EICHLER COHOMOLOGY AND AUTOMORPHIC FORMS

BY

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1. Introduction

This paper is devoted, for the most part, to a new proof of a theorem proved by Gunning [3]. In essence the theorem originates with Eichler [2] who first investigated systematically the cohomology of a Riemann surface R obtained from the generalized periods arising from the integrals of automorphic forms. The automorphic forms in question are of degree ≤ -2 with respect to discontinuous groups related to R by means of uniformization theory. Our method, totally different from that of Gunning, employs only the classical theory of automorphic forms and a device introduced in [4]. Throughout we ignore the Riemann surface and work only with the discontinuous group.

Before we can state the main results we must introduce some definitions and notation. Let $\mathcal{H}$ denote the upper half-plane and let Γ be a discontinuous group of linear fractional transformations acting on $\mathcal{H}$. For convenience we normalize Γ so that an element of Γ has the form $z \to (az + b)/(cz + d)$, with a, b, c, d real and $ad - bc = 1$. We also identify the element $V \in \Gamma$ with the matrices

$$\pm \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

We say that Γ is an *H-group* if

(i) Γ is finitely generated,

(ii) Γ is discontinuous in $\mathcal{H}$ but is discontinuous at no point of the real line,

(iii) Γ contains translations.

The automorphic forms to be considered here are of *integral* degree with multiplier system, are *holomorphic* in $\mathcal{H}$, and are, as usual, restricted to those which are meromorphic (in the appropriate uniformizing variables) at all of the parabolic cusps of a fundamental region of Γ. The characteristic functional equation satisfied by an automorphic form F of degree r, with multiplier system v, with respect to Γ, is

$$F(Vz) = v(V)(cz + d)^{-r}F(z), \tag{1}$$

for all $V = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$, where $v(V)$ is independent of z and $|v(V)| = 1$. From (1) we can immediately derive a consistency condition for v which reduces in the case when r is an integer to $v(V_1 \cdot V_2) = v(V_1) \cdot v(V_2)$, for all $V_1, V_2 \in \Gamma$. That is, v is a complex character on Γ thought of as a matrix group. We denote the complex vector space of automorphic forms of degree r, with multiplier system v, with respect to Γ by $\{\Gamma, r, v\}$.

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From now on we assume that r is a nonnegative integer and that v is a multiplier system on Γ with respect to the degree r. (Note that v is then a multiplier system with respect to the degree $-r-2$ and $\bar{v}$ is also a multiplier system with respect to the degrees r and $-r-2$.) A well-known result due to Bol [1] states that

$$\frac{d^{r+1}}{dz^{r+1}}\{(cz+d)^r F(Vz)\} = (cz+d)^{-r-2} F^{(r+1)}(Vz), \tag{2}$$

for any $V = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, with $ad - bc = 1$. This can be proved either by induction on r or directly by the use of Cauchy's integral formula. It follows immediately from (2) that if $F \in \{\Gamma, r, v\}$, then

$$\frac{d^{r+1}}{dz^{r+1}} F = F^{(r+1)} \in \{\Gamma, -r-2, v\}.$$

The converse is not quite true. However it is easy to see from (2) that if $f \in \{\Gamma, -r-2, v\}$ and F is any $(r+1)$-fold indefinite integral of f, then F satisfies the following functional equation:

$$\bar{v}(V)(cz+d)^r F(Vz) = F(z) + p_V(z), \tag{3}$$

for all $V = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$. Here $p_V(z)$ is a polynomial in z of degree $\leq r$ which depends on V. If it should happen that $p_V(z) \equiv 0$ for all $V \in \Gamma$, then in fact (3) reduces to (1) and $F \in \{\Gamma, r, v\}$. In keeping with recent usage, a function satisfying (3), which is meromorphic in $\mathcal{H}$ and meromorphic in the appropriate variables at all of the parabolic cusps of a fundamental region of Γ, will be called an *automorphic integral* of degree r, with multiplier system v and period polynomials p_V, with respect to Γ. The polynomials p_V are also called the *period polynomials* of the automorphic form f.

If we put $(F \mid V)(z) = \bar{v}(V)(cz+d)^r F(Vz)$, for $V = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$, then (3) becomes $F \mid V = F + p_V$ and we conclude from this that

$$p_{V_1 V_2} = p_{V_1} \mid V_2 + p_{V_2}, \tag{4}$$

for $V_1, V_2 \in \Gamma$. For the moment we will concentrate our attention upon (4). Suppose $\{p_V \mid V \in \Gamma\}$ is any collection of polynomials of degree $\leq r$ satisfying (4); then we call $\{p_V \mid V \in \Gamma\}$ a *cocycle*. A *coboundary* is a set $\{p_V \mid V \in \Gamma\}$ of polynomials of degree $\leq r$ such that

$$p_V = p \mid V - p \quad \text{for all } V \in \Gamma,$$

with p a *fixed* polynomial of degree $\leq r$. With these definitions every coboundary is a cocycle. The *cohomology group* $H_v^1(\Gamma, P_r)$ is defined as usual to be the vector space obtained by forming the quotient of the cocycles by the coboundaries. Here P_r is the vector space of polynomials of degree $\leq r$. It is of interest to note that if we begin with an automorphic form f of degree $-r-2$ and attach to f the cocycle of period polynomials $\{p_V\}$ by means of (3), this cocycle is not uniquely determined by f. For the indefinite integral

F is determined only up to a polynomial p of degree r. Replacing F by $F + p$, we find that $\{p_V\}$ is replaced by $\{p_V^*\}$, where $p_V^* = p_V - (p \mid V - p)$. The important feature here is that the cocycle $\{p_V^*\}$ *is in the same cohomology class as is* $\{p_V\}$. Thus f uniquely determines an element of $H_v^1(\Gamma, P_r)$ by means of (3).

