THE STRUCTURE OF WEAKLY MIXING MINIMAL TRANSFORMATION GROUPS

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1. Introduction

In this paper, we give a variety of characterizations of weakly mixing minimal transformation groups. This class of transformation groups was studied in [13] with respect to various dynamical and eigenfunction properties under the assumption of a compact metric phase space. We extend these results to transformation groups with compact Hausdorff phases spaces by noting that a large class of point-transitive transformation groups with such phase spaces are an inverse limit of transformation groups with compact metric phase spaces (Theorem 2.2). This "approximation" theorem enables us to lift various properties which are true for the metric case and which are preserved by inverse limits. In particular, we can then answer a conjecture of Wu on the density of the proximal relation [1, p. 518].

Two other types of characterizations are given. It is shown that the weakly mixing minimal transformation groups are precisely those totally minimal transformation groups with a zero-dimensional structure transformation group. The other characterization is given in terms of disjointness (cf. [7]). In [7, Problem G], a description of the discrete flows disjoint from the distal flows is sought. It is shown in [17] that these are the weakly mixing minimal flows. Here we derive this result via our techniques for a large class of transformation groups. We then prove a theorem, suggested by R. Ellis, characterizing disjointness of minimal transformation groups in terms of their associated groups (Theorem 4.5). This enables us to extend the above results to larger classes than the distal transformation groups.

The last section is concerned with density problems of integer sets associated with weakly mixing minimal discrete flows. In [7], the problem of whether a product of a weakly mixing flow with an ergodic flow is ergodic was raised. In general the answer is no [14]. A certain class of weakly mixing minimal flows is distinguished (Theorem 5.3) which has the property that its product with an ergodic flow on a compact metric supporting an invariant Borel probability on the whole space is also ergodic. This result applies to a larger class than the minimal flows.

Unless otherwise stated, we will consider transformation groups (Z, T, ρ), where Z is compact Hausdorff and T is abelian. We recall that (Z, T, ρ) is *ergodic* (i.e., $(Z, T, \rho) \in \mathcal{E}$) if every proper closed invariant set is nowhere

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dense. When Z is metric, this is equivalent to having a point with dense orbit (*point-transitive*). Also, (Z, T, ρ) is *weakly mixing* (i.e., $(Z, T, \rho) \in \mathcal{W}$) if $(Z \times Z, T, \rho \times \rho) \in \mathcal{E}$. Since weakly mixing and minimality are independent of the topology on T, we assume unless otherwise stated that T is discrete. If T has a non-trivial topological group topology, then ρ is assumed continuous. We will frequently surpress ρ. Finally, we denote a surjective transformation group homomorphism φ from (Z, T) to another transformation group (W, T) by $\varphi : (Z, T) \to (W, T)$.

We shall assume familiarity with the algebraic theory of point-transitive transformation groups. See [1, pp. 165–184] or [3] for a general reference and for notation. We will use extensively the notion of a *quasi-separable* algebra: the T-subalgebra $\mathcal{A}$ is *quasi-separable* if there exists a subset $\mathcal{G}$ of $\mathcal{A}$ which separates points of $|\mathcal{A}|$ and such that the T-subalgebra $\mathcal{A}(f)$ generated by f is separable $(f \in \mathcal{G})$. This is a slight modification of the definition given in [5]. The point transitive transformation group (Z, T, ρ) is quasi-separable if an associated algebra $\mathcal{A}$ is quasi-separable.

A general reference for the notions and the notation used is [13].

2. Approximation by metric transformation groups

For completeness, we include the following lemma (cf. [5, Definition 3.12]).

2.1 LEMMA. *Let (Z, T, ρ) be point-transitive and $\mathcal{A}$ a T-subalgebra for which $Z = |\mathcal{A}|$. Suppose that T can be provided with a σ-compact topology for which ρ is continuous. Then (Z, T, ρ) is quasi-separable.*

Proof. Recall that $\mathcal{A}$ is isomorphic to $\mathcal{C}(|\mathcal{A}|)$, the continuous functions on $|\mathcal{A}|$, by $f \leftrightarrow \bar{f}$, where $\langle \bar{f}, \varphi \rangle = \langle \varphi(f), e \rangle$. It is direct to verify that $\overline{tf} = t\bar{f}$ $(f \in \mathcal{A}, t \in T)$.

Now fix $g \in \mathcal{A}$ and consider $\psi : T \to \mathcal{C}(|\mathcal{A}|)$ where $\psi(t) = \overline{tg}$ $(= t\bar{g})$. Now by [10, Definition 1.66], $(\mathcal{C}(|\mathcal{A}|), T)$ with action $(\bar{f}, t) \to t^{-1}\bar{f}$ is a transformation group, where T is provided with its σ-compact topology. Thus, $\psi(T) = \{t\bar{g} \mid t \in T\}$ is σ-compact and hence separable. But then $\{tg \mid t \in T\}$, which together with $\{1\}$ algebraically generates a dense subalgebra of $\mathcal{A}(g)$, is also separable. Thus $\mathcal{A}(g)$ is separable, and the result follows.

We note that 2.1 is applicable to a large class of transformation groups. For example, if the topology on T is connected, locally compact or separable, locally compact, then it must be σ-compact. Moreover we obtain as a corollary that if T has a separable topology for which ρ is continuous then (Z, T, ρ) is quasi-separable. For let S be a countable dense subset of T, and choose $g \in \mathcal{A}$. If $t \in T$ and $\varepsilon > 0$, then by [10, Remark 4.39], there exists a neighborhood N of t such that $|\bar{g}(zu) - \bar{g}(zv)| < \varepsilon$ $(z \in Z, v \in N)$. Hence, if $s \in S \cap N$, $|\bar{g}(zt) - \bar{g}(zs)| < \varepsilon$ $(z \in Z)$. Thus $\{sg \mid s \in S\}$, which is countable, is dense in $\{tg \mid t \in T\}$. It then follows that $\mathcal{A}(g)$ is separable.

Note that the commutativety of T was not used in the above argument. Hence 2.1 and the following theorem still hold in the case that T is nonabelian.

