NON-ARCHIMEDIAN ANALYTIC FUNCTIONS TAKING THE SAME VALUES AT THE SAME POINTS

BY

W. W. ADAMS AND E. G. STRAUS[1]

A well known theorem of R. Nevanlinna (see e.g. [2, Theorem 2.6]) states that two nonconstant meromorphic functions of a complex variable which attain five distinct values at the same points must be identical. The functions e^z and e^{-z} and the values 0, ∞, 1, -1 show that four distinct values are not enough even when the multiplicities are the same. On the other hand, *two nonconstant polynomials f, g over an algebraically closed field of characteristic zero are identical if for two distinct (finite) values a, b we have $f(x) = a \Leftrightarrow g(x) = a$ and $f(x) = b \Leftrightarrow g(x) = b$.* To see this let the two polynomials be f and g and assume (without loss of generality) that the two values are 0 and 1. So $f(x) = 0 \Leftrightarrow g(x) = 0$ and $f(x) = 1 \Leftrightarrow g(x) = 1$. Suppose that $n = \deg f \geq \deg g > 0$. Now f divides $f'(f - g)$ since every zero of f is a zero of $f - g$. Also $f - 1$ divides $f'(f - g)$, and thus, since f and $f - 1$ are relatively prime, $f(f - 1)$ divides $f'(f - g)$. But $\deg f(f - 1) = 2n$ and $\deg f'(f - g) \leq 2n - 1$, and so $f - g \equiv 0$.

This result does not remain valid if the condition that the field is of characteristic zero is dropped. For example, x and x^q attain each value in $GF(q)$ at the same unique point.

Entire functions over a (complete, algebraically closed) non-Archimedian field of characteristic zero behave, in many ways, more like polynomials than like entire functions of a complex variable. This is also true in connection with the problem under discussion here. In the following it is assumed that the non-Archimedian variable ranges over a complete algebraically closed field of characteristic zero.

THEOREM 1. *Let f, g be two nonconstant entire functions of a non-Archimedian variable, so that for two distinct (finite) values a, b we have $f(x) = a \Leftrightarrow g(x) = a$ and $f(x) = b \Leftrightarrow g(x) = b$. Then $f \equiv g$.*

It may be desirable to define our concepts of analyticity to the extent that they are used here. By an *analytic function* we mean the values of a convergent Laurent series in some domain. A *meromorphic function* is the ratio of two analytic functions. A function analytic in a punctured neighborhood of a point has an *essential singularity* at that point if the Laurent series about the point contains an infinite number of negative powers. For further reference

see, for example, the appendix of [1]. The facts about analytic functions that we need are summarized in the following lemma.

LEMMA. *Let*

$$f(x) = \sum_{n=-\infty}^{\infty} c_n x^n$$

be an analytic function of a non-Archimedian variable in the domain

$$R_0 < (=) \, |x| < (=) \, R_1$$

where $0 \leq R_0 < R_1 \leq \infty$, and define

$$M_f(r) = \sup_{|x|=r} |f(x)|, R_0 < (=) \, r < (=) \, R_1.$$

Then

(i) *$M_f(r) = \max_n |c_n| \, r^n$ and hence the maximum-modulus principle holds and $M_f(r)$ is continuous;*

(ii) *the maximum on the right of (i) is attained for a unique value of n except for a discrete sequence of values $\{r_\nu\}$ in the open interval (R_0, R_1);*

(iii) *if $r \notin \{r_\nu\}$ and $R_0 < |x| = r < R_1$ then $|f(x)| = M_f(r)$;*

(iv) *if f is a nonconstant entire function then $M_f(r) \to \infty$ as $r \to \infty$;*