Let $C^+(\Gamma, -r - 2, v)$ denote the subspace of $\{\Gamma, -r - 2, v\}$ consisting of *entire* automorphic forms, that is, those which are holomorphic in $\mathcal{H}$ and holomorphic at all of the parabolic cusps of a fundamental region. Let $C^0(\Gamma, -r - 2, v)$ be the subspace of *cusp forms*, that is, those entire automorphic forms which vanish at all of the parabolic cusps of a fundamental region. We are now in a position to state our main results.

THEOREM 1. *Let Γ be an H-group, r a positive integer, and v a multiplier system on Γ corresponding to the degree r. Then as vector spaces,*

$$C^0(\Gamma, -r - 2, \bar{v}) \oplus C^+(\Gamma, -r - 2, v) \quad \textit{and} \quad H_v^1(\Gamma, P_r)$$

are isomorphic under a mapping which is "canonical" in the sense that its construction is independent of Γ, r, and v.

THEOREM 2. *Let Γ, r, and v be as in Theorem 1. Then given a cohomology class in $H_v^1(\Gamma, P_r)$ there exists an automorphic form h in $\{\Gamma, -r - 2, v\}$ whose period polynomials are in the given cohomology class. In fact h can be so chosen that it is holomorphic in $\mathcal{H}$ and at all of the parabolic cusps except for the cusp at $i\infty$.*

Remarks. 1. Theorem 1 was stated by Gunning [3, Theorem 5] as follows: there exists an exact sequence of spaces and maps of the form

$$0 \to C^+(\Gamma, -r - 2, v) \to H_v^1(\Gamma_1, P_r) \to C^0(\Gamma, -r - 2, \bar{v}) \to 0.$$

Gunning assumes that the multiplier system v consists entirely of roots of unity, while here we make no such assumption on v. On the other hand Gunning assumes only that Γ is a finitely generated Fuchsian group of the first kind, not necessarily an H-group.

2. Eichler's version of Theorem 1 [2, p. 283] (the original version) deals not with $H_v^1(\Gamma, P_r)$ but rather with a modification of $H_v^1(\Gamma, P_r)$ which we will denote $\tilde{H}_v^1(\Gamma, P_r)$. $\tilde{H}_v^1$ does not contain all of the elements of H_v^1, but only those whose cocycles $\{p_V \mid V \in \Gamma\}$ satisfy the following condition:

(5) Let $Q_1, \cdot, Q_t$ represent all of the parabolic classes in Γ. Then for each h, $1 \leq h \leq t$, there exists a polynomial p_h of degree $\leq r$ such that

$$p_{Q_h} = p_h \mid Q_h - p_h .$$

Eichler's theorem can be stated as

COROLLARY 1. *With Γ, r, and v as in Theorem 1,*

$$C^0(\Gamma, -r - 2, \bar{v}) \oplus C^0(\Gamma, -r - 2, v)$$

is isomorphic to

$$\tilde{H}^1_v(\Gamma, P_r).$$

3. In [2], Corollary 1 is proved only for r even and $v \equiv 1$. In [3, pp. 61-2] it was proved under the assumption that v consists entirely of roots of unity. In [2], the case $r = 0$, $v \equiv 1$ is included. As we have stated Corollary 1, the case $r = 0$ is *not* included. However in the Appendix we give a proof of Theorem 1 for $r = 0$. (The case $r = 0$ is treated in an appendix as it requires, at least at present, a proof different from that for $r > 0$.) In §6 we present a deduction of Corollary 1 from Theorem 1 that is valid for $r \geq 0$. Thus Corollary 1 for $r = 0$ is actually included among our results here.

2. Cusp forms and the supplementary function

The key to our proof of Theorems 1 and 2 is the use of the "supplementary function". This is very nearly the same concept as the "supplementary series" introduced in [4, pp. 183–184].

Let

$$S = \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}, \qquad \lambda > 0,$$

be the minimal positive translation in Γ, let $v(S) = e^{2\pi i x}$, $0 \leq x < 1$, and let ν be an integer and r a positive integer. Consider the Poincaré series

$$g_\nu(z, \bar{v}) = \sum_V \frac{\exp\{2\pi i(\nu + x)Vz/\lambda\}}{\bar{v}(V)(cz + d)^{r+2}}$$

where $V = \begin{pmatrix} * & * \\ c & d \end{pmatrix}$ runs through a complete set of elements of Γ with *distinct lower row.* The following facts concerning the Poincaré series are well known [6, 272–289].

(i) $g_\nu(z, \bar{v}) \in \{\Gamma, -r - 2, \bar{v}\}$.

(ii) $g_\nu(z,\bar{v})$ vanishes at all cusps of Γ except possibly at $i\infty$. At $i\infty$ it has an expansion of the form

$$g_\nu(z, \bar{v}) = 2e^{2\pi i(\nu+x)z/\lambda} + 2\sum_{m+x>0} a_m(\nu, \bar{v})e^{2\pi i(m+x)z/\lambda}.$$

Thus if $\nu + x > 0$, $g_\nu(z, \bar{v}) \in C^0(\Gamma, -r - 2, \bar{v})$.