2.2 THEOREM. *Let (Z, T, ρ) be point-transitive. Then (Z, T, ρ) is quasi-separable* iff *there is an inverse system $((Z_\alpha, T, \rho_\alpha); \varphi_\alpha^\beta, \beta \geq \alpha)$ of point-transitive transformation groups with compact metric phase spaces such that (Z, T, ρ) is the inverse limit transformation group of $((Z_\alpha, T, \rho_\alpha))$.*

Proof. Suppose (Z, T, ρ) is quasi-separable. Let $\mathfrak{A}$ be a T-subalgebra such that $Z = |\mathfrak{A}|$, and $\mathcal{G} \subset \mathfrak{A}$ such that $\mathcal{G}$ separates points and $\mathfrak{A}(f)$ is separable $(f \in \mathcal{G})$. Let $\mathfrak{F}$ be the finite subsets of $\mathcal{G}$ directed by $\alpha \leq \beta$ iff $\alpha \subset \beta$ and $\mathfrak{A}(\alpha)$ the T-subalgebra generated by $\alpha (\alpha \in \mathfrak{F})$. Let $(Z_\alpha, T, \rho_\alpha) = (|\mathfrak{A}(\alpha)|, T, \rho_\alpha)$ $(\alpha \in \mathfrak{F})$. Since $\mathfrak{A}(f)$ is separable $(f \in \mathcal{G})$, it follows that $\mathfrak{A}(\alpha) \simeq \mathcal{C}(|\mathfrak{A}(\alpha)|)$ is separable, whence Z_α is compact metric $(\alpha \in \mathfrak{F})$. Moreover, $\alpha \leq \beta$ implies $\mathfrak{A}(\alpha) \subset \mathfrak{A}(\beta)$ and thus the restriction map induces

$$\varphi_\alpha^\beta : (|\mathfrak{A}(\beta)|, T, \rho_\beta) \to (|\mathfrak{A}(\alpha)|, T, \rho_\alpha).$$

It is direct to verify that $\varphi_\alpha^\alpha = \text{id}$ and $\varphi_\alpha^\beta \varphi_\beta^\gamma = \varphi_\alpha^\gamma$ if $\alpha \leq \beta \leq \gamma$. Moreover, since $\mathcal{G}$ separates points, then the T-subalgebra generated by $\bigcup \{\mathfrak{A}(\alpha) \mid \alpha \in \mathfrak{F}\}$ is dense in $\mathfrak{A}$. Thus, $(Z, T, \rho) = (|\mathfrak{A}|, T, \rho)$ is isomorphic to the inverse limit transformation group of $((Z_\alpha, T, \rho_\alpha))$, and this implication follows.

Conversely, suppose $(Z, T, \rho) = \text{invlim}_\alpha (Z_\alpha, T, \rho_\alpha)$ where Z_α is compact metric. Let $\mathfrak{A}, \mathfrak{A}_\alpha$ $(\alpha \in A)$ be T-subalgebras for which $|\mathfrak{A}| = Z$, $|\mathfrak{A}_\alpha| = Z_\alpha (\alpha \in A)$. Let $\varphi_\alpha : (|\mathfrak{A}|, T) \to (|\mathfrak{A}_\alpha|, T)$ $(\alpha \in A)$ be the cannonical homomorphisms. Then $\mathfrak{A}_\alpha p_\alpha \subset \mathfrak{A}$ for some $p_\alpha \in \beta T$ and

$$\varphi_\alpha(x) = p_\alpha x \; (x \in |\mathfrak{A}_\alpha|, \quad a \in A).$$

Note that we identify $z \in \beta T$ and $z \mid \mathfrak{A} \in |\mathfrak{A}|$ here.

Now suppose $x, y \in |\mathfrak{A}|$ and $x \neq y$. Then $\varphi_\alpha(x) \neq \varphi_\alpha(y)$, i.e. $p_\alpha x \mid \mathfrak{A}_\alpha \neq p_\alpha y \mid \mathfrak{A}_\alpha$ for some α. If $f \in \mathfrak{A}_\alpha$ and $t \in T$ such that $\langle f p_\alpha x, t\rangle \neq \langle f p_\alpha y, t\rangle$, then

$$\langle t^{-1}(f p_\alpha), x\rangle = \langle (t^{-1} f(p_\alpha, x\rangle \neq \langle (t^{-1} f) p_\alpha), y\rangle = \langle t^{-1}(f p_\alpha), y\rangle.$$

Since $t^{-1} f \in \mathfrak{A}_\alpha$, $\mathcal{G} = \bigcup \{\mathfrak{A}_\alpha p_\alpha \mid \alpha \in A\}$ separates points of $|\mathfrak{A}|$. Moreover, $(\mathfrak{A}_\alpha) p_\alpha = \mathfrak{A} p_\alpha$ implies that the induced map $\bar{p}_\alpha : |\mathfrak{A}_\alpha p_\alpha| \to |\mathfrak{A}_\alpha|$ is one-one. Then $|\mathfrak{A}_\alpha p_\alpha|$ is compact metric and $\mathfrak{A}_\alpha p_\alpha$ separable. Thus, if $f \in \mathfrak{A}_\alpha p_\alpha$, $\mathfrak{A}(f)$ is separable.

One should note that if $\mathfrak{A}$ is a T-subalgebra and $f \in \mathfrak{A}$, then $\mathfrak{A}(f)$ is separable iff there exists a coutable subset T_f of T such that the algebra generated by $\{tf \mid t \in T_f\}$ is $\mathfrak{A}(f)$.

For the rest of this paper, (X, T, π) will denote a quasi-separable point-transitive transformation group with X non-trivial. One application from the metric case to the quasi-separable case is:

2.3 PROPOSITION. *Let (X, T) be weakly mixing minimal. Then the proximal relation $P(X)$ is not an equivalence relation.*

Proof. By 2.2, there exists a transformation group (Z, T) with Z compact metric such that $(X, T) \to (Z, T)$. Thus, (Z, T) is weakly mixing minimal.

By [13, Proposition 3.1], $P(Z)$ is not an equivalence relation. This implies by [4, Lemma 7] that $P(X)$ is not an equivalence relation.

For our extensions, the following technical lemma is needed. If $A, B \subset X$, let $N(A, B) = \{t \mid At \cap B \neq \emptyset\}$. If $x \in X$, we will write $N(x, A)$ for $N(\{x\}, A)$.

2.4 LEMMA. *Let (X, T) be minimal. Then (X, T) is weakly mixing* iff *for every $x \in X$ and for all non-empty open subsets A, B, C of X, $T = N(A, B)N(x, C)$.*

Proof. ($\Leftarrow$) Let A, B, C, D be non-vacuous open subsets of X. Let $x \in B$. Then

$$T = N(C, A)N(x, D) = N(A, C)^{-1}N(x, D).$$

Thus, $e \in N(A, C)^{-1}N(x, D)$ and $N(A, C) \cap N(B, D) \neq \phi$. If

$$t \in N(A, C) \cap N(B, D),$$

$$(A \times B)t \cap (C \times D) \neq \phi$$

and the result follows.