(v) *for two analytic functions f, g,*

$$M_{fg}(r) = M_f(r) M_g(r);$$

(vi) *$M_{f'}(r) \leq M_f(r)/r$ $(r > 0)$.*

Proof. To prove (i) suppose that $n_1 < n_2 < \cdots < n_k$ are all the values of n such that

$$\max_n |c_n| \, r^n = |c_{n_i}| \, r^{n_i} \qquad (i = 1, \cdots, k).$$

Then if

(1) $\sup_{|x|=r} |c_{n_1} x^{n_1} + \cdots + c_{n_k} x^{n_k}| = |c_{n_1}| \, r^{n_1}$

we would have by the ultrametric inequality

$$\max_n |c_n| \, r^n = \sup_{|x|=r} |c_{n_1} x^{n_1} + \cdots + c_{n_k} x^{n_k}|$$
$$= \sup_{|x|=r} \left| \sum_{n=-\infty}^{\infty} c_n x^n \right| = M_f(r)$$

as desired. So we must show (1). If $k = 1$ the result is clear, so assume $k > 1$. Write

$$c_{n_1} x^{n_1} + \cdots + c_{n_k} x^{n_k} = c_{n_1} x^{n_1} P(x)$$

where $P(x)$ is a polynomial of degree $\nu = n_k - n_1$ and we must show that

$$\max_{|x|=r} |P(x)| = 1.$$

If $|x| < r$ then $P(x)$ is dominated by its constant term and if $|x| > r$ then $P(x)$ is dominated by its leading term. Thus $|P(x)| \geq 1$ whenever $|x| \neq r$. So every root ξ of $P(x)$ satisfies $|\xi| = r$. Write

$$P(x) = (1 - x/\xi_1) \cdots (1 - x/\xi_\nu)$$

where $|\xi_i| = r$ $(i = 1, \cdots, \nu)$. If $|\alpha| \leq 1$ denote by $\bar{\alpha}$ the image of α in the residue class field. Then since the residue class field is infinite there is an x_0 such that

$$|x_0| = r \quad \text{and} \quad (\overline{x_0/\xi_1}) \neq (\overline{\xi_i/\xi_1}) \qquad (i = 1, \cdots, \nu).$$

Thus $|1 - x_0/\xi_i| = 1$ for $i = 1, \cdots, \nu$ since $|1 - x_0/\xi_i| < 1$ implies

$$|\xi_i/\xi_1 - x_0/\xi_1| < 1$$

contradicting the choice of x_0. Hence $|P(x_0)| = 1$ and (i) is proved. In (ii) the sequence $\{r_\nu\}$ consists of the real numbers r such that $M_f(r) = |c_n| r^n = |c_m| r^m$ for some $m \neq n$ and this clearly is a discrete set. So (iii) is obvious from the definition of the r_ν and (iv) follows from (i) and (ii). Further (v) is clear from (iii) for all r outside some discrete set and so follows for all r by (i). Finally to prove (vi) we have by (i)

$$M_{f'}(r) = \max_n |nc_n| r^{n-1} \leq \max |c_n| r^{n-1} = M_f(r)/r.$$

Proof of Theorem 1. Assume without loss of generality that $f(x) = 0 \Leftrightarrow g(x) = 0$ and $f(x) = 1 \Leftrightarrow g(x) = 1$. Then $f - g$ has a zero wherever $f(f - 1)$ has a zero. Thus for every zero of $f(f - 1)$ the function $f'(f - g)$ has a zero of multiplicity at least as high. In other words, there is an entire function F such that

$$(2) \qquad\qquad f'(f - g) = Ff(f - 1).$$

Now we may assume that $M_f(r) \geq M_g(r)$ for some arbitrarily large values of r, and we restrict our attention to these values of r. Assume r is chosen so large that $M_f(r) > 1$. Thus $M_{f-1}(r) = M_f(r)$. Then parts (v) and (vi) of the lemma yield from (2),

$$M_F(r)M_f(r)^2 = M_{f-g}(r)M_{f'}(r)$$

$$\leq M_f(r)M_{f'}(r) \leq M_f(r)^2/r.$$

So $M_F(r) \leq 1/r$ or $F \equiv 0$. Since $f' \not\equiv 0$ we have $f \equiv g$.

If we analyze the proof of Theorem 1 we can get a quantitative result.

