_{\Omega^+} | p_{r,y,s}(x, t) | \, dy \, ds$$

$$= \iint_{\Omega^+} dy \, ds \iint | p_{r,y,s}(x, t) | \, dx \, dt$$

$$\leq 2 \iint_{\Omega^+} \| H_\alpha \|_1 \, dy \, ds = C \| H_\alpha \|_1 .$$

Hence for $\phi \in L^p$, the convolution $k_r * \phi$ converges absolutely a.e. By the above calculation, we may change the order of integration so that

$$k_r * \phi(x, t) = \iint_{\Omega^+} p_{r,y,s} * \phi(x, t) \, dy \, ds \quad \text{a.e.}$$

Let H be the Hilbert space of functions defined on $(0, \infty)$ whose modulus is square integrable with respect to the measure $r^{-1-2\alpha} dr$. Let $B\phi(x, r) = k_r * \phi(x, t)$. Then

$$\| B\phi(x, t) \|_H^2 = \int_0^\infty | k_r * \phi(x, t) |^2 r^{-1-2\alpha} \, dr$$

$$= \int_0^\infty \left| \iint_{\Omega^+} p_{r,y,s} * \phi(x, t) \, dy \, ds \right|^2 r^{-1-2\alpha} \, dr$$

$$= \int_0^\infty \left| \iint_{\Omega^+} [f(x - ry, t - r^2 s) - f(x, t)] \, dy \, ds \right|^2 r^{-1-2\alpha} \, dr$$

$$= T_\alpha f(x, t)^2$$

Hence $B\phi(x, t) \in H$ a.e. and

$$\| B\phi \|_{L^p(H)} = \| T_\alpha f \|_p \leq \| S_\alpha f \|_p \leq C_{p,\alpha} \| \phi \|_p .$$

For $\phi \in L^\infty_{\text{com}}$, $(B\phi)^\wedge(\xi, \tau) = \hat{\phi}(\xi, \tau)\hat{k}_r(\xi, \tau)$. We compute

$$\hat{k}_r(\xi, \tau) = (2\pi)^{-(n+1)/2} \iint e^{-ix\cdot\xi - it\tau} k_r(x, t)\, dx\, dt$$

$$= \iint_{\Omega^+} dy\, ds \left[(2\pi)^{-n+1/2} \iint e^{-ix\cdot\xi - it\tau} p_{r,y,s}(x, t)\, dx\, dt \right]$$

$$= \iint_{\Omega^+} p_{r,y,s}(\xi, \tau)\, dy\, ds$$

$$= \hat{H}_\alpha(\xi, \tau) \iint_{\Omega^+} (e^{-iry\cdot\xi - ir^2 s\tau} - 1)\, dy\, ds$$

$$= C(|\xi|^2 + i\tau)^{-\alpha/2} \iint_{\Omega^+} (e^{-iry\cdot\xi - ir^2 s\tau} - 1)\, dy\, ds$$

Thus

$$\| \hat{k}_r(\xi, \tau) \|_H^2 = C \| \xi |^2 + i\tau |^{-\alpha} \int_0^\infty \left| \iint_{\Omega^+} (e^{-iry\cdot\xi - ir^2 s\tau} - 1)\, dy\, ds \right|^2 \cdot r^{-1-2\alpha}\, dr$$

$$= C \int_0^\infty \left| \iint_{\Omega^+} \left(\exp\left\{ \frac{-iry\cdot\xi}{\| \xi |^2 + i\tau |^{1/2}} - \frac{ir^2 s\tau}{\| \xi |^2 + i\tau |} \right\} - 1 \right) dy\, ds \right| r^{-1-2\alpha}\, dr$$

Using the mean value theorem to estimate the integrand for $0 < r < 1$, we see that this integral converges absolutely for $0 < \alpha < 1$. Consequently $\| \hat{k}_r(\xi, \tau) \|_H$ is a continuous function away from $(\xi, \tau) = (0, 0)$. As

$$\| \hat{k}_r(\lambda\xi, \lambda^2\tau) \|_H = \| \hat{k}_r(\xi, \tau) \|_H \quad \text{for } \lambda > 0$$

and

$$\| \hat{k}_r(\xi, \tau) \|_H \neq 0 \quad \textit{for } (\xi, \tau) \neq (0, 0),$$

we have $\| \hat{k}_r(\xi, \tau) \|_H \geq C$ for $(\xi, \tau) \neq (0, 0)$. Consequently

$$\| B\phi \|_{L^2(H)} \geq C \| \phi \|_2, \quad \text{all } \phi \in L^2.$$

The equicontinuity condition in (2.4) follows immediately since

$$\| \hat{k}_r(\rho\xi, \rho^2\tau) - \hat{k}_r(\rho\xi', \rho^2\tau') \|_H = \| \hat{k}_r(\xi, \tau) - \hat{k}_r(\xi', \tau') \|_H .$$

Thus (2.4) is applicable and

$$\| T_\alpha f \|_p = \| B\phi \|_{L^p(H)} \geqq C_{p,\alpha} \| \phi \|_p, \quad \text{all } \phi \in L^\infty_{\text{com}},\ 1 < p < \infty .$$

Proof of Theorem (2.2). Let $\phi \in L^1 \cap L^\infty$. Let A be the operator defined in the proof of (2.3). Then we have

$$C \| A\phi \|_{L^p(Y)} \leq \| \phi \|_p \leq C' \| A\phi \|_{L^p(Y)} .$$

Since $H_\alpha \in L^1 + L^\infty$, the convolution

$$\phi * p_{r,y,s} = \phi * (H_\alpha(\cdot - ry, \cdot - r^2 s) - H_\alpha)$$

converges absolutely, so that $A\phi = \phi * p_{r,y,s}$, and for $f = H_\alpha * \phi$ we have

$$C\| S_\alpha f \|_p \leq \| \phi \|_p \leqq C' \| S_\alpha f \|_p .$$

Let $\psi \in L^1 \cap L^\infty$, $f = \mathcal{G}_\alpha * \psi$. Then $f \in \mathfrak{L}_\alpha^p$ and $\hat{f} = (1 + |x|^2 + it)^{-\alpha/2}\hat{\psi}$. By (1.11), there exists a bounded measure μ such that

$$(1 + |x|^2 + it)^{-\alpha/2} = (|x|^2 + it)^{-\alpha/2}\hat{\mu}(x, t).$$

Thus

$$\hat{f}(x, t) = (|x|^2 + it)^{-\alpha/2}\hat{\mu}(x, t)\hat{\phi}(x, t) = (|x|^2 + it)^{-\alpha/2}(\mu * \psi)^\wedge(x, t).$$

But $\mu * \psi \in L^1 \cap L^\infty$; hence $f = CH_\alpha * (\mu * \psi)$ and

$$\| S_\alpha f \|_p \leqq C\| \mu * \psi \|_p \leq C \| \psi \|_p = C \| f \|_{p,\alpha} .$$

By (1.12),

$$\| f \|_{p,\alpha} \leqq C\| f \|_p + C\| \mu * \psi \|_p \leqq C \| f \|_p + C \| S_\alpha f \|_p .$$

Since the functions $\{\mathcal{G}_\alpha * \psi : \psi \in L^1 \cap L^\infty\}$ are dense in $\mathfrak{L}_\alpha^p$, we have

$$C\| f \|_{p,\alpha} \leqq \| f \|_p + \| S_\alpha f \|_p \leqq C' \| f \|_{p,\alpha} \quad \text{for all } f \in \mathfrak{L}_\alpha^p .$$

Suppose now that $f \in L^p$ and $S_\alpha f \in L^p$. We must show that $f \in \mathfrak{L}_\alpha^p$. Let $\{g_n\}_{n=1}^\infty$ satisfy

(i) $g_n \in \mathcal{S}$,
(ii) $g_n \geqq 0$
(iii) $\| g_n \|_1 = 1$
(iv) $\phi * g_n \to \phi$ in L^p for all $\phi \in L^p$.