($\Rightarrow$) Let A, B, C be non-empty, open subsets of X, and $x \in X$. In [14, Remark 2.8], it is shown that $N(A, B)$ is discretely replete. Since (X, T) is minimal, $N(x, C)$ is discretely syndetic and thus $T = N(A, B)N(x, C)$.

2.5 LEMMA. *Let (X, T) be an inverse limit of weakly mixing minimal transformation groups. Then (X, T) is weakly mixing minimal.*

Proof. Suppose (X, T) is the inverse limit of $((X_\alpha, T); \varphi_\alpha^\beta, \beta \geq \alpha)$, where (X_α, T) is weakly mixing minimal. Then the minimality of (X, T) can be directly verified. Now let A, B, C be non-empty open subsets of X and $x \in X$. Then for some α, there exists A_α, B_α, C_α non-empty open in X_α for which

$$\overset{-1}{\varphi_\alpha}(A_\alpha) \subset A, \quad \overset{-1}{\varphi_\alpha}(B_\alpha) \subset B \quad \text{and} \quad \overset{-1}{\varphi_\alpha}(C_\alpha) \subset C,$$

where $\varphi_\alpha : (X, T) \to (X_\alpha, T)$ is the induced homomorphism. Let $x_\alpha = \varphi_\alpha(x)$. Then

$$N(A_\alpha, B_\alpha) \subset N(A, B) \quad \text{and} \quad N(x_\alpha, C_\alpha) \subset N(x, C).$$

Since $T = N(A_\alpha, B_\alpha)N(x_\alpha, C_\alpha)$ by 2.4, then $T = N(A, B)N(x, C)$. Thus, (X, T) is weakly mixing.

For the definitions of $Q(X)$ and $\Gamma(X)$, see [6].

2.6 THEOREM. *Let (X, T) be minimal. Then the following are equivalent:*

(1) *(X, T) is weakly mixing.*

(2) *The regional proximal relation $Q(X) = X \times X$.*

(3) *The structure group $\Gamma(X)$ is trivial.*

Proof. That (1) implies (2) follows directly from the ergodicity of $(X \times X, T)$, and (2) implies (3) is obvious. Suppose (3) holds. Then the equicontinuous structure relation $S(E) = X \times X$. Also, (X, T) is isomorphic to $\text{invlim}_\alpha(X_\alpha, T)$, where X_α compact metric, by 2.2. Now for every α, $S(E_\alpha) = X_\alpha \times X_\alpha$, whence (X_α, T) is weakly mixing minimal by [13, Theorem 3.4]. The result follows by 2.5.

2.7 COROLLARY. *Let (X, T) be minimal. Then the following are equivalent:*

(1) *(X, T) is weakly mixing.*
(2) *$P(X)$ is dense.*
(3) *The distal structure relation $S(D) = X \times X$.*

Proof. Since (2) implies (3) is obvious and (3) implies (1) follows by 2.6, it remains to show (1) implies (2). Suppose (X, T) is weakly mixing, and $(X, T) = \text{invlim}_\alpha(X_\alpha, T)$, where X_α compact metric. For every α, let $\varphi_\alpha : (X, T) \to (X_\alpha, T)$ be the induced homomorphism. Then

$$\varphi_\alpha \times \varphi_\alpha(P(X)) = P(X_\alpha)$$

by [12, Theorem 3.3] and (X_α, T) weakly mixing minimal implies $P(X_\alpha)$ is dense by [13, Remark 3.2]. Let U, V be non-empty, open subsets of $X \times X$. Then $\overset{-1}{\varphi_\alpha}(U_\alpha) \subset$ and $\overset{-1}{\varphi_\alpha}(V_\alpha) \subset V$, where U_α, V_α are non-empty open subsets of X_α for some α. If $(x_\alpha, y_\alpha) \in P(X_\alpha)$ is such that $(x_\alpha, y_\alpha) \in U_\alpha \times V_\alpha$ and $(x, y) \in P(X)$ satisfies $\varphi_\alpha \times \varphi_\alpha(x, y) = (x_\alpha, y_\alpha)$, then $(x, y) \in U \times V$. Thus, $P(X)$ is dense and the result follows.

Notice that 2.7 answers Wu's conjecture [1, p. 518].

We now give the extension of the eigenvalue theorem [13, Corollary 2.11].

2.8 THEOREM. *Let (X, T) be minimal. Then (X, T) is weakly mixing iff the only continuous eigenfunctions are the constants.*

Proof. Suppose (X, T) is not weakly mixing. Since

$$(X, T) \simeq \text{invlim}_\alpha(X_\alpha, T),$$

where X_α compact metric, it follows by 2.5 that (X_α, T) is not weakly mixing for some α. By [13, Corollary 2.11], there exists a non-constant continuous eigenfunction f_α for (X_α, T). If $\varphi_\alpha : (X, T) \to (X_\alpha, T)$ is the cannonical homomorphism then $f = f_\alpha \varphi_\alpha$ is a non-constant continuous eigenfunction for (X, T). Since the other way holds by [13, Theorem 2.3], the result follows.

As a consequence, if X is connected, (X, T, π) is minimal and $H^1(X) = 0$ in Cěch cohomology, then (X, T, π) is weakly mixing ([19, Proposition 11] guarantees a continuous logarithm in this case).

3. Total minimality and weakly mixing

We now characterize weakly mixing in terms of total minimality. In this section, we do *not* necessarily assume that T is discrete but simply that (X, T) is point-transitive, quasi-separable.

For the definition of total minimality, see [9, Definition 1.02].

3.1 THEOREM. *The following are equivalent:*

(1) (X, T) *is weakly mixing minimal.*

(2) (X, T) *is totally minimal and the structure space* $X/S(E)$ *is zero-dimensional.*

Proof. Suppose (1) holds. Since the regional proximal relation

$$Q(X) = \bigcap \{\overline{\alpha T} \mid \alpha \quad \text{an index of} \quad X\},$$

then (X, T) weakly mixing implies $\overline{\alpha T} = X \times X$ for every α. If Λ is the trace relation on X (cf. [9, Definition 1.02]), then $\Lambda \supset S(E)$ [9, Theorem 1.08]. Hence $\Lambda = X \times X$ and the result follows.