(iii) There exist integers $0 \leq \nu_1 < \cdots < \nu_s$ such that $g_{\nu_1}, \cdots, g_{\nu_s}$ form a basis for $C^0(\Gamma, -r - 2, \bar{v})$.

Suppose $g \in C^0(\Gamma, -r - 2, \bar{v})$. By (iii), there exist complex numbers $b_1, \cdots, b_s$ such that $g = \sum_{i=1}^s b_i g_{\nu_i}(z, \bar{v})$. Put $g^* = \sum_{i=1}^s \bar{b}_i g_{\nu_i'}(z,v)$, where

$$\nu' = -\nu \qquad \text{if } x = 0$$
$$= -1 - \nu \quad \text{if } x > 0.$$

Note that with $\bar{v}(S) = e^{2\pi i x}$, $0 \leq x < 1$, we also have $v(S) = e^{2\pi i x'}$, $0 \leq x' < 1$ where

$$x' = 0 \qquad \text{if } x = 0$$
$$= 1 - x \quad \text{if } x > 0.$$

Thus we have the expansion at $i\infty$

$$g_{\nu_i'}(z, v) = 2\exp\{2\pi i(\nu_i' + x')z/\lambda\} + 2\sum_{m+x'>0} a_m(\nu_i', v)\exp\{2\pi i(m + x')z/\lambda\}$$
$$= 2e^{-2\pi i(\nu_i+x)z/\lambda} + 2\sum_{m+x'>0} a_m(\nu_i', v)e^{2\pi i(m+x')z/\lambda}.$$

It follows that $g^* \in \{\Gamma, -r-2, v\}$, g^* has a pole at $i\infty$ with principal part

$$2\sum_{i=1}^{s} \bar{b}_i \exp\{-2\pi i(\nu_i + x)z/\lambda\},$$

and g^* vanishes at all of the other parabolic cusps of Γ. Let G^* be the $(r+1)$-fold indefinite integral of g^*, so normalized that

$$G^*(z + \lambda) = v(S)G^*(z) = e^{2\pi i x'}G^*(z).$$

We call $G^*(z)$ *the function supplementary to* g.

In analogy with (3) we have

$$\bar{v}(V)(cz + d)^r G^*(Vz) = G^*(z) + q_V^*(z), \tag{6}$$

for all $V = \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \Gamma$, where $q_V^*(z)$ is a polynomial in z of degree $\leq r$. Also if we let G be the $(r+1)$-fold integral of g, so normalized that

$$G(z + \lambda) = \bar{v}(S)G(z) = e^{2\pi i x}G(z)$$

and G has no constant term in its expansion at $i\infty$, then

$$v(V)(cz + d)^r G(Vz) = G(z) + q_V(z), \tag{7}$$

for all $V = \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \Gamma$, where $q_V(z)$ is a polynomial in z of degree $\leq r$. The fact upon which our entire proof hinges is that with $q_V^*(z)$, $q_V(z)$ as in (6) and (7), respectively, we have

$$\overline{q_V(\bar{z})} = q_V^*(z) \quad \text{for all } V \in \Gamma. \tag{8}$$

This was proved in [4, §IV] under the assumption $r > 0$.

As an immediate consequence of (8) we have the following result which has already appeared, in a slightly different form, as Theorem (4.9) of [4].

THEOREM 3. *Let r be a positive integer, $g \in C^0(\Gamma, -r-2, \bar{v})$, and G^* the function supplementary to g. Then $g \equiv 0$ if and only if $G^* \in \{\Gamma, r, v\}$.*

Proof. Suppose $g \equiv 0$. Then G, the $(r+1)$-fold integral of g, is also identically 0. Then $q_V(z) \equiv 0$ for all $V \in \Gamma$, where q_V is as in (7). By (8) $q_V^*(z) \equiv 0$ for all $V \in \Gamma$. Thus by (6), we have

$$\bar{v}(V)(cz + d)^r G^*(Vz) = G^*(z),$$

for all $V = \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \Gamma$. There remains only the question of the behavior of G^* at the parabolic cusps. That G^* is meromorphic at the parabolic cusps follows since G^* is an $(r+1)$-fold integral of g^* and g^* as an element of $\{\Gamma, -r-2, v\}$ is meromorphic at the parabolic cusps. Thus $G^* \in \{\Gamma, r, v\}$

Conversely, suppose $G^* \in \{\Gamma, r, v\}$. Then $q_V^*(z) \equiv 0$ for all $V \in \Gamma$. By (8)

$q_V(z) \equiv 0$ for all $V \in \Gamma$. It follows as above that $G \in \{\Gamma, r, \bar{v}\}$. But G, the $(r+1)$-fold integral of a cusp form, is regular in $\mathcal{H}$ and also at all of the parabolic cusps of Γ. It is well known that under these circumstances $G \equiv 0$, since $r > 0$ [6, p. 301]. It follows that $g \equiv 0$.

Remarks. 1. Theorem 3 follows directly from Petersson's "Principal Parts Condition" [9].