Since $\mathcal{S}$ is invariant under $\mathcal{J}_\alpha$, $g_n = \mathcal{G}_\alpha * h_n$ with $h_n \in \mathcal{S}$. We have

$$f * g_n = f * (\mathcal{G}_\alpha * h_n) = \mathcal{G}_\alpha * (f * h_n).$$

Since $f * h_n \in L^p$, we have $f * g_n \in \mathfrak{L}_\alpha^p$ and

$$\| f * g_n \|_{p,\alpha} \leqq C \| f * g_n \|_p + C \| S_\alpha (f * g_n) \|_p .$$

Since $g_n \geqq 0$, Minkowski's inequality gives us $S_\alpha(f * g_n) \leqq g_n * S_\alpha f$. Thus

$$\| f * g_n \|_{p,\alpha} \leqq C \| f * g_n \|_p + C \| g_n * S_\alpha f \|_p \leq C \| f \|_p + C \| S_\alpha f \|_p .$$

Consequently some subsequence $f * g_{n_k}$ converges weakly in $\mathfrak{L}_\alpha^p$. But $f * g_n \to f$ in L^p; therefore $f \in \mathfrak{L}_\alpha^p$.

2.7 *Remark.* Theorem (2.2) remains valid if Ω^+ is replaced by Ω in the definition of S_α; the proof is longer but is essentially the same. Also, if the integrand $f(x - ry, t - r^2 s) - f(x, t)$ is replaced by the mixed second difference $f(x + ry, t - r^2 s) + f(x - ry, t - r^2 s) - 2f(x, t)$ we obtain a characterization of $\mathfrak{L}_\alpha^p$ valid for $0 < \alpha < 2$.

3. Interpolation

In this section we review the definition of complex interpolation of Banach spaces given by Calderón [3], and we state some of his results. We then give an interpolation theorem for $\mathfrak{L}_\alpha^p$ spaces.

3.1 DEFINITION. Let A_0 and A_1 be Banach spaces continuously embedded in a Hausdorff topological vector space V. We assume $A_0 \cap A_1$ is dense in both A_0 and A_1. $A_0 + A_1$ is a Banach space with the norm

$$\| w \|_{A_0+A_1} = \inf \{ \| x \|_{A_0} + \| y \|_{A_1} : x \in A_0, y \in A_1, w = x + y \}.$$

Let $\mathfrak{F}$ be the space of functions f defined on $0 \leqq \text{Re}(z) \leqq 1$ and with values in $A_0 + A_1$ such that

(1) f is bounded and continuous;
(2) f is holomorphic for $0 < \text{Re}(z) < 1$;
(3) for real t, $f(it) \in A_0$ with

$$\sup \| f(it) \|_{A_0} < \infty \quad \text{and} \quad \| f(it) \|_{A_0} \to 0 \quad \text{as } t \to \pm\infty;$$

(4) for real t, $f(1 + it) \in A_1$ with

$$\sup \| f(1 + it) \|_{A_1} < \infty \quad \text{and} \quad \| f(1 + it) \|_{A_1} \to 0 \quad \text{as } t \to \pm\infty.$$

(For a discussion of holomorphic functions taking values in a Banach space see Hille and Phillips [6].)

$\mathfrak{F}$ is a Banach space with respect to the norm

$$\| f \| = \max \{ \sup \| f(it) \|_{A_0}, \sup \| f(1 + it) \|_{A_1} \}.$$

For $0 < s < 1$, let $\mathfrak{N}_s = \{ f \in \mathfrak{F} : f(s) = 0 \}$. Then $\mathfrak{N}_s$ is a closed subspace of $\mathfrak{F}$. We define $A_s = [A_0, A_1]_s = \mathfrak{F}/\mathfrak{N}_s$; i.e., $A_s = \{ f(s) : f \in \mathfrak{F} \}$ with the norm

$$\| x \|_{A_s} = \inf \{ \| f \|_{\mathfrak{F}} : f \in \mathfrak{F} \text{ and } f(s) = x \}.$$

(A_0, A_1) is called an interpolation pair; A_s is called an intermediate space.

3.2. THEOREM (Multilinear Interpolation). *Let* $(A_0^{(k)}, A_1^{(k)})$ $(k = 1, \cdots, m)$ *and* (B_0, B_1) *be interpolation pairs. Let L be a multilinear map from* $\prod_{k=1}^{m} A_0^{(k)} \cap A_1^{(k)}$ *into* $B_0 \cap B_1$ *such that*

$$\| L(x_1, \cdots, x_m) \|_{B_i} \leqq M_i \prod_{k=1}^{m} \| x_k \| \; A_i^{(k)} \quad \textit{for } i = 0, 1.$$

Then L can be extended uniquely to a multilinear map from $\prod_{k=1}^{m} A_s^{(k)}$ *into* B_s *satisfying*

$$\| L(x_1, \cdots, x_m) \|_{B_s} \leqq M_0^{1-s} M_1^{s} \prod_{k=1}^{m} \| x_k \| \; A_s^{(k)}.$$

3.3 THEOREM (Duality). *Let* A_0, A_1 *be reflexive Banach spaces. Then* $[A_0, A_1]_s' = [A_0', A_1']_s$.

3.4 THEOREM. *Let* $1 < p_0 < \infty$, $1 < p_1 < \infty$. *Let* α_0, α_1 *be any real numbers. Then* $[\mathfrak{L}_{\alpha_0}^{p_0}, \mathfrak{L}_{\alpha_1}^{p_1}]_s = \mathfrak{L}_{\alpha}^{p}$ *where* $0 < s < 1$, $1/p = (1 - s)/p_0 + s/p_1$, *and* $\alpha = (1 - s)\alpha_0 + s\alpha_1$.

Proof. By (1.8), $\mathfrak{L}_{\alpha}^{p}$ is reflexive for $1 < p < \infty$. Hence if we prove $\mathfrak{L}_{\alpha}^{p} \subset [\mathfrak{L}_{\alpha_0}^{p_0}, \mathfrak{L}_{\alpha_1}^{p_1}]_s$ with the inclusion map continuous, then by duality we have also

$$\mathfrak{L}_{\alpha}^{p} = (\mathfrak{L}_{-\alpha}^{p'})' \supset [\mathfrak{L}_{-\alpha_0}^{p_0'}, \mathfrak{L}_{-\alpha_1}^{p_1'}]_s' = [\mathfrak{L}_{\alpha_0}^{p_0}, \mathfrak{L}_{\alpha_1}^{p_1}]_s$$

and therefore $\mathfrak{L}_{\alpha}^{p} = [\mathfrak{L}_{\alpha_0}^{p_0}, \mathfrak{L}_{\alpha_1}^{p_1}]_s$.

Let $f = \mathcal{J}_\alpha \psi$, where ψ is simple. Since simple functions are dense in L^p and $\mathcal{J}_\alpha$ is an isometric isomorphism of L^p onto $\mathcal{L}^p_\alpha$, the class of all such functions f is dense in $\mathcal{L}^p_\alpha$. To prove the theorem we need only to find a function $F \in \mathfrak{F}$ such that $F(s) = f$,

$$\| F(it) \|_{p_0,\alpha_0} \leqq C \| f \|_{p,\alpha} \quad \text{and} \quad \| F(1 + it) \|_{p_1,\alpha_1} \leqq C \| f \|_{p,\alpha},$$

where C is independent of f.

Let us note some properties of the operator valued function $\mathcal{J}_z$.

1°. For $\operatorname{Re} z \geqq 0$ and $1 < q < \infty$, $\mathcal{J}_z : L^q \to L^q$ continuously with $\| \mathcal{J}_z \|_{\mathcal{L}(L^q)} \leqq C_q \, e^{(\pi/2)\operatorname{Im} z} \, | P(z) |$ where P is a polynomial determined by n.

2°. For $\operatorname{Re} z > 0$ and $1 < q < \infty$, $\mathcal{J}_z$ is a holomorphic $\mathcal{L}(L^q)$-valued function.

3°. For each $f \in L^q$ $(1 < q < \infty)$, $\mathcal{J}_z f$ is a continuous L^q-valued function on $\operatorname{Re} z \geqq 0$.

Statement 1° was noted after (1.5). To prove 2°, since $\mathcal{S}$ is dense in both L^q and $(L^q)'$ it suffices to prove that for each $\phi, \psi \in \mathcal{S}$ the function

$$z \to \iint \phi(x, t) \mathcal{J}_z \psi(x, t) \, dx \, dt$$

is holomorphic. But it follows immediately from Parseval's formula that the above function is entire.