Now suppose (2) holds. Since $X/S(E)$ is zero-dimensional, then $(X/S(E), T)$ is regularly almost periodic by [9, Remark 1.13] and hence pointwise regularly almost periodic. Now $(X/S(E), T)$ has singleton traces by [9, Remark 1.10]. But (X, T) totally minimal implies $(X/S(E), T)$ is totally minimal. Thus $S(E) = X \times X$, and the structure group $\Gamma(X)$ is trivial. By 2.6, (X, T) is weakly mixing, where T is provided with the discrete topology. Since weakly mixing is independent of the topology on T, (X, T) is weakly mixing under the original topology on T. The result follows.

The implication (1) implies (2) is not new for discrete or continuous flows (see [17] or [15, Theorem 1.1]). Indeed, [15, Theorem 1.3] shows that for continuous minimal flows, weakly mixing and total minimality are equivalent. Moreover, since discrete or continuous minimal flows satisfy our standing hypothesis by 2.1, one may regard the assertion following 2.8 as a generalization of [2, Theorem, p. 377].

3.2 COROLLARY. *Suppose* (X, T) *is minimal distal and* X *is zero-dimensional. Then* (X, T) *is not totally minimal.*

Proof. Consider the canonical homomorphism $\varphi : (X, T) \sim (X/S(E), T)$. By [8, Theorem 8.1] (stated here for compact metric spaces but also valid for compact Hausdorff spaces), φ is open. Thus, $X/S(E)$ is zero-dimensional. If (X, T) were totally minimal, then (X, T) is weakly mixing by 3.1. But then by 2.3, (X, T) cannot be distal. Thus, (X, T) is not totally minimal.

3.3 PROPOSITION (*Inheritance*). *Suppose* (X, T) *is weakly mixing minimal and* S *is a closed syndetic subgroup of* T. *Then* (X, S) *is weakly mixing minimal.*

Proof. It follows by 3.1 that (X, S) is minimal. Moreover, (X, T) quasi-separable implies (X, S) is quasi-separable since the relevant algebras for (X, S) are contained in those for (X, T). Now it is easy to see that $P(X, T) = P(X, S)$, where obvious notation is used for the proximal relations. Then by 2.7, $P(X, T)$ is dense. Another application of 2.7 yields the desired result.

4. Disjointness and weakly mixing

Recall [7] that two transformation groups (Z, T) and (W, T) are disjoint if whenever A is a closed invariant subset of $Z \times W$ such that $\rho_1 A = Z$, $\rho_2 A = W$, where ρ_1, ρ_2 are the projections, then $A = Z \times W$. (This is a characterization of disjointness, cf. [7, Lemma II.1]). If (Z, T) and (W, T) are minimal, then this is equivalent to $(Z \times W, T)$ being minimal. Again, the topology of T is unessential in disjointness.

In this section, we impose the assumption on T (discrete for simplicity) that every point-transitive transformation group (Z, T) is quasi-separable. For brevity, we use the following notation (cf. [7]): $\mathcal{K}$ will denote the almost periodic minimal transformation groups (cf. [10, Theorem 4.48]) $\mathcal{A}_0$ the almost periodic transformation groups, $\mathcal{D}$ the distal transformation groups, $\mathcal{C}_0$ the transformation groups with proximal a closed equivalence relation, $\mathcal{M}$ the minimal transformation groups and $\mathcal{W}$ the weakly mixing transformation groups. All transformation groups have T as their phase group. If $\mathcal{F}$ is a class of transformation groups, then $\mathcal{F}^{\perp}$ will denote the transformation groups disjoint from $\mathcal{F}$.

We first show that $\mathcal{A}_0^{\perp} = \mathcal{D}^{\perp} = \mathcal{W} \cap \mathcal{M}$. This result has been shown in [17] for discrete flows with compact metric spaces.

4.1 Lemma. $\mathcal{A}_0^{\perp} \subset \mathcal{W} \cap \mathcal{M}$.

Proof. Let $(Y, T) \in \mathcal{K}^{\perp} \cap \mathcal{M}$ and suppose $(Y, T) \notin \mathcal{W}$. Then there exists a non-constant eigenfunction f for (Y, T) by 2.8. By [13, Lemma 3.3], we have $f : (Y, T) \to (Z, T)$, where Z is non-trivial and $(Z, T) \in \mathcal{K}$. But by [7, Proposition II.2], this is a contradiction, since $(Y, T) \in \mathcal{K}^{\perp}$. Thus, $(Y, T) \in \mathcal{W} \cap \mathcal{M}$.

Now suppose $(Y, T) \in \mathcal{A}_0^{\perp}$. Since there exist non-minimal transformation groups in $\mathcal{A}_0$, then $(Y, T) \in \mathcal{M}$ by [7, Theorem II.1]. Thus, $\mathcal{K} \subset \mathcal{A}_0$ implies

$$\mathcal{A}_0^{\perp} = \mathcal{A}_0^{\perp} \cap \mathcal{M} \subset \mathcal{K}^{\perp} \cap \mathcal{M} \subset \mathcal{W} \cap \mathcal{M}.$$

4.2 Lemma. $\mathcal{D} \cap \mathcal{M} \perp \mathcal{W} \cap \mathcal{M}$.

Proof. First let $(Y, T) \in \mathcal{D} \cap \mathcal{M}$, $(Z, T) \in \mathcal{W} \cap \mathcal{M}$ with Y, Z compact metric. Then the proof of [7, Theorem II.3], suitably modified for arbitrary transformation groups, shows that $(Y, T) \perp (Z, T)$.

We note here a short version of the fact that $(W, T) \in \mathcal{W} \cap \mathcal{M}$, $(W, T) \perp (V, T)$, (V_1, T) a minimal G-group extension of (V, T) [5, Definition 4.1] implies $(W, T) \perp (V_1, T)$. Note that $(G, W \times V_1, T)$ is a G-group extension of $(W \times V, T)$ in a canonical way:

$$(W \times V_1)/G \simeq W \times (V_1/G) \simeq W \times V.$$

Hence $(W \times V, T)$ minimal implies $(W \times V_1, T)$ is pointwise almost periodic as in [7, Proposition II.10]. But $(W \times V_1, T)$ has a transitive point [14, Corollary (2.9)] and hence is minimal.

