ISOMETRIC IMMERSIONS INTO SPACES OF CONSTANT CURVATURE

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1. Introduction

In this paper we first give a unified account of invariant tensor methods in Riemannian geometry and their application to the study of isometric immersions of Riemannian manifolds. The ideas presented here are more or less implicit in classical work, but the details are usually erroneously oversimplified or treated incorrectly in the more recent literature. For this reason, we give our treatment in some detail.

We first discuss, in Section 2, covariant differentiation of tensor fields over mappings, proving the generalized structural equations in detail. This material is indispensable to our treatment of the immersion theory in Section 3, where we develop analogues of the classical Gauss, Codazzi, and Ricci equations. Our version of these equations is, on the one hand, completely index-free, so the roles of the various operators (connection, normal connection, second fundamental form, etc.) in describing the geometry of the immersion is made clear. At the same time, these equations are valid for vector fields over arbitrary mappings; thus they retain all the flexibility of the classical equations.

We then use these equations to prove the following theorem, pausing to note the most general conditions needed at each stage of the proof.

THEOREM. *Let $I : M^d \to \bar{M}^{d+k}$ be an isometric immersion of a complete d-dimensional Riemannian manifold M^d in a $(d + k)$-dimensional Riemannian manifold $\bar{M}^{d+k}$ of constant curvature K. Then there exists a complete n-dimensional totally geodesic submanifold L of M which has constant curvature K in the induced metric, and which is totally geodesically immersed in $\bar{M}$ by I. Here n is the minimal value of the index of relative nullity.*

This theorem generalizes a result of B. O'Neill and E. Stiel [7], where both M and $\bar{M}$ have constant curvature K. P. Hartman [5, Lemma 3.1 (v)] proved our theorem for the $K = 0$ case, generalizing the previous result of O'Neill [6] for M and $\bar{M}$ flat (see also S. B. Alexander [1]). The index of relative nullity ν was defined by Chern and Kuiper [3], who also showed, essentially, that if M and $\bar{M}$ both have constant curvature K, then $\nu \geq d - k$, so the theorem is not vacuous for $d - k$ positive.

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2. Tensor fields over mappings

In this section we develop standard material on covariant differentiation of vector fields over mappings. For alternative treatments, see Gromoll, Klingenberg, and Meyer [4, p. 46] or Bishop and Goldberg [2, p. 227].

In the sequel, all manifolds will be of class C^∞. $\mathfrak{F}(M)$ denotes the set of C^∞ functions on M.

Let $F : M \to N$ be a C^∞ mapping of a manifold M into a manifold N which has a linear connection operator ∇. A *vector field Y over F* (or a *vector field parametrized by F*) is a C^∞ mapping $Y : M \to T(N)$ such that $\pi \circ Y = F$, where $T(N)$ is the tangent bundle of N and π the bundle projection. We denote the set of vector fields over F by $\mathfrak{X}_F(N)$.

If Y is a vector field on N, then $Y \circ F \in \mathfrak{X}_F(N)$. The set of vector fields over F of form $Y \circ F$, denoted $\mathfrak{X} \circ F$, forms a vector space over $\mathbf{R}$ and a module over $\mathfrak{F} \circ F$, the ring of C^∞ functions on M of form $g \circ F$, $g \in \mathfrak{F}(N)$. $\mathfrak{X} \circ F$ also has a Lie algebra structure over $\mathbf{R}$, with the bracket defined by

$$[X \circ F, Y \circ F] = [X, Y] \circ F.$$

Also, if A is a vector field on M, then $F_*A \in \mathfrak{X}_F(N)$. Vector fields of form F_*A are said to be *tangent* to F, and the set of such vector fields we denote by $\mathfrak{X}_F(M)$. $\mathfrak{X}_F(M)$ and $\mathfrak{X}_F(N)$ have natural module structures over $\mathfrak{F}(M)$, and vector space structures over $\mathbf{R}$.

If $Y_1, \cdots, Y_n$ form a local field of bases for the tangent spaces on a neighborhood U of N, then on $V = F^{-1}(U)$ any vector field Y over F has a local representation $Y|_V = y^i(Y_i \circ F)$, where $y^i \in \mathfrak{F}(M)$, and we are using the summation convention on repeated indices.