For 3°, note that for $\operatorname{Re} z \geqq 0$, $\mathcal{J}_z$ is uniformly bounded in $\mathcal{L}(L^q)$ for z in $N(z_0) \cap \{z : \operatorname{Re} z \geqq 0\}$, where $N(z_0)$ is a neighborhood of z_0. Hence it suffices to prove that $\mathcal{J}_z \phi$ is a continuous L^q-valued function for each $\phi \in S$. As above, $\mathcal{J}_z \phi$ is an entire L^q-valued function and hence continuous.

Express $\psi = \sum_{k=1}^n a_k \chi_{E_k}$, where $a_k \in \mathbf{C}$, $a_k \neq 0$, χ_{E_k} is the characteristic function of a set E_k of finite measure, and the sets $\{E_k\}$ are pairwise disjoint.

Define

$$g(z) = \sum_{k=1}^n | a_k |^{p((1-z)/p_0 + z/p_1)} \operatorname{sgn}(a_k) \chi_{E_k}.$$

For $1 < q < \infty$, $g(z)$ is a bounded and continuous L^q-valued function on $0 \leqq \operatorname{Re} z \leqq 1$ which is also holomorphic in $0 < \operatorname{Re} z < 1$. Moreover

$$g(s) = \sum_{k=1}^N | a_k |^{p((1-s)/p_0 + s/p_1)} \operatorname{sgn}(a_k) \chi_{E_k} = \psi,$$

$$\| g(it) \|_{p_0}^{p_0} = \sum_{k=1}^N | a_k |^p | E_k | = \| \psi \|_p^p$$

and

$$\| g(1 + it) \|_{p_1}^{p_1} = \sum_{k=1}^N | a_k |^p | E_k | = \| \psi \|_p^p.$$

Define

$$F(z) = \| \psi \|_p^{1 - p((1-z)/p_0 + z/p_1)} e^{z^2 - s^2} \mathcal{J}_{\alpha_0(1-z) + \alpha_1 z} \, g(z).$$

Then

$$F(s) = \| \psi \|_p^{1 - p((1-s)/p_0 + s/p_1)} \mathcal{J}_{\alpha_0(1-s) + \alpha_1 s} \, g(s) = \mathcal{J}_\alpha \psi = f.$$

$$F(it) = \| \psi \|_p^{1 - p((1-it)/p_0 + it/p_1)} e^{-t^2 - s^2} \mathcal{J}_{\alpha_0(1-it) + \alpha_1 it} \, g(it).$$

$F(it) \in \mathcal{L}_{\alpha_0}^{p_0}$ with

$$\| F(it) \|_{p_0,\alpha_0} = \| \psi \|_p^{1-p/p_0} \| e^{-t^2-s^2} \mathcal{J}_{(\alpha_1-\alpha_0)it}\, g(it) \|_{p_0}.$$

Hence by 1° above, $\| F(it) \|_{p_0,\alpha_0} \to 0$ as $t \to \pm\infty$ and

$$\| F(it) \|_{p_0,\alpha_0} \leqq C \| \psi \|_p^{1-p/p_0} \| g(it) \|_{p_0}$$
$$= C \| \psi \|_p^{1-p/p_0} \| \psi \|_p^{p/p_0} = C \| \psi \|_p = C \| f \|_{p,\alpha}.$$

Similarly $F(1 + it) \in \mathcal{L}_{\alpha_1}^{p_1}$, $\| F(1 + it) \|_{p_1,\alpha_1} \to 0$ as $t \to \pm\infty$, and

$$\| F(1 + it) \|_{p_1,\alpha_1} \leq C \| f \|_{p,\alpha}.$$

For convenience, assume $\alpha_0 \leqq \alpha_1$. Then $e^{z^2-s^2} \mathcal{J}_{\alpha_0(1-z)+\alpha_1 z}$ is a uniformly bounded operator from L^{p_0} to $\mathcal{L}_{\alpha_0}^{p_0}$ for $0 \leqq \operatorname{Re} z \leqq 1$, holomorphic for $0 < \operatorname{Re} z < 1$. Consequently $F(z)$ is bounded as a function with values in $\mathcal{L}_{\alpha_0}^{p_0}$ (and hence as a function with values in $\mathcal{L}_{\alpha_0}^{p_0} + \mathcal{L}_{\alpha_1}^{p_1}$) for $0 \leqq \operatorname{Re} z \leqq 1$, holomorphic for $0 < \operatorname{Re} z < 1$. Since

$$\mathcal{J}_{\alpha_0(1-z)+\alpha_1 z}\, g(z) = \sum_{k=1}^{N} | a_k |^{p((1-z)/p_0+z/p_1)} \operatorname{sgn} (a_k) \mathcal{J}_{\alpha_0(1-z)+\alpha_1 z}\, \chi_{E_k},$$

it follows from 3° above that $F(z)$ is a continuous $\mathcal{L}_{\alpha_0}^{p_0}$-valued function for $0 \leqq \operatorname{Re} z \leqq 1$.

Thus $F \in \mathfrak{F}$, $F(s) = f$, and $\| F \|_{\mathfrak{F}} \leqq C \| f \|_{p,\alpha}$.
The theorem is proved.

4. Multipliers on $\mathcal{L}_\alpha^p$ spaces

In this chapter we use the results of the previous two chapters to determine conditions for the product of two functions to be in an $\mathcal{L}_\alpha^p$ space.

The results are analogous to those obtained by Strichartz [13]; the only real difference is that we lack a suitable characterization of $\mathcal{L}_\alpha^p$ for $1 \leqq \alpha \leqq 2$. This problem has been circumvented in Theorem 4.5, but it has prevented us from obtaining localization results analogous to those of Strichartz [13].

4.1 Definition. A function ϕ is called a *multiplier* on $\mathcal{L}_\alpha^p$ if $\phi f \in \mathcal{L}_\alpha^p$ whenever $f \in \mathcal{L}_\alpha^p$ and $\| \phi f \|_{p,\alpha} \leqq K \| f \|_{p,\alpha}$ for some K independent of $f \in \mathcal{L}_\alpha^p$. The space of multipliers on $\mathcal{L}_\alpha^p$ is denoted $M\mathcal{L}_\alpha^p$.

4.2 Proposition. *$M\mathcal{L}_\alpha^p \subset M\mathcal{L}_\beta^p$ if $\alpha \geqq \beta \geqq 0$. In particular, $M\mathcal{L}_\alpha^p \subset L^\infty$ if $\alpha \geqq 0$.*

Proof. Let $f \in M\mathcal{L}_\alpha^p$, $\alpha \geq 0$. Let $1/p + 1/q = 1$. Then by duality,

$$\| f\phi \|_{q,-\alpha} \leqq K \| \phi \|_{q,-\alpha} \quad \text{as well as} \quad \| f\phi \|_{p,\alpha} \leqq K \| \phi \|_{p,\alpha}$$

for all $\phi \in \mathcal{L}_\alpha^p \cap \mathcal{L}_{-\alpha}^q$. Interpolating according to (3.2) and identifying the interpolated spaces according to (3.4), we see that $\| f\phi \|_2 \leqq K \| \phi \|_2$ for all $\phi \in L^2$, and hence $f \in L^\infty$. But then $f \in M\mathcal{L}_0^p$. Interpolating again, $f \in M\mathcal{L}_\beta^p$ if $0 \leqq \beta \leqq \alpha$.

4.3 LEMMA. *Let* $0 < \alpha < 1, f \in L^\infty$. *Then* $S_\alpha(fg) \leqq \|f\|_\infty S_\alpha g + |g| S_\alpha f$.

Proof. Noting that

$$f(x - ry, t - r^2 s)g(x - ry, t - r^2 s) - f(x, t)g(x, t)$$
$$= f(x - ry, t - r^2 s)[g(x - ry, t - r^2 s) - g(x, t)]$$
$$+ g(x, t)[f(x - ry, t - r^2 s) - f(x, t)]$$

and that the functional

$$\phi \rightarrow \left(\int_0^\infty \left[\iint_{\Omega+} |\phi| \, dy \, ds\right]^2 r^{-1-2\alpha} \, dr\right)^{1/2}$$

is a semi-norm, the result follows immediately.

4.4 LEMMA. *Let* $1 < p < \infty, \alpha > (n + 2)/p$. *Suppose k is an integer such that* $2k < \alpha < 2k + 1$, *and let* $0 \leqq j \leqq 2k$. *Let* $f \in \mathfrak{L}^p_{\alpha-j}, g \in \mathfrak{L}^p_{\alpha-(2k-j)}$. *Then* $fg \in \mathfrak{L}^p_{\alpha-2k}$ *and* $\|fg\|_{p,\alpha-2k} \leqq C \|f\|_{p,\alpha-j} \|g\|_{p,\alpha-(2k-j)}$.