2. The mapping from g to the cocycle $\{q_V^* \mid V \in \Gamma\}$ appears from our construction to depend upon the choice of a basis for $C^0(\Gamma, -r-2, \bar{v})$ from among the functions $g_\nu(z, \bar{v})$; $\nu = 1, 2, \cdots$. However our mapping is in fact *independent* of the choice of the basis, The point is that if $g \in C^0(\Gamma, -r-2, \bar{v})$ is expressed in any way at all as a finite sum

$$g = \sum_{\nu=1}^{N} b_\nu g_\nu(z, \bar{v}),$$

then the periods of $g^* = \sum_{\nu=1}^{N} \bar{b}_\nu\, g_{\nu'}(z, v)$ are related to those of g by means of the equation (8). Thus, although g^* depends not upon g but upon a particular representation of g in the form $\sum_{\nu=1}^{N} b_\nu\, g_\nu(z, \bar{v})$ the corresponding cocycle $\{q_V^* \mid V \in \Gamma\}$ depends only upon g. Another way of stating this is that to each cusp form $g \in C^0(\Gamma, -r-2, \bar{v})$ there corresponds not a *single* supplementary function but rather an *infinite class* of supplementary functions, all with the same cocycle of periods. In this context we may expand Theorem 3 to

THEOREM 3′. *Let r be a positive integer and $g \in C^0(\Gamma, -r-2, \bar{v})$. Then $g \equiv 0$ if and only if $G^* \in \{\Gamma, r, v\}$ for every function G^* supplementary to g. This in turn holds if and only if $G^* \in \{\Gamma, r, v\}$ for a single function G^* supplementary to g. Furthermore with G^* defined as in Theorem 3, with respect to a fixed basis of $C^0(\Gamma, -r-2, \bar{v})$, we have $g = 0$ if and only if $G^* = 0$.*

The last statement follows immediately, since $\sum_{i=1}^{s} b_i\, g_{\nu_i}(z, \bar{v}) = 0$, with $g_{\nu_1}, \cdots, g_{\nu_s}$ a basis, of course implies $b_i = 0$ for $1 \leq i \leq s$. Thus $g^* = 0$ and consequently G^* is constant. Since $G^* \in \{\Gamma, r, v\}$ and $r > 0$, it follows that $G^* = 0$.

3. The mapping into $H_v^1(\Gamma, P_r)$

We now exhibit explicitly the mapping referred to in Theorem 1. Let $f \in C^+(\Gamma, -r-2, v)$. Put $\beta(f)$ equal to the cohomology class of the cocycle $\{p_V \mid V \in \Gamma\}$ of period polynomials of F, an $(r+1)$-fold integral of f (refer to equation (3)). For $g \in C^0(\Gamma, -r-2, \bar{v})$ put $\alpha(g)$ equal to the cohomology class of the cocycle $\{q_V^* \mid V \in \Gamma\}$ of period polynomials of G^*. Here G^* is the function supplementary to g, and q_V^* are the polynomials occurring in (6).

For $(g, f) \in C^0(\Gamma, -r-2, \bar{v}) \times C^+(\Gamma, -r-2, v)$ put $\mu(g, f) = \alpha(g) + \beta(f)$ Since α and β are linear maps, so is μ. We now show that μ is $1-1$. For this is sufficient to prove that the kernel of μ is $(0, 0)$. With this in mind suppose $\mu(g, f) = 0$. This implies that there exists a polynomial $p(z)$ of degree $\leq r$ such that $F + G^* + p \in \{\Gamma, r, v\}$. Here F is an $(r+1)$-fold integral of f

and G^* is the function supplementary to g. Now $F + G^* + p$ is regular in $\mathfrak{H}$ and at all of the cusps of Γ except at the cusp $i\infty$. The principal part of $F + G^* + p$ at $i\infty$ agrees with that of G^* at $i\infty$. Hence by a well-known formula for the Fourier coefficients of automorphic forms of positive dimension on H-groups, obtained first by Petersson [8] and later by Lehner using the circle method [7], it follows that $F + G^* + p = G^*$. Hence $F = -p$, so that $f = D^{(r+1)}F = 0$. Also $G^* = F + G^* + p \in \{\Gamma, r, v\}$. Thus by Theorem 3 $g = 0$. We have proved that the kernel of μ is $(0, 0)$, so that μ is $1 - 1$.

4. Completion of the proof of Theorem 1

In section 3 we showed how to imbed $C^0(\Gamma, -r-2, \bar{v}) \oplus C^+(\Gamma, -r-2, v)$ isomorphically into $H_v^1(\Gamma, P_r)$ via the linear mapping μ. The proof of Theorem 1 will be complete if we show that μ is *onto* $H_v^1(\Gamma, P_r)$. To accomplish this we will prove that

$$\dim C^0(\Gamma, -r-2, \bar{v}) + \dim C^+(\Gamma, -r-2, v) = \dim H_v^1(\Gamma, P_r). \tag{9}$$

Put $D_1 = \dim C^0(\Gamma, -r-2, \bar{v})$ and $D_2 = \dim C^+(\Gamma, -r-2, v)$. The equality (9) is correct for $r \geq 0$, not merely for $r > 0$, and we prove it under the assumption that $r \geq 0$ and v is a multiplier system on Γ for the degree $-r-2$. The case $r = 0$, $v \equiv 1$ is slightly exceptional.