If $Y \circ F \in \mathfrak{X} \circ F$, then $Y \circ F|_V = (y^i \circ F)\ (Y_i \circ F)$ where now $y^i \in \mathfrak{F}(N)$. Hence we see that $\mathfrak{X} \circ F$ is generated locally by the $Y_i \circ F$, using coefficients from $\mathfrak{F} \circ F$, while $\mathfrak{X}_F(N)$ is obtained by enlarging the coefficient ring to $\mathfrak{F}(M)$.

The elements of $\mathfrak{X}_F(N)$ can be defined to operate on $\mathfrak{F}(N)$ as derivations: if $X \in \mathfrak{X}_F(N)$, $g \in \mathfrak{F}(N)$, $p \in M$, then set $X(g)(p) = X_p(g)$. The elements of $\mathfrak{X} \circ F$ operate on $\mathfrak{F} \circ F$ as well: set $Y \circ F(g \circ F) = Y(g) \circ F$.

Note that if $M = N$, and F is the identity, then vector fields over F are simply vector fields on N. The covariant derivative $\nabla : \mathfrak{X}(N) \times \mathfrak{X}(N) \to \mathfrak{X}(N)$ will now be extended to a function, also denoted by ∇, mapping $\mathfrak{X}(M) \times \mathfrak{X}_F(N) \to \mathfrak{X}_F(N)$, the two ∇'s agreeing when F is the identity.

We first define ∇ for fields of form $Y \circ F$, $Y \in \mathfrak{X}(N)$. If $p \in M$, $A \in \mathfrak{X}(M)$ set $\nabla_A(Y \circ F)_p = \nabla_{F_*A_p} Y$. Now if $X \in \mathfrak{X}_F(N)$ is arbitrary, and X has a local representation $X|_V = x^i(Y_i \circ F)$, then on V we define

$$\nabla_A X = A(x^i)(Y_i \circ F) + x^i \nabla_A(Y_i \circ F).$$

The usual computation shows this definition is independent of local representation.

PROPOSITION 2.1. *∇ has the following properties, for A, $B \in \mathfrak{X}(M)$; X, $Y \in \mathfrak{X}_F(N)$; $f \in \mathfrak{F}(M)$.*

(i) $\nabla_A(X + Y) = \nabla_A X + \nabla_A Y$.
(ii) $\nabla_A fY = f\nabla_A Y + (Af)Y$.
(iii) $\nabla_{A+B} Y = \nabla_A Y + \nabla_B Y$.
(iv) $\nabla_{fA} Y = f\nabla_A Y$.

Now let $\mathfrak{X}_F^*(N)$ denote the dual module of $\mathfrak{X}_F(N)$. An (r, s)*-tensor field over F* is an $\mathfrak{F}(M)$-multilinear mapping $T_s^r : \{\mathfrak{X}_F(N)\}^r \times \{\mathfrak{X}_F(N)\}^s \rightarrow \mathfrak{F}(M)$. T_s^r is a function of $r + s$ variables; by choosing fixed values for certain of these variables, tensors of lower degree can be defined. Also if S_s^r is a tensor field on N then $S_s^r \circ F$ defines a tensor field over F in a natural manner:

$$S_s^r \circ F(X^1, \cdots, X^r, X_1, \cdots, X_s)(p) = S_s^r(X^1(p), \cdots, X^r(p), X_1(p), \cdots, X_s(p)),$$

for $X^i \in \mathfrak{X}_F^*(N)$, $X_j \in \mathfrak{X}_F(N)$, and $p \in M$. Finally, a tensor field over F is completely determined by its local coordinates

$$T_{j_1, \cdots, j_s}^{i_1, \cdots, i_r} = T_s^r(Y^{i_1} \circ F, \cdots, Y^{i_r} \circ F, Y_{j_1} \circ F, \cdots, Y_{j_s} \circ F)$$

where $\{Y_j\}$, $\{Y^i\}$ are dual local base fields for $\mathfrak{X}(N)$ and $\mathfrak{X}^*(N)$ respectively.