Proof. First assume $j = 0$. Since $0 < \alpha - 2k < 1$, we may use (2.2). We have

$$\|fg\|_p \leqq \|f\|_\infty \|g\|_p \leqq C \|f\|_{p,\alpha} \|g\|_{p,\alpha-2k}$$

by (1.7), since $\alpha > (n + 2)/p$. By (4.3),

$$\|S_{\alpha-2k}(fg)\|_p \leqq \|f\|_\infty \|S_{\alpha-2k}\, g\|_p + \|gS_{\alpha-2k} f\|_p ;$$

by (1.7) and (2.2),

$$\|f\|_\infty \|S_{\alpha-2k}\, g\|_p \leqq C \|f\|_{p,\alpha} \|g\|_{p,\alpha-2k} .$$

To estimate $\|gS_{\alpha-2k} f\|_p$, we find $q, r \in (1, \infty)$ such that

(i) $1/q + 1/r = 1/p$
(ii) $\|g\|_q \leqq C \|g\|_{p,\alpha-2k}$
(iii) $\|S_{\alpha-2k} f\|_r \leqq C \|f\|_{p,\alpha}$.

The result will then follow from Hölder's inequality.

By (1.7), (ii) is satisfied if

$$(*) \qquad 1/q \leqq 1/p < 1/q + (\alpha - 2k)/(n + 2).$$

Also, $\|S_{\alpha-2k} f\|_r \leqq C \|f\|_{r,\alpha-2k}$ so that (iii) is satisfied if

$$(**) \qquad 1/r \leqq 1/p < 1/r + 2k/(n + 2).$$

Combining $(*)$ and $(**)$, we see that we may pick q, r such that (ii) and (iii) are satisfied and such that $1/q + 1/r$ is any positive number between $2/p - \alpha/(n + 2)$ and $2/p$. As $\alpha > (n + 2)/p$, $1/p$ lies in this range.

Hence by (2.2),

$$\|fg\|_{p,\alpha-2k} \leqq C(\|fg\|_p + \|S_{\alpha-2k}(fg)\|_p) \leqq C \|f\|_{p,\alpha} \|g\|_{p,\alpha-2k} .$$

We have now shown that multiplication defines a continuous bilinear map from $\mathcal{L}_\alpha^p \times \mathcal{L}_{\alpha-2k}^p$ into $\mathcal{L}_{\alpha-2k}^p$ and therefore also from $\mathcal{L}_{\alpha-2k}^p \times \mathcal{L}_\alpha^p$ into $\mathcal{L}_{\alpha-2k}^p$. Hence by (3.2)

$$\| fg \|_{p,\alpha-2k} \leqq C \| f \|_{[\mathcal{L}_\alpha{}^p, \mathcal{L}_\alpha{}^{p-2k}]_s} \| g \|_{[\mathcal{L}_\alpha{}^{p-2k}, \mathcal{L}_\alpha{}^p]_s} .$$

Choosing $s = 1 - j/2k$, by (3.4), $[\mathcal{L}_\alpha^p, \mathcal{L}_{\alpha-2k}^p]_s = \mathcal{L}_{\alpha-j}^p$ and $[\mathcal{L}_{\alpha-2k}^p, \mathcal{L}_\alpha^p]_s = \mathcal{L}_{\alpha-(2k-j)}^p$, so the lemma is proved.

4.5 THEOREM. *Let* $1 < p < \infty$, $\alpha > (n + 2)/p$. *Let* $f, g \in \mathcal{L}_\alpha^p$. *Then* $fg \in \mathcal{L}_\alpha^p$ *and*

$$\| fg \|_{p,\alpha} \leqq C \| f \|_{p,\alpha} \| g \|_{p,\alpha} .$$

Proof. Case (i). Suppose some integer k satisfies $2k < \alpha < 2k + 1$. By (1.9), $fg \in \mathcal{L}_\alpha^p$ if $D_x^\gamma D_t^j (fg) \in \mathcal{L}_{\alpha-2k}^p$ for every nonnegative integer j and multi-index γ such that $|\gamma| + 2j \leqq 2k$; moreover

$$\| fg \|_{p,\alpha} \leqq C \sum_{|\gamma|+2j \leqq 2k} \| D_x^\gamma D_t^j (fg) \|_{p,\alpha-2k} .$$

By Leibnitz's rule,

$$D_x^\gamma D_t^j (fg) = \sum_{\beta \leqq \gamma, l \leqq j} C(\beta, \gamma, l, j) (D_x^\beta D_t^l f)(D_x^{\gamma-\beta} D_t^j g).$$

Again by (1.9),

$$\| D_x^\beta D_t^l f \|_{p,\alpha-|\beta|-2l} \leqq C \| f \|_{p,\alpha} \quad \text{and} \quad \| D_x^{\gamma-\beta} D_t^{j-l} g \|_{p,\alpha-|\gamma-\beta|-2(j-l)} \leqq C \| g \|_{p,\alpha}.$$

Hence by (4.4),

$$(D_x^\beta D_t^l f)(D_x^{\gamma-\beta} D_t^{j-l} g) \in \mathcal{L}_{\alpha-|\gamma|-2j}^p$$

and

$$\| (D_x^\beta D_t^l f)(D_x^{\gamma-\beta} D_t^{j-l} g) \|_{p,\alpha-|\gamma|-2j} \leqq C \| f \|_{p,\alpha} \| g \|_{p,\alpha} .$$

As $|\gamma| + 2j \leqq 2k$, $\mathcal{L}_{\alpha-|\gamma|-2j}^p \subset \mathcal{L}_{\alpha-2k}^p$ and the result follows.

Case (ii). Arbitrary $\alpha > (n + 2)/p$. Applying interpolation theory to the bilinear operator $(f, g) \to fg$, we see that

$$\{ (x, y) \in E^2 : 0 < x < 1, \text{ and } \| fg \|_{1/x,y} \leqq C_{x,y} \| f \|_{1/x,y} \| g \|_{1/x,y} \text{ for all } f, g \in \mathcal{L}_y^{1/x} \}$$

is convex. Since the convex hull of

$$\{ (1/p, \alpha) : 1 < p < \infty, \alpha > (n + 2)/p, 2k < \alpha < 2k + 1 \text{ for some integer } k \}$$

is the set $\{ (x, y) : 0 < x < 1, y > (n + 1)x \}$ the result follows for all p, α such that $1 < p < \infty$ and $\alpha > (n + 2)/p$.

4.6 *Remark.* If $0 < \alpha \leqq (n + 2)/p$, we no longer have $\mathcal{L}_\alpha^p \subset L^\infty$. Since $M\mathcal{L}_\alpha^p \subset L^\infty$ by (4.2), the above theorem fails in this case. However, some substitute results are available.

4.7 THEOREM. *Let* $f \in L^\infty \cap \mathcal{L}_{(n+2)/p}^p$, *where* $1 < p < \infty$. *Then* $f \in M\mathcal{L}_\alpha^q$ *if* $1 < q < \infty$, $\alpha < (n + 2)/q$, $\alpha \leq (n + 2)/p$, *and* $0 < \alpha < 1$.

Proof. The restriction $0 < \alpha < 1$ allows to use (2.2). As in (4.4), the problem reduces to showing that $|g| S_\alpha f \in L^q$. Again we find r, s such that $g \in L^r$, $S_\alpha f \in L^s$, and $1/r + 1/s = 1/q$.

By (1.7), $g \in L^r$ for $1/r = 1/q - \alpha/(n+2)$. $S_\alpha f \in L^s$ if $f \in \mathfrak{L}_\alpha^s$; again by (1.7), $f \in \mathfrak{L}_\alpha^s$ for

$$1/p = 1/s + ((n+2)/p - \alpha)/(n+2) = 1/s + 1/p - \alpha/(n+2)$$

or $s = (n+2)/\alpha$. But then $1/r + 1/s = 1/q - \alpha/(n+2) + \alpha/(n+2) = 1/q$, and the theorem follows.