COROLLARY. *Let f, g be different nonconstant entire functions of a non-Archimedian variable and let a, b be distinct values. For any x_0 for which the expression on the right is defined let*

$$R(x_0) = \max\left\{\frac{|f(x_0) - a||f(x_0) - b|}{|f(x_0) - g(x_0)||f'(x_0)|}, \frac{|g(x_0) - a||g(x_0) - b|}{|g(x_0) - f(x_0)||g'(x_0)|}\right\}.$$

Then there exists an x such that $|x - x_0| \leq R(x_0)$ and one of $f(x), g(x)$ is a or b while $f(x) \neq g(x)$.

Proof. We may assume without loss of generality that $x_0 = 0, a = 0, b = 1$. We may further assume that the first term in the definition of $R(0)$ is the larger. Suppose the result is false, so that $f(x) = g(x)$ whenever $f(x)$ or $g(x)$ is 0 or 1

and $|x| < R$ where R is the least absolute value of a 0- or 1-point of f or g outside the disc $|x| \leq R(0)$. Let $R > r > R(0)$. Define F by equation (2); so $F(x)$ is analytic for $|x| < R$. Moreover as in Theorem 1 $M_F(r) \leq 1/r$. But by definition

$$|F(0)| = 1/R(0) > 1/r.$$

This gives the desired contradiction.

The method of proof of Theorem 1 allows us to prove the following more general theorem.

THEOREM 2. *Let f, g be nonconstant entire functions of a non-Archimedian variable. Suppose that $M_g(r) = o(rM_f(r))$ for an infinite sequence of values of r tending to ∞. Let a, b be two distinct values. Assume that $f(x) = a \Rightarrow g(x) = a$ and $f(x) = b \Rightarrow g(x) = b$. Then $f \equiv g$.*

Proof. Reasoning as in Theorem 1 we obtain an entire function F such that

$$f'(f - g) = F(f - a)(f - b)$$

and our hypothesis suffices to prove that $F \equiv 0$.

We have thus shown that whenever the distinct a-points and b-points of a nonconstant entire function f are contained among the a-points and b-points respectively of a different entire function g, the function g must have a significantly higher growth-rate than f.

In trying to generalize Theorem 1 to meromorphic functions of a non-Archimedian variable, we are again guided by the analogy with rational functions over an algebraically closed field of characteristic zero. A first guess might be that a nonconstant rational function is determined by the pre-images of three distinct values. However this is not the case. For example the functions

$$f(x) = x/(x^2 - x + 1), \qquad g(x) = x^2/(x^2 - x + 1)$$

attain the values 0, 1, ∞ at the same points. The multiplicities of the values 0 and 1 are not the same.

A degree argument very similar to the one used for polynomials shows that a nonconstant rational function over an algebraically closed field of characteristic zero is indeed determined by the pre-images of four distinct values. The analogous theorem holds for meromorphic functions of a non-Archimedian variable.

THEOREM 3. *Let f, g be two nonconstant meromorphic functions of a non-Archimedian variable so that for four distinct values a_1, a_2, a_3, a_4 we have $f(x) = a_i \Leftrightarrow g(x) = a_i$; $i = 1, 2, 3, 4$. Then $f \equiv g$.*

Proof. Let $f = f_1/f_2$ and $g = g_1/g_2$ where f_1, f_2, g_1, g_2 are entire functions and, f_1 and f_2 as well as g_1 and g_2 have no common zeros. We may assume 0, ∞ are two of the given values; that is we assume $f_1 = 0 \Leftrightarrow g_1 = 0, f_2 = 0 \Leftrightarrow g_2 = 0$,

$f_1 = af_2 \Leftrightarrow g_1 = ag_2$ and $f_1 = bf_2 \Leftrightarrow g_1 = bg_2$ for values a, b different from 0, ∞. Further, without loss of generality, we may assume that

$$M_{f_1}(r) \geq \max \{M_{f_2}(r), M_{g_1}(r), M_{g_2}(r)\}$$

for some arbitrarily large values of r. From now on we restrict our attention to these values of r.