We may now extend ∇_A to be a derivation on the tensor algebra over F, since we know $\nabla_A Y$ for $Y \in \mathfrak{X}_F(N)$, and we set $\nabla_A(f) = A(f)$ by definition, for $A \in \mathfrak{X}(M)$ and $f \in \mathfrak{F}(M)$. Then we have a unique extension of ∇_A commuting with contractions, and satisfying

$$\nabla_A(T \otimes S) = \nabla_A T \otimes S + T \otimes \nabla_A S.$$

An important special case of this theory is where $F = \alpha : (a, b) \rightarrow N$ is a C^∞ curve. Then if t denotes the canonical coordinate function of the interval (a, b), and X is a vector field over α, we have the covariant derivative $\nabla_{d/dt} X$ of X along α. We will sometimes write $\nabla_{\alpha'} X$ for $\nabla_{d/dt} X$. If $\nabla_{d/dt} X = 0$ (i.e. is the zero vector field over α) then X is said to be *parallel* along α, and $X(t_2)$ is called the *parallel translate* of $X(t_1)$ from t_1 to t_2 along α, for t_1, t_2 in (a, b). The theory of ordinary linear differential equations guarantees a uniquely defined smooth parallel vector field X along α, for each choice of vector $X(t_1)$ at $\alpha(t_1)$, and parallel translation from t_1 to t_2 along α defines a nonsingular linear transformation from $N_{\alpha(t_1)}$ to $N_{\alpha(t_2)}$. See for example [2, pg. 224] for more details.

The following is a useful characterization of $\nabla_{d/dt} X$. If $(e_1(t), \cdots, e_n(t))$ is a parallel frame field along α, and $X(t) = x^i(t)e_i(t)$, then $\nabla_{d/dt} X = d/dt(x^i)e_i(t)$. Hence $\nabla_{d/dt} X = 0$ if and only if the $x^i(t)$ are constant.

It is worth remarking that the definition of ∇ as a map $\mathfrak{X}(N) \times \mathfrak{X}(N) \rightarrow \mathfrak{X}(N)$ is inadequate even to define $\nabla_{d/dt} X$. The usual idea is to attempt to extend α' and X to vector fields Y and $\tilde{X}$ respectively on a neighborhood of

$\alpha(t_0)$, and set $\nabla_{d/dt} X |_{t_0} = (\nabla_Y \tilde{X})(\alpha(t_0))$. But, for instance, α might be a constant curve $\alpha(t) = p$, in which case $X(t)$ is a parametrized curve in M_p, which cannot in general be extended to a neighborhood of p. It is easy to see that $X(t)$ is parallel along α in this case if and only if $X(t)$ is a constant curve in M_p. Hence we may well have $\nabla_{\alpha'} X \neq 0$ even though $\alpha' = 0$.

We are now in a position to extend the structural equations to vector fields over mappings. Let

$$R_{XY}(Z) = \nabla_{[X,Y]}Z - \nabla_X \nabla_Y Z + \nabla_Y \nabla_X Z$$

denote the curvature tensor of N, and

$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]$$

the torsion tensor. To simplify notation, we will adopt the following conventions, to remain in force in the sequel wherever needed. We write A_*, B_*, etc. instead of $F_* A$, $F_* B$, for $A, B \in \mathfrak{X}(M)$. Similarly, we write S^* (or simply S, if no confusion is likely) in place of $S \circ F$, where S is a tensor field on N. However, if $Y \in \mathfrak{X}(N)$ and $f \in \mathfrak{F}(N)$ then we write $\tilde{Y}$ and $\tilde{f}$ in place of $Y \circ F$ and $f \circ F$, respectively.

PROPOSITION 2.2 (generalized structural equations).

(i) $R^*_{A_* B_*} Y = \nabla_{[A,B]} Y - \nabla_A \nabla_B Y + \nabla_B \nabla_A Y$
(ii) $T^*(A_*, B_*) = \nabla_A B_* - \nabla_B A_* - [A, B]_*$

where $A, B \in \mathfrak{X}(M)$, $Y \in \mathfrak{X}_F(N)$.