We also note that if (W, T) is minimal, $(V, T) = \text{invlim}_\alpha(V_\alpha, T)$ is minimal and $(W, T) \perp (V_\alpha, T)$ for every α, then $(W, T) \perp (V, T)$. For the inverse system $((W \times V_\alpha, T); \text{id} \times \pi_\alpha^\beta)$ satisfies

$$(W \times V, T) = \text{invlim}_\alpha (W \times V_\alpha, T),$$

and the minimality of $(W \times V_\alpha, T)$ implies $(W \times V, T)$ is minimal.

Suppose now $(Y, T) \in \mathfrak{D} \cap \mathfrak{M}$, $(Z, T) \in \mathfrak{W} \cap \mathfrak{M}$ are arbitrary. Then $(Y, T) = \text{invlim}_\alpha(Y_\alpha, T)$, $(Z, T) = \text{invlim}_\beta(Z_\beta, T)$, where

$$(Y_\alpha, T) \in \mathfrak{D} \cap \mathfrak{M}, \qquad (Z_\beta, T) \in \mathfrak{W} \cap \mathfrak{M}$$

and Y_α, Z_β are compact metric. Then by the above comment, $(Y_\alpha, T) \perp (Z, T)$ for every α, and another application yields $(Y, T) \perp (Z, T)$. The result follows.

The following lemma was shown in [17].

4.3 LEMMA. *Suppose (Y, T) is minimal and (Z, T) satisfies $Z = U_\alpha Z_\alpha$, where (Z_α, T) is minimal. If $(X, T) \perp (Z_\alpha, T)$ for every α, then $(X, T) \perp (Z, T)$.*

4.4 THEOREM. $\mathfrak{A}_0^\perp = \mathfrak{D}^\perp = \mathfrak{W} \cap \mathfrak{M}$.

Proof. Let $(Y, T) \in \mathfrak{D}$. Then $Y = UY_\alpha$, where $(Y_\alpha, T) \in \mathfrak{D} \cap \mathfrak{M}$. By 4.2, $\mathfrak{W} \cap \mathfrak{M} \perp (Y_\alpha, T)$ for every α. Hence $\mathfrak{W} \cap \mathfrak{M} \perp (Y, T)$ by 4.3, whence $\mathfrak{W} \cap \mathfrak{M} \subset \mathfrak{D}^\perp$. Since clearly $\mathfrak{D}^\perp \subset \mathfrak{A}_0^\perp$ and $\mathfrak{A}_0^\perp \subset \mathfrak{W} \cap \mathfrak{M}$ by 4.1, the result follows.

The question arises from 4.4 as to just how large a class $\mathfrak{F} \supset \mathfrak{A}_0$ will satisfy $\mathfrak{F}^\perp = \mathfrak{W} \cap \mathfrak{M}$. In general, the decomposition of a transformation group into minimal transformation groups will not occur. To circumvent this problem, we will only consider classes of minimal transformation groups. The following result, suggested by R. Ellis, shows that disjointness in these classes can be easily described in terms of the groups associated with the minimal sets. For this analysis, we recall the following (cf. [5]). Let M be a minimal subset of βT, and u an indempotent of M. Every minimal T-subalgebra $\mathfrak{A}$ can be regarded as a subset of $\mathfrak{A}(u) = \{f \mid f \in \mathfrak{C}, fu = f\}$. Moreover, if $\varphi \in |\mathfrak{A}|$, then $\varphi = p \mid \mathfrak{A}$ for some $p \in M$. We write p for $p \mid \mathfrak{A}$ when no confusion arises.

In the following theorem, only the commutativity of T is needed; the assumption on quasi-separability is not needed.

4.5 THEOREM. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be minimal T-subalgebras contained in $\mathfrak{A}(u)$. Let $A = \mathfrak{G}(\mathfrak{A})$, $B = \mathfrak{G}(\mathfrak{B})$ be the assotiated groups. Then $(|\mathfrak{A}| \times |\mathfrak{B}|, T)$ is minimal* iff $AB = BA = G$.

Proof. Assume $AB = BA = G$. Consider the set

$$X = \{(p \mid \mathfrak{A}, q \mid \mathfrak{B}) \mid p, q \in M, \text{ and } pu = p, qu = q\}.$$

We claim X is dense in $|\mathfrak{A}| \times |\mathfrak{B}|$. For $u \in |\mathfrak{A}|$, $u \in |\mathfrak{B}|$ and by commutativity $utu = uut = u^2t = ut$ for every $t \in T$. Since $uT\ [= (u \mid \mathfrak{A})T]$ is dense in $|\mathfrak{A}|$, uT is dense in $|\mathfrak{B}|$ and $(uT) \times (uT) \subset X$, this shows our assertion.

Now suppose $(p \mid \mathfrak{A},\ q \mid \mathfrak{B}) \in X$. Since $pu = p$ and $qu = q$, then $p \mid \mathfrak{A} = pu \mid \mathfrak{A}$, $q \mid \mathfrak{B} = qu \mid \mathfrak{B}$. Since pu, $qu \in G = Mu$, this means that

$$X \subset \{(r \mid \mathfrak{A},\quad s \mid \mathfrak{B}) \mid r,\quad s \in G\} = Y.$$

Finally, consider $(u, u) \in |\mathfrak{A}| \times |\mathfrak{B}|$. Since $(u, u)u = (u^2, u^2) = (u, u)$, then (u, u) is an almost periodic point of $(|\mathfrak{A}| \times |\mathfrak{B}|, T)$. The assertion will be shown if $\overline{(u, u)T} \supset Y$, for then $\overline{(u, u)T} = |\mathfrak{A}| \times |\mathfrak{B}|$ by the above. So let $(r \mid \mathfrak{A}, s \mid \mathfrak{B}) \in Y$. Choose $w \in G$ for which $wr = s$. Since $G = BA$, then $w = ba$ for some $a \in A$, $b \in B$. Let $c = ar = b^{-1}s \in G$. Now

$$(u, u)c = (uc, uc) \in \overline{(u, u)T},$$

and $uc \mid \mathfrak{A} = uar \mid \mathfrak{A} = r \mid \mathfrak{A}$ since $ua \in A$, $uc \mid \mathfrak{B} = ub^{-1}s \mid \mathfrak{B} = s \mid \mathfrak{B}$ since $ub^{-1} \in B$. Thus

$$(r \mid \mathfrak{A}, s \mid \mathfrak{B}) = (uc \mid \mathfrak{A}, uc \mid \beta)$$

and the result follows.