Now the function $f_2 g_2 (f - g) = f_1 g_2 - f_2 g_1$ vanishes whenever $f = 0$, ∞, a or b. Thus the function $(f_1' f_2 - f_1 f_2')(f_1 g_2 - f_2 g_1)$ has zeros at every zero of $f_1 f_2 (f_1 - af_2)(f_1 - bf_2)$ of multiplicity at least as great. In other words there exists an entire function F such that

$$(3) \qquad (f_1' f_2 - f_1 f_2')(f_1 g_2 - f_2 g_1) = F f_1 f_2 (f_1 - af_2)(f_1 - bf_2).$$

Now let

$$M(r) = \min \{M_{f_1}(r), M_{f_1-af_2}(r), M_{f_1-bf_2}(r)\}.$$

Since

$$
\begin{aligned}
f_1' f_2 - f_1 f_2' &= (f_1 - af_2)' f_2 - (f_1 - af_2) f_2' \\
&= (f_1 - bf_2)' f_2 - (f_1 - bf_2) f_2'
\end{aligned}
$$

we obtain from parts (v) and (vi) of the lemma

$$(4) \qquad\qquad M_{f_1' f_2 - f_1 f_2'}(r) \leq \frac{M(r) M_{f_2}(r)}{r}.$$

Thus again using the lemma and equations (3), (4) we see

$$M_F(r) M_{f_2}(r) M_{f_1}(r)^2 M(r) \leq M(r) M_{f_2}(r) \cdot \frac{M_{f_1}(r)^2}{r},$$

so that $M_F(r) \leq 1/r$. So we have $F \equiv 0$ and since $f' \not\equiv 0$ we must have $f_1 g_2 - f_2 g_1 \equiv 0$ or $f \equiv g$.

Our method of proof can also be applied to certain classes of analytic and meromorphic functions in an annulus, although the results may not be best possible.

THEOREM 4.   *Let f be analytic and unbounded in the annulus $A : r_0 < |x| < r_1$ where $0 < r_0 < r_1 < \infty$. Let g be analytic in A. If there are three distinct (finite) values a_1, a_2, a_3 so that $f(x) = a_i \Leftrightarrow g(x) = a_i$, $i = 1, 2, 3$ then $f \equiv g$.*

Proof.   As always we may assume that 0 is one of the values, so assume $f = 0 \Leftrightarrow g = 0$, $f = a \Leftrightarrow g = a$ and $f = b \Leftrightarrow g = b$ where 0, a, b are distinct. We may assume, without loss of generality, that

$$\lim_{r \to r_1} M_f = \infty.$$

We may further assume that $M_f(r) \geq M_g(r)$ through some sequence of r's tending to r_1. From now on we shall restrict our attention to these values of

r. Now we obtain exactly as in the proof of Theorem 1 that

$$(5) \qquad f'(f-g) = Ff(f-a)(f-b)$$

for some F analytic in A. We may choose r close enough to r_1 to guarantee that $M_f(r) > \max \{|a|, |b|\}$. Hence by parts (v) and (vi) of the lemma

$$(6) \qquad M_F(r)M_f(r)^3 \leq M_f(r)^2/r.$$

Thus $M_F(r) \leq (rM_f(r))^{-1} \to 0$ $(r \to r_1)$. From this it follows that $F \equiv 0$. To see this suppose the Laurent series for F has the form

$$(7) \qquad F(x) = \sum_{n=-\infty}^{\infty} C_n x^n.$$

Then by part (i) of the lemma

$$M_F(r) = \max_n |C_n| r^n \geq |C_m| r^m$$

for each fixed m. Thus

$$|C_m| \leq r_1^{-m} \lim_{r \to r_1} M_F(r) = 0.$$

It follows that $F \equiv 0$. Then since $f' \not\equiv 0$ we have $f \equiv g$.