Proof of (i). First note that the right side of the equation is linear in A and B, by Proposition 2.1, so we can assume that A and B are local coordinate vector fields, or $[A, B] = 0$. Now let $\{y^i\}$ denote local coordinates on a neighborhood in N, so $\{Y_i = \partial/\partial y^i\}$ denote a local base for $\mathfrak{X}(N)$, and $\{\tilde{Y}_i\}$ is a local base for $\mathfrak{X}_F(N)$. Then $A_* = a^i \tilde{Y}_i$, $B_* = b^j \tilde{Y}_j$ where $a^i = A(\tilde{y}^i)$, $b^j = B(\tilde{y}^j)$, and we are using the Einstein summation convention as usual. We also have

$$(\nabla_A \tilde{Y}_k)_p = \nabla_{A_*(p)} Y_k = (a^i (\nabla_{Y_i} Y_k)^\sim)(p) = (a^i \tilde{\Gamma}^\nu_{ik} \tilde{Y}_\nu)(p),$$

where $\Gamma^\nu_{ik} Y_\nu = \nabla_{Y_i} Y_k$ by definition.

(a) Now

$$\begin{aligned} \nabla_B \nabla_A (\tilde{Y}_k) &= \nabla_B (a^i \tilde{\Gamma}^\nu_{ik}) \tilde{Y}_\nu \\ &= B(a^i \tilde{\Gamma}^\nu_{ik})(\tilde{Y}_\nu) + a^i \tilde{\Gamma}^\nu_{ik} (\nabla_B \tilde{Y}_\nu) \\ &= (B(a^i) \tilde{\Gamma}^\mu_{ik} + a^i b^j (Y_j \Gamma^\mu_{ik})^\sim + a^i (\tilde{\Gamma}^\nu_{ik}) b^j \tilde{\Gamma}^\mu_{j\nu}) \tilde{Y}_\mu . \end{aligned}$$

(b) Interchanging the roles of A and B, we find

$$\nabla_A \nabla_B (\tilde{Y}_k) = (A(b^i) \tilde{\Gamma}^\mu_{ik} + b^i a^j (Y_j \Gamma^\mu_{ik})^\sim + b^i \tilde{\Gamma}^\nu_{ik} a^j \tilde{\Gamma}^\mu_{j\nu}) \tilde{Y}_\mu .$$

(c) Therefore

$$(\nabla_B \nabla_A - \nabla_A \nabla_B)\tilde{Y}_k$$
$$= (B(a^i) - A(b^i))\tilde{\Gamma}^{\mu}_{ik} + a^i b^j (Y_j \Gamma^{\mu}_{ik} - Y_i \Gamma^{\mu}_{jk} + \Gamma^{\nu}_{ik}\Gamma^{\mu}_{j\nu} - \Gamma^{\nu}_{jk}\Gamma^{\mu}_{i\nu}) \circ F)\tilde{Y}_{\mu}.$$

But $B(a^i) - A(b^i) = (BA - AB)\tilde{y}^i = 0$ since $[A, B] = 0$.

(d) On the other hand,

$$(R^*_{A_* B_*}\tilde{Y}_k)(p) = (a^i b^j (R_{Y_i Y_j} Y_k) \circ F)(p)$$
$$= a^i b^j ((Y_j \Gamma^{\mu}_{ik} + \Gamma^{\nu}_{ik}\Gamma^{\mu}_{j\nu} - Y_i \Gamma^{\mu}_{jk} - \Gamma^{\nu}_{jk}\Gamma^{\mu}_{i\nu}) \circ F)\tilde{Y}_{\mu}(p),$$

as follows from the definitions of R and Γ. This formula agrees with (c).

Proof of (ii). (a)

$$T^*(A_*, B_*) = a^i b^j (\tilde{\Gamma}^k_{ij} - \tilde{\Gamma}^k_{ji})\tilde{Y}_k.$$

(b)

$$\nabla_A B_* - \nabla_B A_* - [A, B]_*$$
$$= (A(b^k) + b^j a^i \tilde{\Gamma}^k_{ij})\tilde{Y}_k - (B(a^k) + a^i b^j \tilde{\Gamma}^k_{ji})\tilde{Y}_k - ((AB - BA)\tilde{y}^k)\tilde{Y}_k$$
$$= a^i b^j (\tilde{\Gamma}^k_{ij} - \tilde{\Gamma}^k_{ji})\tilde{Y}_k.$$

Remarks. (1) We apologize for this proof, but a simpler one in this context does not seem to be possible. See (2, pg. 227) for an alternative approach using differential forms.