4.8 Remark. As in Strichartz [13, II 3.6 and II 3.7], this result can be strengthened. Virtually the same arguments show $f \in M\mathfrak{L}_\alpha^p$ if $1 < p < \infty$, $0 < \alpha < 1$, $\alpha < (n+2)/p$, $f \in L^\infty$, and

$$|\{(x, t) : S_\alpha f(x, t) > \lambda\}| \leqq (K/\lambda)^{(n+2)/\alpha} \quad \text{for all } \lambda > 0.$$

Appendix

Here we perform the calculations to prove

$$\iint_{C\Omega_{2a}} dx\, dt \left(\int_0^\infty \left[\iint_{\Omega^+} | p_{r,y,s}(x - z, t - u) - p_{r,y,s}(x, t) |\, dy\, ds \right]^2 r^{-1-2\alpha}\, dr \right)^{1/2} \leqq C$$

independently of $a > 0$, $(z, u) \in \Omega_a$. Recall

$$p_{r,y,s}(x, t) = H_\alpha(x - ry, t - r^2 s) - H_\alpha(x, t),$$

$$H_\alpha(x, t) \begin{array}{ll} = t^{(\alpha-n)/2-1} \exp\{-|x|^2/4t\}, & t > 0 \\ = 0, & t \leqq 0. \end{array}$$

Note that it suffices to prove the estimate for the case $a = 1$; the change of variables $x = a^{-1}x'$, $t = a^{-2}t'$, $r = a^{-1}r'$ then establishes the estimate for all other values of $a > 0$.

To simplify notation, let

$$I(E) = \iint_E | p_{r,y,s}(x - z, t - u) - p_{r,y,s}(x, t) |\, dy\, ds$$

for E any measurable subset of E^{n+1}. Of course, $I(E)$ depends on (x, t), (z, u), and r.

Step 1. We estimate $\iint_{|t| \geqq 4} dx\, dt (\int_0^\infty I(\Omega^+)^2 r^{-1-2\alpha}\, dr)^{1/2}$. For $t \leqq -4$ and $(z, u) \in \Omega$, $I(\Omega^+) \equiv 0$. For $t \geqq 4$, we have

$$\left(\int_0^\infty I(\Omega^+)^2 r^{-1-2\alpha}\, dr \right)^{1/2} \leqq \left(\int_0^{\frac{1}{2}t^{1/2}} I(\Omega^+)^2 r^{-1-2\alpha}\, dr \right)^{1/2} + \left(\int_{\frac{1}{2}t^{1/2}}^\infty I(\Omega^+)^2 r^{-1-2\alpha}\, dr \right)^{1/2}.$$

(a) First we show $\iint_{t\geqq 4} dx\, dt (\int_0^{\frac{1}{2}t^{1/2}} I(\Omega^+)^2 r^{-1-2\alpha}\, dr)^{1/2} \leqq C$.

Since $t \geqq 4$, $|u| \leqq 1$, $0 \leqq s \leqq 1$, and $0 \leqq r^2 \leqq \frac{1}{4}t$ we have t, $t-u$, $t-r^2 s$, and $t-u-r^2 s \geqq 2$; hence $p_{r,y,s}$ is a C^∞ function. By the mean value theorem

$$p_{r,y,s}(x-z, t-u) - p_{r,y,s}(x,t) = -\sum_{i=1}^n z_i D_{x_i} p_{r,y,s}(\xi,\tau) - uD_t p_{r,y,s}(\xi,\tau)$$

for some (ξ, τ) on the line from (x, t) to $(x-z, t-u)$. In full detail,

$$\begin{aligned}
&p_{r,y,s}(x-z, t-u) - p_{r,y,s}(x,t)\\
&= -\sum_{i=1}^n z_i[-\tfrac{1}{2}(\tau - r^2 s)^{(\alpha-n)/2}(\xi_i - ry_i)\exp\{-|\xi - ry|^2/4(\tau - r^2 s)\}\\
&\quad + \tfrac{1}{2}\tau^{(\alpha-n)/2-2}\xi_i \exp\{-|\xi|^2/4\tau\}]\\
&\quad - u[((\alpha-n)/2 - 1)(\tau - r^2 s)^{(\alpha-n)/2-2}\exp\{-|\xi - ry|^2/4(\tau - r^2 s)\}\\
&\quad - ((\alpha-n)/2 - 1)\tau^{(\alpha-n)/2-2}\exp\{-|\xi|^2/4\tau\}]\\
&\quad - u[(\tau - r^2 s)^{(\alpha-n)/2-3}\tfrac{1}{4}|\xi - ry|^2 \exp\{-|\xi - ry|^2/4(\tau - r^2 s)\}\\
&\quad - \tau^{(\alpha-n)/2-3}\tfrac{1}{4}|\xi|^2 \exp\{-|\xi|^2/4\tau\}]\\
&= -\sum_{i=1}^n z_i I_i - uJ - uK.
\end{aligned}$$

Recall $|z_i| \leqq 1$ and $|u| \leqq 1$. Each of the terms I_i, J, and K is treated separately; for brevity only the calculations for J will be given. Exactly the same techniques are used to treat I_i and K.

Again applying the mean value theorem,

$$(*)\quad \begin{aligned} J = ((\alpha-n)/2 - 1)\exp\{-|\xi'|^2/4\tau'\}[-\tfrac{1}{2}\tau'^{(\alpha-n)/2-3}\sum_{j=1}^n ry_j \xi_j^i\\
- r^2 s((\alpha-n)/2 - 2)\tau'^{(\alpha-n)/2-3} + \tfrac{1}{4}r^2 s|\xi'|^2\tau'^{(\alpha-n)/2-4}]\end{aligned}$$

where (ξ', τ') is on the line from (ξ, τ) to $(\xi - ry, \tau - r^2 s)$ and hence lies in the rectangle with vertices

$$(x,t),\quad (x-z, t-u), (x-ry, t-r^2 s) \quad \text{and} \quad (x-z-ry, t-u-r^2 s).$$

Note that $\frac{1}{2}t \leqq \tau' \leqq 2t$. To estimate $|\xi'|$, we consider separately the cases $|x| \leqq 2t^{1/2}$ and $|x| \geqq 2t^{1/2}$.

For $|x| \leqq 2t^{1/2}$, we have $|\xi'| \leqq 3t^{1/2}$. Estimating the exponential by 1, we have from $(*)$,

$$|J| \leqq C(rt^{(\alpha-n)/2-5/2} + r^2 t^{(\alpha-n)/2-3}) \leqq Crt^{(\alpha-n)/2-5/2},$$

since $r < \frac{1}{2}t^{1/2}$.

Treating I_i and K similarly, we have

$$|p_{r,y,s}(x-z, t-u) - p_{r,y,s}(x,t)| \leqq Crt^{(\alpha-n)/2-5/2}$$

for $r \leqq \frac{1}{2}t^{1/2}$, $|x| \leqq 2t^{1/2}$. Thus we have

$$\begin{aligned}
&\iint_{t\geqq 4, |x| \leqq 2t^{1/2}} dx\, dt \left(\int_0^{\frac{1}{2}t^{1/2}} I(\Omega)^2 r^{-1-2\alpha}\, dr\right)^{1/2}\\
&\leqq C \iint_{t\geqq 4, |x|\leqq 2t^{1/2}} \left(\int_0^{\frac{1}{2}t^{1/2}} r^{1-2\alpha} t^{\alpha-n-5}\, dr\right)^{1/2} dx\, dt\\
&= C\iint_{t\geqq 4, |x|\leqq 2t^{1/2}} t^{-n/2-2}\, dx\, dt = C\int_4^\infty t^{-2}\, dt = C.
\end{aligned}$$

For $|x| \geqq 2t^{1/2}$ and $0 \leqq r \leqq \frac{1}{2}t^{1/2}$, we have $\frac{1}{2}|x| \leqq |\xi'| \leqq 2|x|$. Thus from (*),

$$|J| \leqq Ce^{-|x|^2/ct}[r\,|x|\,t^{(\alpha-n)/2-3} + r^2 t^{(\alpha-n)/2-3} + r^2\,|x|^2\,t^{(\alpha-n)/2-4}]$$
$$\leqq Crt^{(\alpha-n)/2-7/2}\,|x|^2\,e^{-|x|^2/ct}.$$