Now suppose $(|\mathfrak{A}| \times |\mathfrak{B}|, T)$ is minimal and let $r \in G$. Then

$$(u \mid \mathfrak{A}, r \mid \mathfrak{B}) \in Y \quad \text{and} \quad \overline{(u, u)T} \supset Y.$$

Since M is minimal, $up \mid \mathfrak{A} = u \mid \mathfrak{A}$, $up \mid \mathfrak{B} = r \mid \mathfrak{B}$ for some $p \in M$. Now $q = pu \in G$ and $uq \mid \mathfrak{A} = u \mid \mathfrak{A}$, $uq \mid \mathfrak{B} = ru \mid \mathfrak{B}$. If $f \in \mathfrak{A}$, then $fq = fuq = fu = f$, while if $g \in \mathfrak{B}$, then $gq = guq = gru$, whence $g = gruq^{-1} = grq^{-1}$ (since $ru = r$). Thus, $q \in A$, $rq^{-1} \in B$ and $r = rq^{-1}q \in BA$. This shows $G = BA$. Similarly $G = AB$. The proof is completed.

One immediate corollary of 4.5 is the following extension of 4.2.

4.6 Corollary. $\mathcal{C}_0 \cap \mathfrak{M} \perp \mathcal{W} \cap \mathfrak{M}$.

Proof. Let (Z, T) be the universal minimal distal transformation group. Then the subgroup of G associated with the T-subalgebra of $\mathfrak{A}(u)$ corresponding to (Z, T) is denoted by D in [1, pp. 165–184]. Let $(Y, T) \in \mathcal{W} \cap \mathfrak{M}$. Since $(Y, T) \perp (Z, T)$ by 4.2, then $AD = DA = G$ by 4.5, where $A = \mathfrak{G}(\mathfrak{A})$, and $\mathfrak{A}$ is a T-subalgebra of $\mathfrak{A}(u)$ for which $(Y, T) \simeq (|\mathfrak{A}|, T)$. Suppose $(W, T) \in \mathcal{C}_0 \cap \mathfrak{M}$, $\mathfrak{B}$ is a T-subalgebra of $\mathfrak{A}(u)$ corresponding to (W, T) and $B = \mathfrak{G}(\mathfrak{B})$. Since proximal is closed, some results of Horelick shows that $B \supset D$ [11, Theorem 4.7 and 4.10]. Then $AB = BA = G$ and $(Y, T) \perp (W, T)$ by 4.5.

An immediate consequence of 4.6 is that $(\mathcal{C}_0 \cap \mathfrak{M})^{\perp} \cap \mathfrak{M} = \mathcal{W} \cap \mathfrak{M}$. For by 4.6, $(\mathcal{C}_0 \cap \mathfrak{M})^{\perp} \cap \mathfrak{M} \supset \mathcal{W} \cap \mathfrak{M}$, while

$$(\mathcal{C}_0 \cap \mathfrak{M})^{\perp} \cap \mathfrak{M} \subset \mathfrak{D}^{\perp} = \mathcal{W} \cap \mathfrak{M}$$

by 4.4.

An equivalent characterization of 2.6 is that $(|\mathfrak{A}|, T)$ is weakly mixing minimal iff $AE = EA = G$, where $E \subset G$ is the group for the universal minimal equicontinuous algebra in $\mathfrak{A}(u)$. For by 4.4, $AE = EA = G$ when $(|\mathfrak{A}|, T)$ is weakly mixing minimal. Conversely, if $(|\mathfrak{B}|, T) \in \mathfrak{K}$ and $\mathfrak{B} \subset \mathfrak{A}(u)$, then $\mathfrak{G}(\mathfrak{B}) = B \supset E$ [1, p. 177] and $AB = BA = G$. The result follows by 4.5, and 4.4.

An interesting problem is to describe $(\mathfrak{W} \cap \mathfrak{M})^{\perp}$. By 4.5 and 4.6,

$$(\mathfrak{W} \cap \mathfrak{M})^{\perp} \supset \mathfrak{D} \cup (\mathfrak{C}_0 \cap \mathfrak{M}).$$

Notice that $(\mathfrak{W} \cap \mathfrak{M})^{\perp\perp} = \mathfrak{W} \cap \mathfrak{M}$. For if $(Y, T) \in (\mathfrak{W} \cap \mathfrak{M})^{\perp\perp}$, then (Y, T) has no non-trivial factors in $\mathfrak{K}$, since $\mathfrak{K} \subset (\mathfrak{W} \cap \mathfrak{M})^{\perp}$ by 4.1. Hence $(Y, T) \in \mathfrak{W} \cap \mathfrak{M}$ by 2.6. Since obviously $(\mathfrak{W} \cap \mathfrak{M})^{\perp\perp} \supset \mathfrak{W} \cap \mathfrak{M}$, the result follows.

Continuing in this direction, we have as another application of 4.6 the following result.

4.7 THEOREM. *Let $(X, T) \in \mathfrak{M}$, $(Y, T) \in \mathfrak{K}$. Then $(X, T) \perp (Y, T)$ iff (X, T) and (Y, T) have no non-trivial common factors.*

Proof. Suppose (X, T) and (Y, T) have no non-trivial common factors. If $\mathfrak{A}$, $\mathfrak{B}$ are T-subalgebras of $\mathfrak{A}(u)$ for which $|\mathfrak{A}| = X$, $|\mathfrak{B}| = Y$, we need only show $AB = BA = G$, where $A = \mathfrak{G}(\mathfrak{A})$, $B = \mathfrak{G}(\mathfrak{B})$, by 4.5. Since $(|\mathfrak{B}|, T)$ is equicontinuous, $(|\mathfrak{B}|, T)$ is regular and B is a normal subgroup of G for which G/B is a compact Hausdorff topological group [1, p. 177], Since $\mathfrak{A}$ and $\mathfrak{B}$ have no non-trivial common factors, $\mathfrak{A} \cap \mathfrak{B} = \mathbf{R}$. Moreover, $\mathfrak{B} \geq \mathbf{R}$ since $\mathfrak{B}$ is distal [5]. Then

$$G = \mathfrak{G}(\mathbf{R}) = \mathfrak{G}(\mathfrak{A} \cap \mathfrak{B}) = \{\mathfrak{G}(\mathfrak{A}) \cup \mathfrak{G}(\mathfrak{B})\} = \{A \cup B\},$$

where $\{\cdots\}$ means the least closed generated subgroup [1, Proposition, p. 178]. Since the cannonical map $\pi : G \to G/B$ is closed, $\pi^{-1}\pi(A) = AB$ is a closed subgroup of G. Hence $AB = \{A \cup B\} = G$ and by normality of B, $BA = G$. The other way follows by [7, Proposition II.2].