(2) One important advantage of our notational conventions is that the map F does not appear explicitly in equations (i) and (ii). Hence those equations are valid for *arbitrary* parametrization, with the understanding that the variables A and B are chosen tangent to the parametrizing manifold, and the *'s are interpreted appropriately.

If ∇ is the Levi-Cività connection corresponding to a Riemannian metric $\langle\ ,\ \rangle$ on N, then $T^*(A_*, B_*) = 0$ and $\nabla_A \langle X, Y\rangle^* = \langle \nabla_A X, Y\rangle^* + \langle X, \nabla_A Y\rangle^*$ for $A, B \in \mathfrak{X}(M)$ and $X, Y \in \mathfrak{X}_F(N)$, as is easily verified. The second equality is equivalent to: parallel translation along a curve α from t_1 to t_2 is an isometry.

For simplicity, we will only consider Riemannian manifolds N with Levi-Cività connection from now on.

PROPOSITION 2.3. *The curvature tensor has the following properties.*

(i) $\langle R^*_{A_* B_*}(X), Y\rangle = -\langle R^*_{A_* B_*}(Y), X\rangle$,

(ii) $R^*_{A_* B_*} = -R^*_{B_* A_*}$,

(iii) $\langle R^*_{A_* B_*}(C_*), D_*\rangle = \langle R^*_{C_* D_*}(A_*), B_*\rangle$,

(iv) $\mathfrak{S} R^*_{A_* B_*}(C_*) = 0$,

(v) $\mathfrak{S}(\nabla_A R^*)_{B_* C_*} = 0$, *where* $A, B, C, D \in \mathfrak{X}(M)$; $X, Y \in \mathfrak{X}_F(N)$; *and* $\mathfrak{S}$ *denotes cyclic summation over* A, B, *and* C.

Proof. (i) and (v) follow from the generalized structural equations; (ii), (iii), and (iv) are trivial consequences of the corresponding facts for the curvature tensor on N. As a sample, we prove (v) in detail. First we may assume that A, B, and C are local coordinate vector fields on M, since (v) is $\mathfrak{F}(M)$-linear in A, B, and C. Thus $[A, B] = [A, C] = [B, C] = 0$. Now

$$\mathfrak{S}(\nabla_A R^*)_{B_* C_*} Y = \mathfrak{S}(\nabla_A(R^*_{B_* C_*} Y) - R^*_{\nabla_A B_*, C_*} - R^*_{B_*, \nabla_A C_*} - R^*_{B_* C_*} \nabla_A Y).$$

Writing out and regrouping the terms on the right, using the structural equations, gives

$$\mathfrak{S}(\nabla_A \nabla_C \nabla_B Y - \nabla_A \nabla_B \nabla_C Y) - (R_{A_*, [C,B]_*} Y + R_{B_*, [A,C]_*} Y + R_{C_*, [B,A]_*} Y) - \mathfrak{S}R_{B_* C_*} \nabla_A Y.$$

The terms involving brackets vanish by assumption, and by writing out all terms in the first cyclic sum and regrouping, we see it is just $\mathfrak{S}R_{B_* C_*} \nabla_A Y$.

We conclude this section by remarking that given an arbitrary (real) vector bundle E over a manifold N, a connection on E can be defined as an operator $\nabla : \mathfrak{X}(N) \times \Gamma(E) \rightarrow \Gamma(E)$, where $\Gamma(E)$ denotes the module of sections of E. ∇ is assumed to have the properties of Proposition 2.1. Then if $F : M \rightarrow N$ is a mapping of a manifold M into N, ∇ can be pulled back to define a connection $F^*(\nabla)$ in the induced bundle $F^*(E)$ over M, in essentially the way we proceeded above. An analogue of the first structural equation can be proved, but torsion is not definable in this context. The pullback $F^*(\nabla)$ has the following crucial functorial property: if $G : K \rightarrow M$ is a map of another manifold K into M, then $(F \circ G)_* \nabla = G^*(F^* \nabla)$.