Treating I_i and K similarly, we have

$$|p_{r,y,s}(x-z, t-u) - p_{r,y,s}(x,t)| \leqq crt^{(\alpha-n)/2-7/2}\,|x|^2\,e^{-|x|^2/ct}$$

for $|x| \geqq 2t^{1/2}$ and $0 \leqq r \leqq \frac{1}{2}t^{1/2}$. Hence

$$\iint_{t\geqq 4\,|x|\geqq 2t^{1/2}} dx\,dt\left(\int_0^{\frac{1}{2}t^{1/2}} I(\Omega^+)^2 r^{-1-2\alpha}\,dr\right)^{1/2}$$
$$\leqq C\iint_{t\geqq 4,\,|x|\geqq 2t^{1/2}} t^{(\alpha-n)/2-7/2}\,|x|^2 e^{-|x|^2/ct}\,dx\,dt\left(\int_0^{\frac{1}{2}t^{1/2}} r^{1-2\alpha}\,dr\right)^{1/2}$$
$$\leqq C\iint_{t\geqq 4} t^{-n/2-3}\,|x|^2 e^{-|x|^2/ct}\,dx\,dt$$
$$= C\int_4^\infty t^{-2}\,dt = C.$$

(b) Now we show $\iint_{t\geqq 4} dx\,dt(\int_{\frac{1}{2}t^{1/2}}^\infty I(\Omega^+)^2 r^{-1-2\alpha}\,dr)^{1/2} \leqq C$.

Express $\Omega^+ = E_1 \cup E_2 \cup E_3$ where $r^2 s \leqq t-2$, $t-2 \leqq r^2 s \leqq t+2$, and $t+2 \leqq r^2 s$ respectively. We estimate the terms $\iint_{t\geqq 4} dx\,dt(\int_{\frac{1}{2}t^{1/2}}^\infty I(E_k)^2 r^{-1-2\alpha}\,dr)^{1/2}$ separately.

(i) The term in $I(E_1)$.

$$|p_{r,y,s}(x-z, t-u) - p_{r,y,s}(x,t)|$$
$$\leqq |H_\alpha(x-z, t-u) - H_\alpha(x,t)| + |H_\alpha(x-z-ry, t-u-r^2 s) - H_\alpha(x-ry, t-r^2 s)|$$
$$= P + Q.$$

By the mean value theorem,

$$P \leqq C(\tau^{(\alpha-n)/2-2}\,|\xi| + \tau^{(\alpha-n)/2-2} + \tau^{(\alpha-n)/2-3}\,|\xi|^2)\exp\{-|\xi|^2/4\tau\}$$

for some (ξ, τ) on the line from (x,t) to $(x-z, t-u)$. Note $\frac{1}{2}t < \tau < 2t$.

For $|x| \leqq 2$, we estimate $|\xi|$ and the exponential term by constants to obtain

$$P \leqq C(\tau^{(\alpha-n)/2-2} + \tau^{(\alpha-n)/2-2} + \tau^{(\alpha-n)/2-3}) \leqq Ct^{(\alpha-n)/2-2}.$$

For $|x| \geqq 2$, we have $\frac{1}{2}|x| \leqq |\xi| \leqq 2|x|$. Thus

$$P \leqq C(t^{(\alpha-n)/2-2}\,|x| + t^{(\alpha-n)/2-2} + t^{(\alpha-n)/2-3}\,|x|^2)e^{-|x|^2/ct}.$$

It follows readily that

$$\iint_{t\geqq 4} dx\,dt\left(\int_{\frac{1}{2}t^{1/2}}^\infty\left[\iint_{E_1} P\,dy\,ds\right]^2 r^{-1-2\alpha}\,dr\right)^{1/2} \leqq C.$$

For the term in Q, we again have by the mean value theorem

$$Q \leqq C(\tau^{(\alpha-n)/2-2} |\xi| + \tau^{(\alpha-n)/2-2} + \tau^{(\alpha-n)/2-3} |\xi|^2) e^{-|\xi|^2/4\tau}$$

where (ξ, τ) is on the line from $(x - ry, t - r^2 s)$ to $(x - z - ry, t - u - r^2 s)$. Since $t - r^2 s \geqq 2$ in E_1, we have $\frac{1}{2}(t - r^2 s) \leqq \tau \leqq 2(t - r^2 s)$. In order to estimate ξ, we must consider several cases separately.

First we estimate for $|x| \leqq 2$. Since

$$|\xi| \exp\{-|\xi|^2/4\tau\} \leqq C\tau^{1/2} \quad \text{and} \quad |\xi|^2 \exp\{-|\xi|^2/4\tau\} \leqq C\tau$$

$$Q \leqq C\tau^{(\alpha-n)/2-3/2} \leqq C(t - r^2 s)^{(\alpha-n)/2-3/2}.$$

Thus

$$\iint_{E_1} Q\, dy\, ds \leqq C \int_0^{(t-2)r^{-2}} (t - r^2 s)^{(\alpha-n)/2-3/2}\, ds \leqq Cr^{-2}$$

and so

$$\left(\int_{\frac{1}{2}t^{1/2}}^{\infty} \left[\iint_{E_1} Q\, dy\, ds\right]^2 r^{-1-2\alpha}\, dr\right)^{1/2} \leqq C\left(\int_{\frac{1}{2}t^{1/2}}^{\infty} r^{-5-2\alpha}\, dr\right)^{1/2} = Ct^{-1-\alpha/2}.$$

This is integrable over $\{(x, t) \mid |x| \leqq 2, t \geqq 4\}$.

For $|x| \geqq 2$, our estimates must be more delicate. We write $E_1 = F_1 \cup F_2 \cup F_3$, where $|x - ry| \leqq \frac{3}{2}$, $\frac{3}{2} \leqq |x - ry| \leqq \frac{3}{4}|x|$, and $\frac{3}{4}|x| \leqq |x - ry|$ respectively. Note that $F_1 = F_2 = \emptyset$ unless $r \geqq \frac{1}{4}|x|$ and hence unless $r \geqq \frac{1}{8}|x| + \frac{1}{4}t^{1/2}$.

For $(y, s) \in F_1$ we have $|\xi| \leqq C$. Thus

$$Q \leqq C(\tau^{(\alpha-n)/2-2} + \tau^{(\alpha-n)/2-2} + \tau^{(\alpha-n)/2-3}) \leqq C(t - r^2 s)^{(\alpha-n)/2-2}.$$

Noting that $|\{y \mid |x - ry| \leqq \frac{3}{2}\}| = Cr^{-n}$,

$$\iint_{F_1} Q\, dy\, ds \leqq Cr^{-n} \int_0^{(t-2)r^{-2}} (t - r^2 s)^{(\alpha-n)/2-2}\, ds \leqq Cr^{-n-2}.$$

Hence

$$\left(\int_{\frac{1}{2}t^{1/2}}^{\infty} \left[\iint_{F_1} Q\, dy\, ds\right]^2 r^{-1-2\alpha}\, dr\right)^{1/2} \leqq C\left(\int_{\frac{1}{8}|x|+\frac{1}{4}t^{1/2}} r^{-2n-5-2\alpha}\, dr\right)^{1/2}$$

$$= C(\tfrac{1}{2}|x| + t^{1/2})^{-n-2-\alpha}.$$

This is integrable over $\{(x, t) \mid |x| \geqq 2, t > 4\}$.