To calculate the left-hand side of (9) we apply Petersson's generalized Riemann-Roch Theorem [10, Theorem 9]. It is a familiar fact that Γ can be presented in terms of generators and relations as follows:

$$\begin{gathered} A_1, B_1, \cdots, A_p, B_p, E_1, \cdots, E_s, Q_1, \cdots, Q_t, \\ E_j^{l_j} = -I \quad \text{for } 1 \leq j \leq s, \\ \gamma_1 \cdots \gamma_p \cdot E_1 \cdots E_s \cdot Q_1 \cdots Q_t = (-I)^{s+t} \quad \text{with } \gamma_i = A_i B_i A_i^{-1} B_i^{-1}. \end{gathered} \tag{10}$$

Here $I = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, the A_i and B_i are hyperbolic matrices, the E_j are elliptic matrices, and the Q_h are parabolic matrices. Also every elliptic element in Γ is conjugate to one of the E_j and every parabolic element to one of the Q_h. Following Petersson [10] we put

$$v(Q_h) = e^{2\pi i x_h}, \quad 0 \leq x_h < 1 \qquad (1 \leq h \leq t)$$

and

$$v(E_j) = \exp\{\pi i (r + 2 + 2a_j)/l_j\} \qquad (1 \leq j \leq s),$$

where a_j is an integer such that $0 \leq a_j \leq l_j - 1$. Also define

$$\begin{aligned} \vartheta_h &= 1 \quad \text{if } x_h = 0 \\ &= 0 \quad \text{if } x_h > 0, \end{aligned}$$

put $q = t + \sum_{j=1}^{s} (1 - 1/l_j)$, and let

$$\begin{aligned} \delta &= 1 \quad \text{if } r = 0 \text{ and } v \equiv 1 \\ &= 0 \quad \text{otherwise.} \end{aligned}$$

Then by [10, Theorem 9] we have

$$D_1 = -\sum_{h=1}^{t} \vartheta_h + (r+2)(p-1+q/2) - \sum_{h=1}^{t} x_h - \sum_{j=1}^{s} a_j/l_j - p + 1 + \delta,$$

and

$$D_2 = (r+2)(p-1+q/2) - \sum_{h=1}^{t} x_h' - \sum_{j=1}^{s} a_j'/l_j - p + 1.$$

Here x_h' and a_j' are defined by means of

$$\bar{v}(Q_h) = \exp\{2\pi i x_h'\}, \quad 0 \le x_h' < 1 \qquad (1 \le h \le t)$$

and

$$\bar{v}(E_j) = \exp\{\pi i(r + 2 + 2a_j')/l_j\} \qquad (1 \le j \le s),$$

respectively, with a_j' an integer such that $0 \le a_j' \le l_j - 1$. It is clear that $x_h + x_h' = 1 - \vartheta_h$. Then we have

$$\begin{aligned} D_1 + D_2 &= 2(r+2)(p-1+q/2) - \sum_{h=1}^{t} (\vartheta_h + 1 - \vartheta_h) \\ &\qquad - 2p + 2 - \sum_{j=1}^{s} (a_j + a_j')/l_j + \delta \\ &= (r+1)(2p-2+t) + (r+2)\sum_{j=1}^{s} (1 - 1/l_j) \\ &\qquad - \sum_{j=1}^{s} (a_j + a_j')/l_j + \delta. \end{aligned} \tag{11}$$

To calculate $\dim H_v^1(\Gamma, P_r)$ we put D_3 equal to the dimension of the space of cocycles and D_4 equal to the dimension of the space of coboundaries. We can take any polynomial $p(z)$ of degree $\le r$ and form the coboundary $\{(p \mid V - p) \mid V \in \Gamma\}$. For such a coboundary to vanish identically means that $p(z) \in \{\Gamma, r, \bar{v}\}$. Among other things this implies that $p(z + \lambda) = e^{2\pi i x} p(z)$ which unless $r = 0$ implies that $p(z) \equiv 0$. If $r = 0$, but $\bar{v} \neq 1$ again $p(z) \in \{\Gamma, r, \bar{v}\}$ is impossible unless $p(z) \equiv 0$. Thus except in the case $r = 0$ $v \equiv 1$ it turns out that $D_4 = r + 1$. In the exceptional case $p \mid V - p \equiv 0$ always, so that $D_4 = 0$. In general, then, $D_4 = r + 1 - \delta$.

In the calculation of D_3 we first observe that because of the condition (4) satisfied by a cocycle we need only assign polynomials of degree $\le r$ to the generators of Γ in such a way that the assignment is consistent with the relations given in (10). We now make use of the fact that, since Γ is an H-group, $t \ge 1$. We may arbitrarily assign polynomials to $A_1, B_1, \cdots, A_p, B_p, Q_1, \cdots, Q_{t-1}$. This contributes to D_3 the number $(r+1)(2p+t-1)$. Then once an assignment of polynomials is made to $E_1, \cdots, E_s$, the polynomial for Q_t will be determined by the relation $\gamma_1 \cdots \gamma_p \cdot E_1 \cdots E_s \cdot Q_1 \cdots Q_t = (-I)^{s+t}$.

It remains only to calculate the contribution made to D_3 by the polynomials assigned to E_j, $1 \le j \le s$. In this calculation we follow Eichler [2, pp. 274–276]. Let p_{E_j} be the polynomial assigned to E_j in an arbitrary cocycle. From (4) and the relation $E_j^{l_j} = -I$ it follows that there exists a polynomial p_j of degree $\le r$ such that $p_{E_j} = p_j \mid E_j - p_j$, for $1 \le j \le s$. Hence the number of linearly independent polynomials we can attach to E_j is the dimension of the vector space

$$V_j = \{(p \mid E_j - p) \mid p \text{ is a polynomial of degree } \leq r\}.$$

But dim V_j is the number of linearly independent elements among $z^m \mid E_j - z^m$, $0 \leq m \leq r$. Normalize E_j to the form

$$E_j = \begin{pmatrix} e^{\pi i/l_j} & 0 \\ 0 & e^{-\pi i/l_j} \end{pmatrix}.$$

Then

$$\begin{aligned} z^m \mid E_j - z^m &= \bar{v}(E_j)(e^{-\pi i/l_j})^r(e^{2\pi i/l_j}\cdot z)^m - z^m \\ &= z^m[\bar{v}(E_j)\exp\{2\pi i(m - r/2)/l_j\} - 1], \end{aligned}$$