More explicitly, note that if $Y \in \Gamma(E)$, $Y \circ F$ can be identified with an element in $\Gamma(F^*E)$. Now we proceed exactly as before, defining $F^*\nabla_A(Y \circ F)_p = \nabla_{F_* A_p} Y$ (we are using the general fact that a connection $\nabla_A Y$ in a vector bundle over a manifold M can always be evaluated at a point $p \in M$ to give an element $\nabla_{A_p} Y$ in the fibre over p, by $\mathfrak{F}(M)$-linearity in A). Now every section X of F^*E is locally of form $x^i(Y_i \circ F)$ where $\{Y_i\}$ denote a local trivialization of E, and $x^i \in \mathfrak{F}(M)$. Now $F^*\nabla_A$ is defined on X using the same local formula as before. To prove the functoriality property, it suffices to consider $(F \circ G)^* \nabla_A$ evaluated on sections of $(F \circ G)^*E$ of form $(Y \circ F \circ G)$. Then

$$((F \circ G)^* \nabla_A(Y \circ F \circ G))_p = \nabla_{(F \circ G)_* A_p} Y = \nabla_{F_*(G_* A_p)} Y = F^* \nabla_{G_* A_p}(Y \circ F) = (G^* \circ F^*) \nabla_{A_p} Y$$

where $A \in \mathfrak{X}(K)$ and $p \in K$.

If we define the curvature operator of ∇ by $R_{AB} = \nabla_{[A,B]} - [\nabla_A, \nabla_B]$ for A, $B \in \mathfrak{X}(N)$, then the generalized structural equation says that given $F : M \rightarrow N$, then $F^*R_{AB} = (R \circ F)_{A_* B_*}$ where F^*R is the curvature operator of $F^*\nabla$ and here A, $B \in \mathfrak{X}(M)$. If E is of the form $F^*T(N)$ for a map $F : M \to N$, then we can define the torsion tensor of a connection in E by

$$T(A, B) = \nabla_A B_* - \nabla_B A_* - [A, B]_*,$$

and we have a second generalized structural equation: if $G : K \to M$, then $G^*T = T \circ G$, where G^*T is the torsion tensor of $G^*\nabla$, a connection in $G^*E = (F \circ G)^*T(N)$. These structural equations are proved precisely as in the case already handled above, by replacing the local coordinate vector fields $\{Y_i\}$ appearing in the calculations there systematically by local trivializations of E.

It is important to note the following fact. If $F : M \to N$ is an inclusion map of a submanifold M into N, then $F^*\nabla_{A_p}(Y \circ F) = \nabla_{A_p}(Y) = ((\nabla_A Y) \circ F)_p$ for $A, Y \in \mathfrak{X}(N)$, $p \in M$; i.e. the restriction of $\nabla_A Y$ to M is equal to $F^*\nabla$ applied to the restrictions of A and Y to M.

As before, we will write simply ∇_A in place of $F^*\nabla_A$ whenever it is safe to do so.

3. Isometric immersions of Riemannian manifolds

Let $I : M^d \to \bar{M}^{d+k}$ be an isometric immersion of the d-dimensional Riemannian manifold M into the $d + k$-dimensional Riemannian manifold $\bar{M}$. Let $\bar{\nabla}$ and $\langle\ ,\ \rangle^-$ (resp. ∇ and $\langle\ ,\ \rangle$) denote the (unique) Levi-Cività connection and Riemannian inner product on $\bar{M}$ (resp. M). According to the notational conventions of Section 2, the isometric character of I is expressed by the equation $\langle X, Y\rangle = \langle X_*, Y_*\rangle^{-*}$, for $X, Y \in \mathfrak{X}(M)$. Furthermore $\mathfrak{X}_I(\bar{M})$ denotes the set of $\bar{M}$-vector fields parametrized by $I : M \to \bar{M}$, while $\mathfrak{X}_I(M)$ denotes the set of vector fields tangent to I. In order to simplify notation, we will identify $\mathfrak{X}_I(M)$ with $\mathfrak{X}(M)$ whenever convenient, under the identification $X \leftrightarrow X_*$. This correspondence is well defined since X and X_* are both parametrized by M, and $I_*(p)$ is bijective for each p in M. We may also sometimes omit the bar over $\langle X_*, Y_*\rangle^{-*}$. Now we can define tensor fields over I, P and $P^\perp$, such that for each $p \in M$,