For $(y, s) \in F_2$,

$$|\xi| \leqq |x - ry| + 1 \leqq C|x - ry| \quad \text{and} \quad |\xi| \geqq |x - ry| - 1 \geqq C|x - ry|,$$

so we have

$$Q \leqq C((t - r^2)^{(\alpha-n)/2-2} |x - ry| + (t - r^2 s)^{(\alpha-n)/2-2}$$

$$+ (t - r^2 s)^{(\alpha-n)/2-3} |x - ry|^2) \exp\{-|x - ry|^2/C(t - r^2 s)\}.$$

Making the change of variable $y' = (t - r^2 s)^{-1/2}(x - ry)$ and enlarging the y integration to E^n, we see

$$\iint_{F_2} Q\,dy\,ds \leqq Cr^{-n} \int_0^{(t-2)r^{-2}} [(t - r^2 s)^{\alpha/2-3/2} + (t - r^2 s)^{(\alpha-n)/2-2}]\,ds$$
$$\leqq Cr^{-n-2}$$

Exactly as for F_1, we see

$$\left(\int_{\frac{1}{2}t^{1/2}}^{\infty} \left[\iint_{F_2} Q\,dy\,ds\right]^2 r^{-1-2\alpha}\,dr\right)^{1/2} \leqq C(\tfrac{1}{2}\,|x| + t^{1/2})^{-n-2-\alpha}.$$

For $(y, s) \in F_3$ we have

$$|\xi| \geqq |x - ry| - 1 \geqq \tfrac{3}{4}\,|x| - 1 \geqq \tfrac{1}{4}\,|x|$$

and thus $|\xi|^2 \geqq \frac{1}{2}\,|\xi|^2 + \frac{1}{32}\,|x|^2$. Hence

$$|\xi| \exp\{-|\xi|^2/4\tau\} \leqq |\xi| \exp\{-|\xi|^2/8\tau\} \exp\{-|x|^2/128\tau\}$$
$$\leqq C\tau^{1/2} \exp\{-|x|^2/128\tau\}$$
$$\leqq C(t - r^2 s)^{1/2} \exp\{-|x|^2/c(t - r^2 s)\}.$$

Similarly,

$$|\xi|^2 \exp\{-|\xi|^2/4\tau\} \leqq C(t - r^2 s) \exp\{-|x|^2/c(t - r^2 s)\}.$$

Thus

$$Q \leqq C(t - r^2 s)^{(\alpha-n)/2-3/2} \exp\{|x|^2/c(t - r^2 s)\}$$

and

$$\textstyle\iint_{F_3} Q\,dy\,ds \leqq c \int_0^{(t-2)r^{-2}} (t - r^2 s)^{(\alpha-n)/2-3/2} \exp\{-|x|^2/c(t - r^2 s)\}\,ds$$
$$\leqq c\,|x|^{\alpha-n-1} r^{-2} \textstyle\int_0^{\infty} s^{(\alpha-n)/2-3/2} e^{-1/s}\,ds$$
$$= c\,|x|^{\alpha-n-1} r^{-2}.$$

Thus

$$\left(\int_{\frac{1}{2}t^{1/2}}^{\infty} \left[\iint_{F_3} Q\,dy\,ds\right]^2 r^{-1-2\alpha}\,dr\right)^{1/2} \leqq c|x|^{\alpha-n-1} \left(\int_{\frac{1}{2}t^{1/2}}^{\infty} r^{-5-2\alpha}\,dr\right)^{1/2}$$
$$= c|x|^{\alpha-n-1} t^{-1-\alpha/2},$$

which is integrable over $\{(x, t) : |x| \geqq 2, t \geqq 4\}$.

We have now shown

$$\iint_{t \geqq 4} dx\,dt \left(\int_{\frac{1}{2}t^{1/2}}^{\infty} I(E_1)^2 r^{-1-2\alpha}\,dr\right)^{1/2} \leqq c.$$

(ii) The term in $I(E_3)$. For $t \geqq 4$ and $t + 2 \leqq r^2 s$ we have

$$p_{r,y,s}(x - z, t - u) - p_{r,y,s}(x, t) = H_\alpha(x, t) - H_\alpha(x - z, t - u).$$

This can be treated exactly as the term P in (i) above.

(iii) The term in $I(E_2)$. In this region both $p_{r,y,s}(x, t)$ and $p_{r,y,s}(x - z, t - u)$ may have a singularity. The two terms are handled separately. We have

$$|p_{r,y,s}(x, t)| \leqq H_\alpha(x, t) + H_\alpha(x - ry, t - r^2 s).$$

Note that

$$\left(\int_{\frac{1}{2}t^{1/2}}^{\infty}\left[\iint_{E_2} H_\alpha(x, t)\, dy\, ds\right]^2 r^{-1-2\alpha}\, dr\right)^{1/2} = ct^{-1/2-\alpha/2}H_\alpha(x, t)$$
$$= ct^{-n/2-3/2}\exp\left\{-\frac{|x|^2}{4t}\right\}.$$

This is integrable over $\{(x, t) : t \geqq 4\}$.

For the other term we estimate separately the r-integration over the intervals $\frac{1}{2}t^{1/2} \leqq r \leqq \frac{1}{4}|x|$ and $r \geqq \max(\frac{1}{2}t^{1/2}, \frac{1}{4}|x|)$.

For $|x| \geqq 2t^{1/2}$ we have

$$\left(\int_{\frac{1}{2}t^{1/2}}^{\frac{1}{4}|x|}\left[\iint_{E_2} H_\alpha(x - ry, t - r^2 s)\, dy\, ds\right]^2 r^{-1-2\alpha}\, dr\right)^{1/2}$$
$$\leqq \left(\int_{\frac{1}{2}t^{1/2}}^{\frac{1}{4}|x|}\left[\int_{(t-2)r^{-2}}^{tr^{-2}} ds \int_{|y|\leqq 1} (t - r^2 s)^{(\alpha-n)/2-1}\right.\right.$$
$$\left.\left.\exp\{-|x - ry|^2/4(t - r^2 s)\}\, dy\right]^2 r^{-1-2\alpha}\, dr\right)^{1/2}$$
$$\leqq c\left(\int_{\frac{1}{2}t^{1/2}}^{\infty}\left[\int_{(t-2)r^{-2}}^{tr^{-2}} ds \int_{|y|\leqq 1} (t - r^2 s)^{(\alpha-n)/2-1}\right.\right.$$
$$\left.\left.\exp\{-|x|^2/16(t - r^2 s)\}\, dy\right]^2 r^{-1-2\alpha} dr\right)^{1/2}$$
$$= c\left(\int_{\frac{1}{2}t^{1/2}}^{\infty}\left[\int_0^2 s^{(\alpha-n)/2-1}e^{-|x|^2/16s}\, ds\right]^2 r^{-5-2\alpha} dr\right)^{1/2}$$
$$\leqq c|x|^{\alpha-n-2}\left(\int_{\frac{1}{2}t^{1/2}}^{\infty} r^{-5-2\alpha}\, dr\right)^{1/2}$$
$$= c|x|^{\alpha-n-2}t^{-1-\alpha/2}$$

since

$$s^{(\alpha-n)/2-1}e^{-|x|^2/16s} \leqq c|x|^{\alpha-n-2}.$$

Of course, $|x|^{\alpha-n-2}t^{-1-\alpha/2}$ is integrable over $\{(x, t) : |x| \geqq 2t^{1/2}, t \geqq 4\}$.

For the second interval,

$$\left(\int_{\max(\frac{1}{4}|x|,\frac{1}{2}t^{1/2})}^{\infty}\left[\iint_{E_2} H_\alpha(x - ry, t - r^2 s)\, dy\, ds\right]^2 r^{-1-2\alpha} dr\right)$$
$$\leqq \left(\int_{\frac{1}{8}|x|+\frac{1}{4}t^{1/2}}^{\infty}\left[\int_{(t-2)r^{-2}}^{tr^{-2}} ds \int (t - r^2 s)^{(\alpha-n)/2-1}\right.\right.$$
$$\left.\left.\exp\{-|x - ry|^2/4(t - r^2 s)\}\, dy\right]^2 r^{-1-2\alpha} dr\right)^{1/2}$$

$$= c \left(\int_{\frac{1}{8}|x| + \frac{1}{4}t^{1/2}}^{\infty} \left[\int_0^2 s^{\alpha/2-1}\, ds \right]^2 r^{-2n-5-2\alpha} dr \right)^{1/2}$$

$$= c \left(\frac{1}{2} \mid x \mid + t^{1/2} \right)^{-n-2-\alpha}.$$

This is integrable over $\{ (x, t) : t \geqq 4\}$.

Treating the term in $p_{r,y,s}(x - z, t - u)$ similarly, we complete Step 1.

Step 2. It remains only to bound

$$\iint_{|t| \leqq 4, |x| \geqq 2} dx\, dt \left(\int_0^\infty I(\Omega^+)^2 r^{-1-2\alpha}\, dr \right).$$

Since the t-integration is over a compact set this is comparatively easy; the crucial thing is to show that $I(\Omega^+) = O(r)$ as $r \to 0$.