Recall that (X, T) is *proximally equicontinuous* or *coterminous* if $P(X)$ is a closed equivalence relation and $(X/P(X), T)$ is equicontinuous.

4.8 COROLLARY. *Suppose $(X, T) \in \mathfrak{M}$, and $(Y, T) \in \mathfrak{M}$ with (Y, T) proximally equicontinuous. Then $(X, T) \perp (Y, T)$ iff (X, T) and (Y, T) have no non-trivial common factors.*

Proof. Again, it is sufficient to show $AB = BA = G$, where $A = \mathfrak{G}(\mathfrak{A})$, $B = \mathfrak{G}(\mathfrak{B})$, $\mathfrak{A}$, $\mathfrak{B}$ T-subalgebras in $\mathfrak{A}(u)$ for which $|\mathfrak{A}| = X$, $|\mathfrak{B}| = Y$. Now $(X, T) \in \mathfrak{M}$, $(Y/P(Y), T) \in \mathfrak{K}$ and by assumption have no non-trivial common factors. If $|\mathfrak{B}_0| = Y/P(Y)$ with $\mathfrak{B}_0$ a T-subalgebra in $\mathfrak{A}(u)$, then $\mathfrak{B}_0$ is similar to $\mathfrak{B}$ by [11, Theorem 3.14]. This means that $B = \mathfrak{G}(\mathfrak{B}_0)$. Hence $AB = BA = G$ by 4.7.

It is not known whether we can replace $\mathcal{H}$ by $\mathfrak{D} \cap \mathfrak{M}$ in 4.7. However, we can relax the condition in 4.8 in the presence of regularity. In [1, Theorem 2.1, p. 311], Knapp showed that if $(X, T) \in \mathfrak{M}$, $(Y, T) \in \mathfrak{D} \cap \mathfrak{M}$ with (Y, T) regular, then 4.7 still holds. We use this to show

4.9 PROPOSITION. *Suppose* $(X, T) \in \mathfrak{M}$, $(Y, T) \in \mathcal{C}_0 \cap \mathfrak{M}$ *with* (Y, T) *regular. Then* $(X, T) \perp (Y, T)$ iff (X, T) *and* (Y, T) *have no non-trivial common factor.*

Proof. Using notation as in 4.8, we have that $B = \mathfrak{G}(\mathcal{B}) = \mathfrak{G}(\mathcal{B}_0)$ is normal in G, since $\mathcal{B}$ is regular. Hence, since $\mathcal{B}_0$ is distal and B is normal, $\mathcal{B}_0$ is regular [11, Corollary 5.16]. Now, if (X, T) and (Y, T) have no non-trivial common factors, the same is true for (X, T) and $(Y/P(Y), T)$. By Knapp's result, $AB = BA = G$. The result follows.

5. Density and weakly mixing

In [14], it was shown in general that the product of a weakly mixing discrete flow and an ergodic discrete flow is not necessarily ergodic. In this section, we consider this problem for weakly mixing minimal discrete flows.

The problem of when the product of such a transformation group with another is weakly mixing is easily answered for arbitrary transformation groups.

5.1 LEMMA. *Let* (Y, T) *be weakly mixing minimal and* T *abelian. Let* (Z, T) *be a transformation group. Then* $(Y \times Z, T)$ *is weakly mixing* iff (Z, T) *is weakly mixing.*

Proof. A basic open set for $(Y \times Z) \times (Y \times Z)$ is $(A \times B) \times (C \times D)$ with A, C open in Y, B, D open in Z. Using the notation of 2.4, then

$$N((A \times B) \times (C \times D), (E \times F) \times (G \times H)) = N(A, E) \cap N(B, F) \cap N(C, G) \cap N(D, H)$$

where all sets involved are open. If (Z, T) is weakly mixing, then there exists L, K open in Y, P, R open in Z such that

$$N(L, K) \subset N(A, E) \cap N(C, G),$$
$$N(P, R) \subset N(B, F) \cap N(D, H).$$

Then $N(L, K)$ is discretely syndectic and thus $N(L, K) \cap N(P, R) \neq \emptyset$ as in [14, Remark 2.8], Thus, $(Y \times Z, T)$ is weakly mixing. The converse is obvious.

Let C be the unit circle, and χ a (discrete) character for T. Then (C, T) with action $(c, t) \to c \cdot \chi(t)$ is a transformation group and if $C_\chi = \overline{O(1)}$, the orbit closure of 1, then (C_χ, T) is equicontinuous minimal.

The next result characterizes weakly mixing by ergodicity of certain products.

5.2 LEMMA. *Let (X, T) be minimal. Then (X, T) is weakly mixing iff $(X \times C_\chi, T)$ is ergodic for every discrete character χ of T.*

Proof. Suppose $(X \times C_\chi, T)$ is ergodic for every discrete character χ of T. Let f be a non-zero continuous eigenfunction for (X, T) with $f : X \to C$. Then $f\pi^t = \chi(t)f$ $(t \in T)$ for some discrete character χ. Define g on $X \times C_\chi$ by $g(x, c) = f(x)c$. Then

$$g(xt, c\cdot\overline{\chi(t)}) = f(xt)\cdot c\cdot\overline{\chi(t)} = f(x)\cdot\chi(t)\cdot c\cdot\overline{\chi(t)} = f(x)c = g(x, c),$$

and g is a continuous invariant function for $(X \times C_\chi, T)$. Hence g is a constant by [13, Theorem 2.2]. This implies that f is a constant, whence (X, T) is weakly mixing by 2.8. The converse follows from [14, Corollary 2.7], since (C_χ, T) is minimal.

For the rest of this section, we let T be the integers and consider the discrete flow (X, φ). Using the above, if $\lambda \in C$, then C_λ is the orbit closure of 1 under the rotation by λ.

We will also use the following notation: if R is a set of integers, then R^+ will denote the non-negative integers in R and R^- the negative integers. In addition, $\bar{D}(S)$, $\underline{D}(S)$ and $D(S)$ will denote the upper density, lower density, and density of a set S of non-negative integers.

5.3 THEOREM. *Suppose (X, φ) is minimal and satisfies the property*

$(*)$ *for all non-empty open subsets A, B of X, $\bar{D}(N^+(A, B)) = 1$.*

Then (X, φ) is weakly mixing.