so that dim V_j is the number of integers m, $0 \leq m \leq r$, such that

$$\exp\{2\pi i(m - r/2)/l_j\} \neq v(E_j).$$

Since $v(E_j) = \exp\{\pi i(r + 2 + 2a_j)/l_j\}$, we consider the equation

$$\exp\{2\pi i(m - r/2)/l_j\} = \exp\{\pi i(r + 2 + 2a_j)/l_j\}. \tag{12}$$

Equation (12) is satisfied if and only if

$$m - r/2 \equiv (r + 2)/2 + a_j \pmod{l_j},$$

and this in turn is equivalent to $m - r \equiv a_j + 1 \pmod{l_j}$. Putting $u = r - m$ we find that the number of solutions of $u \equiv -a_j - 1 \pmod{l_j}$, $0 \leq m \leq r$, is exactly $[(r + a_j + 1)/l_j]$, where as usual $[x]$ denotes the largest integer $\leq x$. Hence

$$\dim V_j = r + 1 - [(r + a_j + 1)/l_j].$$

We conclude finally that

$$D_3 = (r + 1)(2p + t - 1) + \sum_{j=1}^{s}\left(r + 1 - \left[\frac{r + a_j + 1}{l_j}\right]\right),$$

and thus

$$\begin{aligned} D_3 - D_4 &= (r + 1)(2p + t - 1) \\ &\quad + \sum_{j=1}^{s}\left(r + 1 - \left[\frac{r + a_j + 1}{l_j}\right]\right) - (r + 1 - \delta) \\ &= (r + 1)(2p + t - 2) + s(r + 1) - \sum_{j=1}^{s}\left[\frac{r + a_j + 1}{l_j}\right] + \delta. \end{aligned} \tag{13}$$

The proof of (9), and thus of Theorem 1, will be complete if we show that $D_1 + D_2 = D_3 - D_4$. A comparison of (11) and (13) shows that it is sufficient to prove

$$\begin{aligned} (r + 2)\textstyle\sum_{j=1}^{s}(1 - 1/l_j) - \sum_{j=1}^{s}(a_j + a_j')/l_j \\ = s(r + 1) - \textstyle\sum_{j=1}^{s}[(r + a_j + 1)/l_j], \end{aligned}$$

that is,

(14) $s - \sum_{j=1}^{s} (a_j + a_j' + r + 2)/l_j = -\sum_{j=1}^{s} [(r + a_j + 1)/l_j].$

Equation (14) is equivalent to

$$\sum_{j=1}^{s} (a_j + a_j' + r + 2)/l_j = \sum_{j=1}^{s} \left\{ \left[\frac{r + a_j + 1}{l_j} \right] + 1 \right\},$$

which, in turn, will follow from

(15) $(a_j + a_j' + r + 2)/l_j = [(r + a_j + 1)/l_j] + 1$ for $1 \leq j \leq s$.

From the definition of a_j and a_j' it follows that

$$\exp \{2\pi i(a_j + a_j')/l_j\} = \exp \{-2\pi i(r + 2)/l_j\},$$

or $a_j + a_j' + r + 2 = z_j\, l_j$, with z_j an integer. Since $0 \leq a_j' \leq l_j - 1$, we conclude that

$$(a_j + r + 2)/l_j \leq z_j \leq (a_j + l_j + r + 1)/l_j,$$

or

$$1/l_j + (a_j + r + 1)/l_j \leq z_j \leq 1 + (a_j + r + 1)/l_j.$$

Hence $(a_j + a_j' + r + 2)/l_j = z_j = [(r + a_j + 1)/l_j] + 1$, and (15) follows. The proof of Theorem 1 is complete.

5. Proof of Theorem 2

The proof of Theorem 2 is actually contained in the proof of Theorem 1. In Theorem 1 we proved that given a cocycle $\{p_V \mid V \in \Gamma\}$, then there exists

$$(g, f) \in C^0(\Gamma, -r - 2, \bar{v}) \times C^+(\Gamma, -r - 2, v)$$

such that $\mu(g, f) = \alpha(g) + \beta(f) =$ the cohomology class of $\{p_V \mid V \in \Gamma\}$. Let $\{q_V \mid V \in \Gamma\}$ be the cocycle of period polynomials of f and let $\{q_V^* \mid V \in \Gamma\}$ be the cocycle of period polynomials of g^*. Then $\mu(g, f)$ is the cohomology class of the cocycle $\{q_V + q_V^* \mid V \in \Gamma\}$ and $\{q_V + q_V^* \mid V \in \Gamma\}$ is the set of period polynomials of $f + g^* \in \{\Gamma, -r - 2, v\}$. This completes the proof of Theorem 2.