(a) First we estimate $(\int_0^{\frac{1}{4}|x|} I(\Omega^+)^2 r^{-1-2\alpha}\, dr)^{1/2}$.

$$I(\Omega^+) \leqq \iint_{\Omega^+} | p_{r,y,s}(x, t) |\, dy\, ds + \iint_{\Omega^+} | p_{r,y,s}(x - z, t - u) |\, dy\, ds.$$

We treat the two terms separately. Recall

$$p_{r,y,s}(x, t) = H_\alpha(x - ry, t - r^2 s) - H_\alpha(x, t),$$

with H_α a C^∞ function. By the mean value theorem,

$$p_{r,y,s}(x, t) = -r \textstyle\sum_{i=1}^n y_i\, D_{x_i} H_\alpha(\xi, \tau) - r^2 s D_t H_\alpha(\xi, \tau)$$

for some (ξ, τ) on the line from (x, t) to $(x - ry, t - r^2 s)$.

Note that

$$D_{x_i} H_\alpha(\xi, \tau) = -\tfrac{1}{2}\xi_i\, \tau_2^{(\alpha-n)/2-2} \exp \{-| \xi |^2/4\tau\}, \quad \tau > 0$$
$$= 0, \qquad \tau \leqq 0$$

and

$$\sup_{\tau>0} \tau^{(\alpha-n)/2-2} \exp \{-| \xi |^2/4\tau\} = c \mid \xi \mid^{\alpha-n-4}.$$

Also

$$D_t H_\alpha(\xi, \tau)$$
$$= [((\alpha - n)/2 - 1)\tau^{(\alpha-n)/2-2} + \tfrac{1}{4} \mid \xi \mid^2 \tau^{(\alpha-n)/2-3}] \exp \{-| \xi |^2/4\}, \quad \tau > 0$$
$$= 0, \qquad \tau \leqq 0$$

and

$$\sup_{\tau>0} \tau^{(\alpha-n)/2-3} \exp \{-| \xi |^2/4\tau\} = c \mid \xi \mid^{\alpha-n-6}.$$

Hence

$$| p_{r,y,s}(x, t) | \leqq cr \mid \xi \mid^{\alpha-n-3} + cr^2 \mid \xi \mid^{\alpha-n-4}$$
$$\leqq cr \mid x \mid^{\alpha-n-3} + cr^2 \mid x \mid^{\alpha-n-4} \quad \text{since } r \leqq | x |/4$$
$$\leqq cr \mid x \mid^{\alpha-n-3}.$$

Similarly, we obtain $| p_{r,y,s}(x - z, t - u) | \leqq cr \mid x \mid^{\alpha-n-3}$ for $| x | \geqq 2, r \leqq \frac{1}{4} | x |$.

Thus

$$\left(\int_0^{\frac{1}{2}|x|} I(\Omega^+)^2 r^{-1-2\alpha}\, dr\right)^{1/2} \leqq c\,|\,x\,|^{\alpha-n-3}\left(\int_0^{\frac{1}{2}|x|} r^{1-2\alpha}\, dr\right)^{1/2}$$

$$= c\,|\,x\,|^{-n-2},$$

which is integrable over $\{(x, t) : |t| \leqq 4, |x| \geqq 2\}$.

(b) It remains only to estimate $(\int_{\frac{1}{2}|x|}^{\infty} I(\Omega^+)^2 r^{-1-2\alpha}\, dr)^{1/2}$. Here we may use

$$I(\Omega^+) \leqq \iint_{\Omega^+} H_\alpha(x, t)\, dy\, ds + \iint_{\Omega^+} H_\alpha(x - ry, t - r^2 s)\, dy\, ds$$

$$+ \iint_{\Omega^+} H_\alpha(x - z, t - u)\, dy\, ds + \iint_{\Omega^+} H_\alpha(x - z - ry, t - u - r^2 s)\, dy\, ds.$$

First,

$$\left(\int_{\frac{1}{2}|x|}^{\infty}\left[\iint_{\Omega^+} H_\alpha(x, t)\, dy\, ds\right]^2 r^{-1-2\alpha} dr\right)^{1/2} = c\,|\,x\,|^{-\alpha} t^{(\alpha-n)/2-1} e^{-|x|^2/4t}, \quad t > 0$$

$$= 0, \qquad t \leqq 0$$

which is integrable over $\{(x, t) : |t| \leqq 4, |x| \geqq 2\}$.
The term in $H_\alpha(x - z, t - u)$ is handled in the same manner.

Next, we have

$$\left(\int_{\frac{1}{2}|x|}^{\infty}\left[\iint_{\Omega^+} H_\alpha(x - ry, t - r^2 s)\, dy\, ds\right]^2 r^{-1-2\alpha} dr\right)^{1/2}$$

$$\leqq \left(\int_{\frac{1}{2}|x|}^{\infty}\left[\int_{0 \leqq s \leqq tr^{-2}} ds \int (t - r^2 s)^{(\alpha-n)/2-1} \exp\{-|x - ry|^2/4(t - r^2 s)\}\, dy\right]^2 r^{-1-2\alpha} dr\right)^{1/}$$

$$= c\left(\int_{\frac{1}{2}|x|}^{\infty}\left[\int_{0 \leqq s \leqq tr^{-2}} (t - r^2 s)^{\alpha/2-1}\, ds\right]^2 r^{-2n-1-2\alpha} dr\right)^{1/2}$$

$$= \begin{cases} ct^{\alpha/2}\left(\displaystyle\int_{\frac{1}{2}|x|}^{\infty} r^{-2n-5-2\alpha}\, dr\right)^{1/2}, & t > 0 \\ 0, & t \leqq 0 \end{cases}$$

$$= \begin{cases} ct^{\alpha/2}\,|\,x\,|^{-n-2-\alpha}, & t > 0 \\ 0, & t \leqq 0 \end{cases}$$

which is integrable over $\{(x, t) : |x| \geq 2, |t| \leq 4\}$.

The term in $H_\alpha(x - z - ry, t - u - r^2 s)$ is handled in exactly the same manner, and we are done.

Bibliography

1. N. Aronszajn and K. T. Smith, *Theory of Bessel potentials I*, Ann. Inst. Fourier, vol. 11 (1961), pp. 385–475.
2. M. Benedek, A. P. Calderón, and M. Panzone, *Convolution operators on Banach space valued functions*, Proc. Nat. Acad. Sci. U.S.A., vol. 48 (1962), pp. 356–365.
3. A. P. Calderón, *Intermediate spaces and interpolation*, Studia Math., vol. 24 (1964), pp. 113–190.
4. ———, *Lebesgue spaces of differentiable functions and distributions*, Proc. Symp. Pure Math., vol. 5 (1961), pp. 33–49.
5. E. B. Fabes and N. M. Rivière, *Singular integrals with mixed homogeneity*, Studia Math, vol. 27 (1966), pp. 19–38.
6. E. Hille and R. S. Phillips, *Functional analysis and semigroups*, Amer. Math. Soc. Colloq. Publ., vol. XXXI, rev. ed., Amer. Math. Soc., Providence, 1957.
7. L. Hörmander, *Estimates for translation invariant operators in L^p spaces*, Acta Math., vol. 104 (1960), 93–140.
8. B. F. Jones, Jr., *Lipschitz spaces and the heat equation*, J. Math. Mech., vol. 18 (1968), pp. 379–410.
9. J. E. Lewis, *Mixed estimates for singular integrals and an application to initial value problems of parabolic type*, Proc. Symp. Pure Math., vol. 10 (1968), pp. 218–231.
10. R. O'Neil, *Fractional integration in Orlicz spaces*, Trans. Amer. Math. Soc., vol. 115 (1965), pp. 300–328.
11. C. H. Sampson, *A characterization of parabolic Lebesgue spaces*, Dissertation, Rice University, 1968.
12. L. Schwartz, *Theorie des distributions*, rev. ed., Hermann, Paris, 1966.
13. R. S. Strichartz, *Multipliers on fractional Sobolev spaces*, J. Math. Mech., vol. 16 (1967), pp. 1031–1061.
14. M. H. Taibleson, *On the theory of Lipschitz spaces of distributions on Euclidean n-space, II: Translation invariant operators, duality, and interpolation*, J. Math. Mech., vol. 14 (1965), pp. 821–840.
15. A. Zygmund, *Trigonometric series*, vol. I–II, 2nd ed., Cambridge University Press, Cambridge, 1959.
16. ———, *On a theorem of Marcinkiewicz concerning interpolation of operators*, J. Math. Pures Appl., vol. 35 (1956), pp. 223–248.

New Mexico State University
Las Cruces, New Mexico