Proof. Suppose (X, φ) is not weakly mixing. Then there exists a non-constant continuous eigenfunction f. We can assume $f : X \to C$, the unit circle. If $f\varphi = \lambda f$, then $f : (X, \varphi) \sim (C_\lambda, T_\lambda)$. Now it is a simple exercise to show that the homomorphic image of a flow with the property $(*)$ also has the same property. Thus, (C_λ, T_λ) satisfies $(*)$. But the equicontinuity of (C_λ, T_λ) shows that $\operatorname{diam}(\varphi^n W)$ is small for every $n \geq 0$ if $\operatorname{diam}(W)$ is sufficiently small. Thus, if U, V are non-empty open sets sufficiently far apart and U is sufficiently small, then $\varphi^n U \cap U \neq \emptyset$ implies $\varphi^n U \cap V = \emptyset$. Since

$$\{n \mid n \geq 1 \quad \text{and} \quad \varphi^n U \cap U \neq \emptyset\}$$

is syndetic, then $\bar{D}(N^+(U, V)) < 1$. From this contradiction, it follows the (X, φ) is weakly mixing.

The author conjectures that a weakly mixing minimal discrete flow must satisfy property $(*)$.

If one replace $\bar{D}N^+(A, B)$ by $\bar{D}N^-(A, B)$ in $(*)$, one can prove the same result. Moreover, one can replace the one-sided density in 5.3 by bilateral density $\bar{D}_b\, N(A, B)$, which is the natural concept in a flow.

We have as an immediate corollary.

5.4 COROLLARY. *Suppose* (X, φ) *is minimal and satisfies* $(*)$ *of* 5.3. *Let* (Y, ψ) *be such that* $\underline{D}N^{+}(C, D) > 0$ *for all non-empty open subsets* C, D *of* Y. *Then* $(X \times Y, \varphi \times \psi)$ *is ergodic.*

Note that the condition $\underline{D}N^{+}(C, D) > 0$ implies ergodicity and is implied by minimality.

We now give conditions on (Y, ψ) which guarantee the hypothesis of 5.4.

5.5 LEMMA. *Let* $(a_i)_{i \geq 0}$ *be a sequence of real numbers such that*

$$\frac{1}{n+1} \sum_{0}^{n} a_i \rightarrow a > 0$$

and $1 \geq a_i \geq 0$ *for every* i. *Then* $\underline{D}(\{i \mid a_i > 0\}) > 0$.

Proof. Suppose $\underline{D}(\{i \mid a_i > 0\}) = \underline{D}(A) = 0$. Then there exists a sequence (m_n) for which

$$\frac{1}{m_n + 1} \sum_{0}^{m_n} \chi_A(i) \rightarrow 0,$$

where χ_A is the characteristic function for A. Now $\chi_A(i) \geq a_i$ for every i, since $i \notin A$ implies $a_i = 0$. Then

$$\frac{1}{m_n + 1} \sum_{0}^{m_n} a_i \rightarrow 0,$$

which is a contradiction.

5.6 THEOREM. *Suppose* (Y, ψ) *is ergodic with* Y *compact metric, and there exists an invariant Borel probability* μ *for which* supp $\mu = Y$. *Then* $\underline{D}N^{+}(C, D) > 0$ *for all non-empty open subsets* C, D *of* Y.

Proof. First note that if A is open and $\mu(A) > 0$, then $\nu(A) > 0$ for some ergodic probability ν, since $\mu(A) = \int \nu(A)\, dm(\nu)$, where m is a measure on the ergodic measures (cf. [18, p. 82]).

Now let C, D be non-empty open subsets of Y. Then by ergodicity, $\psi^n C \cap D \neq \emptyset$ for some integer n. Since by hypothesis $\mu(\psi^n C \cap D) > 0$, then $\nu(\psi^n C \cap D) > 0$ for some ergodic ν. Thus, $\nu(\psi^n C) = \nu(C) > 0$ and $\nu(D) > 0$. Since ν is ergodic,

$$\frac{1}{n} \sum_{0}^{n-1} \nu(\psi^{-i} D \cap C) \rightarrow \nu(D)\nu(C) > 0.$$

By 5.5 $\underline{D}(\{n \mid \nu(\psi^{-n} D \cap C) > 0\}) > 0$ and thus

$$\underline{D}(\{n \mid n \geq 0 \quad \text{and} \quad \psi^{-n} D \cap C \neq 0\}) > 0.$$

Since $\psi^{-n}D \cap C \neq \emptyset$ iff $\psi^{n}C \cap D \neq \emptyset$, the result follows.

It would be interesting to know if the existence of μ guarantees the existence of an ergodic probability ν for which supp $\nu = Y$.

The existence of such a μ is easily describe in terms of the Kryloff-Bogliouboff theory (cf. [16])

5.7 Lemma. *Let* (Y, ψ) *be a flow with* Y *compact metric. Then there exists an invariant Borel probability* μ *for which* supp $\mu = Y$ iff *the points of density* Q_D *are dense in* Y.

Proof. Suppose Q_D is dense. If $\{U_n \mid n \geq 1\}$ is a countable base for Y, then for every n, there exists $p_n \in Q_D$ such that $p_n \in U_n$. If μ_n is the probability induced by p_n, then $\mu_n(U_n) > 0$. Thus, $\mu = \sum_1^\infty \mu_n/2^n$ is the desired probability. On the other hand, since every set of measure 1 for a measure supported on the whole space must be dense and since Q_D is a Borel set of invariant measure 1 [16, p. 119], the result follows.

5.8 Corollary. *Suppose* (Y, ψ) *is a flow with* Y *compact metric such that the periodic points are dense. Then there exists an invariant Borel probability* μ *for which* supp $\mu = X$.

Proof. By 5.7, it is sufficient to show that each periodic point is a point of density. Let p be a periodic point with period n. Then it is direct to verify that p is a mean point and the measure μ_p is the ergodic measure giving weight $1/n$ to each point in $\mathcal{O}(p)$. Thus, p is a point of density, and the result follows.

Using 5.8, we can describe large classes of non-minimal flows for which 5.6 holds. One class is the $\mathfrak{F}$-flows of Furstenberg [7, p. 27], which includes the symbolic flows. Another class which can be seen to satisfy 5.8 (cf. [20, p. 759]) are Anosov diffeomorphisms of a compact connected manifold which yield a topologically ergodic flow. Finally, if φ is a diffeomorphism of a compact manifold M satisfying [20, Axiom A, p. 777], then certain ergodic subflows of the non-wandering set (cf. [20, Theorem 6.2]) will satisfy 5.8.

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