6. Proof that Theorem 1 implies Corollary 1

In view of Theorem 1 it suffices to prove that, with

$$(g, f) \in C^0(\Gamma, -r - 2, \bar{v}) \times C^+(\Gamma, -r - 2, v), \quad \mu(g, f) = \alpha(g) + \beta(f)$$

satisfies condition (5) if and only if $f \in C^0(\Gamma, -r - 2, v)$. Let $x_1', \cdots, x_t'$ be defined as in §4; suppose $S = Q_t$ so that $x_t' = x'$, with x' as in §2. Further, let q_h, $1 \leq h \leq t$, be the parabolic cusp of Γ left fixed by Q_h. Then $q_t = i\infty$. With these definitions it is known [6, pp. 272–3] that $f \in \{\Gamma, -r - 2, \bar{v}\}$ has expansions at the parabolic cusps q_h of the form

$$f(z) = (z - q_h)^{-r-2} \sum_{m \geq -m_h} b_m(h) \exp \{-2\pi i(m + x_h')(z - q_h)^{-1}/\lambda_h\} \qquad 1 \leq h \leq t - 1, \tag{16}$$

$$f(z) = \sum_{m \geq -m_t} b_m(t) \exp \{2\pi i(m + x_t')z/\lambda_t\}, \qquad h = t.$$

In (16) λ_n, $1 \leq h \leq t$ are certain positive numbers depending on the structure of Γ, and m_h, $1 \leq h \leq t$ are integers. (Note that $\lambda_t = \lambda$.)

Suppose $F(z)$ is an $(r + 1)$-fold integral of $f(z)$. If $1 \leq h \leq t - 1$, then applying (2), we find that $F(z)$ has an expansion at q_h of the form

$$F(z) = (z - q_h)^r (2\pi i/\lambda_h)^{-r-1} \cdot \sum_{m \geq -m_h} b_m(h)(m + x_h')^{-r-1} \exp\{-2\pi i(m + x_h')(z - q_h)^{-1}/\lambda_h\} + \delta_h \cdot \frac{(-1)^{r+1}}{(r+1)!}(z - q_h)^{-1} + p_h(z),$$

where $p_h(z)$ is a polynomial of degree $\leq r$ and $\delta_h = b_0(h)$ or 0 according as the expansion (16) has a term with $m + x_h' = 0$ (i.e. $m = x_h' = 0$) or not. At $q_t = i\infty$, $F(t)$ has the expansion

$$F(z) = (2\pi i/\lambda_t)^{-r-1} \sum_{m \geq -m_t} b_m(t)(m + x_t')^{-r-1} \exp\{2\pi i(m + x_t')z/\lambda_t\} + \delta_t z^{r+1}/(r + 1)! + p_t(z);$$

here δ_t has the same meaning as before and $p_t(z)$ is a polynomial of degree $\leq r$. It follows from these expansions of $F(z)$ that the cocycle of periods of f satisfies (5) if and only if $\delta_h = 0$, for $1 \leq h \leq t$. Thus the cocycle of periods satisfies (5) if and only if none of the expansions (16) of $f(z)$ has a term with $m = x_h' = 0$.

With $f \in C^+(\Gamma, -r - 2, v)$ it follows that $\beta(f)$ satisfies (5) if and only if $f \in C^0(\Gamma, -r - 2, v)$. On the other hand, for

$$g \in C^0(\Gamma, -r - 2, \bar{v}),\ g^* \in \{\Gamma, -r - 2, v\}$$

and g^* has no term with $m + x_h' = 0$, for $1 \leq h \leq t$. Thus $\alpha(g)$ always satisfied (5), so that $\mu(g, f) = \alpha(g) + \beta(f)$ satisfies (5) if and only if $f \in C^0(\Gamma, -r - 2, v)$. The proof is complete.

Appendix. A proof of Theorem 1 for $r = 0$

In this appendix we give a proof of Theorem 1 for $r = 0$. Since equation (8), a key feature of our proof of Theorem 1, depends upon the assumption $r > 0$, we give a different proof for $r = 0$, based upon results of Petersson. Then Theorem 2 and Corollary 1 also follow for $r = 0$.

Since equation (9) is value for $r = 0$, it is sufficient to display a mapping which imbeds $C^0(\Gamma, -2, \bar{v}) \oplus C^+(\Gamma, -2, v)$ isomorphically into $H_v^1(\Gamma, P_0)$, $P_0 =$ complex numbers. In [12], [13], Petersson has carried out a construction of automorphic forms of degree -2 with arbitrary multiplier system v on $H-$ groups. He obtains these automorphic forms from the usual Poincaré series of degree $-r - 2$, $r > 0$, by a passage to the limit as $r \to 0+$. In this way he produces functions $g_\nu(z, \bar{v})$, with ν an arbitrary integer, satisfying conditions (i), (ii), (iii) of §2, but now with $r = 0$.

In [11], Peterson establishes two further results which are essential in our proof. The first of these is the existence of a "gap sequence" in a setting more general than that of the classical gap sequence of Weierstrass [11, p. 207].

We apply only a very special case of this Petersson gap sequence. The second result connects this gap sequence with a basis for cusp forms [11, p. 211, Theorem 9α]. We state both results together under the single title of

PETERSSON GAP THEOREM. *Let s be the dimension over the complex field of the vector space* $C^0(\Gamma, -2, \bar{v})$. *Then there exist exactly s integers* w_i, $0 < w_1 < \cdots < w_s$, *such that there does not exist an element of* $\{\Gamma, 0, v\}$ *having as its only singularity in a fundamental region of* Γ *a pole at* $i\infty$ *of order* $w_i - x'$, $1 \leq i \leq s$. *Furthermore*

$$(17)\quad \begin{aligned} &g_{w_1}, \cdots, g_{w_s} \textit{ form a basis for } C^0(\Gamma, -2, \bar{v}) \textit{ if } x = 0,\\ &g_{w_1-1}, \cdots, g_{w_s-1} \textit{ form a basis for } C^0(\Gamma, -2, \bar{v}) \textit{ if } x \neq 0. \end{aligned}$$