We can extend Theorem 4 to the case where one of the discs in the complement of the annulus A degenerates to a point.

THEOREM 5. *Let f and g be analytic functions of a non-Archimedian variable in a punctured neighborhood N of a point P. Assume f has an essential singularity at P and that for three distinct (finite) values a_1, a_2, a_3 we have $f(x) = a_i \Leftrightarrow g(x) = a_i$; $i = 1, 2, 3$ for all x in N. Then $f \equiv g$ in N.*

Proof. Without loss of generality we may assume $P = \infty$, and $N : r_0 < |x| < \infty$. Now as in the proof of Theorem 4 we get (5) and (6) and hence

$$M_F(r) \leq (rM_f(r))^{-1}$$

over a sequence of values of r tending to ∞. Thus if F is expressed again by (7) we obtain

$$|C_m| \leq r^{-m}M_F(r) \leq (r^{m+1}M_f(r))^{-1} \to 0$$

(the latter following because f has an essential singularity at ∞). In other words $F \equiv 0$ and the theorem follows.

In a manner entirely analogous to that which led from Theorem 1 to Theorem 3 we can prove an analog to Nevanlinna's theorem for meromorphic functions in an annulus or in the neighborhood of an isolated essential singularity.

THEOREM 6. *Let f, g be meromorphic functions of a non-Archimedian variable in the annulus $A : r_0 < |x| < r_1$ so that f cannot be expressed as the ratio of two bounded analytic functions in A. If there are five different values a_i $(i = 1, \cdots, 5)$ so that $f(x) = a_i \Leftrightarrow g(x) = a_i$; $i = 1, \cdots, 5$ for all x in A then $f \equiv g$ in A.*

THEOREM 7. *Let f, g be meromorphic functions of a non-Archimedian variable*

in a punctured neighborhood N of a point P and suppose f has an essential singularity at P. If there are 5 different values a_i $(i = 1, \cdots, 5)$ so that $f(x) = a_i \Leftrightarrow g(x) = a_i$; $i = 1, \cdots, 5$ for all x in N then $f \equiv g$ in N.

The proofs are obvious combinations of the proofs of Theorem 3 and Theorems 4, 5.

While the method of proof in Theorems 4–7 seems to require three and five values respectively we have not been able to prove that two and four values, respectively, would not have sufficed. We should like to pose this as a problem.

Problem. Would the conclusions of Theorems 4–7 remain valid if the hypothesis on the number of values attained at the same points are reduced from three to two and five to four respectively?

The answer to the problem is affirmative if we strengthen the hypotheses to saying that the two functions attain certain values at the same points with the same multiplicities.

THEOREM 8. *Let f, g be analytic functions of a non-Archimedian variable either in an annulus A or in a punctured neighborhood N of a point P. Let f be unbounded in A or have an essential singularity at P respectively. If there are two (finite) values a, b which are attained by f and g at the same points with the same multiplicities then $f \equiv g$.*

If we had assumed only that f and g are meromorphic then we would have needed three values.

Proof. Instead of equation (5) we would obtain

$$f - g = F(f - a)(f - b)$$

and the proof now proceeds as for Theorems 4 and 5 respectively.

To verify the statement for meromorphic functions we note that we may assume ∞, a, b to be the common values and thus assume that $f = f_1/h$, $g = g_1/h$ where f_1, g_1, h are analytic in the domains in question and $f_1 g_1$ has no zero in common with h. Our hypotheses now show that $(f_1 - ah)(f_1 - bh)$ divides $f_1 - g_1$ and we can argue as before.

REFERENCES

1. W. W. ADAMS, *Transcendental numbers in the P-adic domain*, Amer. J. Math., vol. 88 (1966), pp. 279–308.
2. W. K. HAYMAN, *Meromorphic functions*, Oxford Univ. Press, Oxford, 1964.

UNIVERSITY OF CALIFORNIA,
    BERKELEY, CALIFORNIA
UNIVERSITY OF CALIFORNIA
    LOS ANGELES, CALIFORNIA