We are now in a position to describe the mapping into $H_v^1(\Gamma_1\, P_0)$. For $f \in C^+(\Gamma, -2, v)$, $\beta(f)$ is as described in §3; that is, $\beta(f)$ is the cohomology class of the cocycle of periods of F, an indefinite integral of f. Suppose $g \in C^0(\Gamma, -2, \bar{v})$. From (17) and the definition of ν' given in §2 it follows that the functions $g_{(-w_i)'}$, $1 \leq i \leq s$, form a basis for $C^0(\Gamma, -2, \bar{v})$ whether $x = 0$ or $x > 0$. Thus there exist complex numbers $b_1, \cdots, b_s$ such that $g = \sum_{i=1}^s b_i\, g_{(-w_i)'}(z, \bar{v})$. Put

$$g^* = \sum_{i=1}^s \bar{b}_i\, g_{(-w_i)}(z, v) \in \{\Gamma, -2, v\}$$

and let $\alpha(g)$ be the cohomology class of the cocycle of periods of G^*, an indefinite integral of g^* so normalized that $G^*(z + \lambda) = e^{2\pi i x'} G^*(z)$. Note that the principal part of g^* at $i\infty$ is

$$2 \sum_{i=1}^s \bar{b}_i \exp\{+2\pi i(-w_i + x')z/\lambda\},$$

so that the principal part of G^* at $i\infty$ is

$$2 \sum_{i=1}^s \bar{b}_i \{2\pi i(x' - w_i)/\lambda\}^{-1} \exp\{2\pi i(-w_i + x')z/\lambda\}.$$

Since g^* is regular at all points of a fundamental region other than the point at $i\infty$, the same is true of G^*, so that if G^* were in $\{\Gamma, 0, v\}$ it would contradict the Petersson Gap Theorem, unless $b_i = 0$ for $1 \leq i \leq s$. Thus $G^* \in \{\Gamma, 0, v\}$ if and only if $g \equiv 0$. This is Theorem 3 for the case $r = 0$.

For $(g, f) \in C^0(\Gamma, -2, \bar{v}) \times C^+(\Gamma, -2, v)$ put $\mu(g, f) = \alpha(g) + \beta(f)$. Then μ is a linear map and we want to show that μ is 1-1. Suppose $\mu(g, f) = 0$. Then there exists a complex number c such that $F + G^* + c \in \{\Gamma, 0, v\}$. Now $F + G^* + c$ is regular in $\mathcal{H}$ and at all of the cusps of Γ except at the cusp $i\infty$; at $i\infty$ the principal part of $F + G^* + c$ agrees with that of G^*. Thus $F + G^* + c$ is an element of $\{\Gamma, 0, v\}$, with a singularity of the type excluded by the Petersson Gap Theorem, unless $b_i = 0$ for $1 \leq i \leq s$. Since all $b_i = 0$, it follows that $g \equiv 0$ and G^* is a constant. Thus $F + G^* + c$ is an everywhere regular element of $\{\Gamma, 0, v\}$. By the result of [5], $F + G^* + c$ is constant. Thus F is constant and $f = F' = 0$. Therefore the kernel of μ is $(0, 0)$, μ is 1-1, and Theorem 1 is proved for the case $r = 0$.

REFERENCES

1. G. BOL, *Invarianten linearer differentialgleichungen*, Abh. Math. Sem. Univ. Hamburg, vol. 16 (1949), pp. 1–28.
2. M. EICHLER, *Eine Verallgemeinerung der Abelschen Integrale*, Math. Zeitschrift, vol. 67 (1957), pp. 267–298.
3. R. C. GUNNING, *The Eichler cohomology groups and automorphic forms*, Trans. Amer. Math. Soc., vol. 100 (1961), pp. 44–62.
4. M. I. KNOPP, *Construction of automorphic forms on H-groups and supplementary Fourier series*, Trans. Amer. Math. Soc., vol. 103 (1962), pp. 168–188.
5. M. I. KNOPP, J. LEHNER, AND M. NEWMAN, *A bounded automorphic form of dimension zero is constant*, Duke Math. J., vol. 32 (1965), pp. 457–460.
6. J. LEHNER, *Discontinuous groups and automorphic functions*, Math. Surveys, no. 8, American Math. Soc., Providence, R. I., 1964.
7. ———, *The Fourier coefficients of automorphic forms on horocyclic groups, II*, Michigan Math. J., vol. 6 (1959), pp. 173–193.
8. H. PETERSSON, *Konstruktion der Modulformen und zu gewissen Grenzkreisgruppen gehörigen automorphen Formen von positive reeller Dimension und die vollständige Bestimmung ihrer Fourierkoeffizienden*, S.-B. Heidelberger Akad. Wiss. Math.-Natur. Kl., 1950, pp. 417–494.
9. ———, *Über automorphe Formen mit Singuläritaten im Diskontinuitätsgebeit*, Math. Ann., vol. 129 (1955), pp. 370–390.
10. ———, *Zur analytischen Theorie der Grenzkreisgruppen II*, Math. Ann., vol. 115 (1937–38), pp. 175–204.
11. ———, *Automorphe Formen als metrische Invarianten, I*, Math. Nachr., vol. 1 (1948), pp. 158–212.
12. ———, *Automorphe Formen als metrische Invarianten, II*, Math. Nachr., vol. 1 (1948), pp. 218–257.
13. ———, *Explizite Konstruktion der automorphen Orthogonalfunktionen in den multiplikativen Differentialklassen*, Math. Nachr., vol. 16 (1957), pp. 343–368.

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