$$P(p) : \bar{M}_{I(p)} \to I_* M_p \quad \text{and} \quad P^\perp(p) : \bar{M}_{I(p)} \to M^\perp_{I(p)}$$

are orthogonal projection operators; here $M^\perp_{I(p)}$ denotes the orthogonal complement of $I_* M_p$ in $\bar{M}_{I(p)}$. If we set

$$\mathfrak{X}_I^\perp(M) = \{X \in \mathfrak{X}_I(\bar{M}) : P^\perp X = X\},$$

the set of *normal* vector fields over I, we can write $\mathfrak{X}_I(\bar{M}) = \mathfrak{X}_I(M) \oplus \mathfrak{X}_I^\perp(M)$.

We will need to define the following operators on $\mathfrak{X}_I(M)$ and $\mathfrak{X}_I^\perp(M)$. Here $X, Y, Z \in \mathfrak{X}(M)$, $N \in \mathfrak{X}_I^\perp(M)$.

1. *Operators on* $\mathfrak{X}_I(M)$. (a′) $P\bar{\nabla}_X Y_* = (\nabla_X Y)_*$. This identity can be seen as follows. If we define $\tilde{\nabla}_X Y$ by $(\tilde{\nabla}_X Y)_* = P\bar{\nabla}_X Y_*$ (using invertibility of $I_*(p)$ for each p in M), we find that $\tilde{\nabla}_X Y$ is a connection on M. Furthermore

$$P\bar{\nabla}_X Y_* - P\bar{\nabla}_Y X_* = P[X, Y]_* = [X, Y]_*.$$

Hence $\tilde{\nabla}_X Y - \tilde{\nabla}_Y X = [X, Y]$, or $\tilde{\nabla}$ is torsion-free. Finally

$$\begin{aligned} X\langle Y, Z\rangle &= X\langle Y_*, Z_*\rangle^* \\ &= \langle \bar{\nabla}_X Y_*, Z_*\rangle^{-*} + \langle Y_*, \bar{\nabla}_X Z_*\rangle^{-*} \\ &= \langle P\bar{\nabla}_X Y_*, Z_*\rangle^* + \langle Y_*, P\bar{\nabla}_X Z_*\rangle^* \\ &= \langle \tilde{\nabla}_X Y, Z\rangle + \langle Y, \tilde{\nabla}_X Z\rangle. \end{aligned}$$

Hence $\tilde{\nabla}$ is compatible with the metric on M, or $\tilde{\nabla} = \nabla$ is the unique Levi-Cività connection on M.

Using the identification of $\mathfrak{X}(M)$ with $\mathfrak{X}_I(M)$, we can simplify (a′) to read (a) $P\bar{\nabla}_X Y = \nabla_X Y$.

(b′) $P^{\perp}\bar{\nabla}_X Y_* = T_X(Y_*)$, by definition. Under the identification, this can be simplified to (b) $P^{\perp}\bar{\nabla}_X Y = T_X(Y)$.

Hence as an operator on $\mathfrak{X}(M)$, $\bar{\nabla}_X Y = \nabla_X Y + T_X(Y)$.

2. *Operators on* $\mathfrak{X}^{\perp}(M)$. We define (a) $P^{\perp}\bar{\nabla}_X N = \nabla_X^{\perp} N$. It is immediate that $\nabla_X^{\perp}$ is a connection on the normal bundle of M. We set (b) $P\nabla_X^{\perp} N = T_X N$. (This double use of the symbol T_X is justified by Proposition 3.1 below.) Hence as operators on $\mathfrak{X}^{\perp}(M)$, (c) $\bar{\nabla}_X N = T_X N + \nabla_X^{\perp} N$.

1(b) and 2(b) define the second fundamental form operator T_X. T_X is related to the classical second fundamental form $S : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}^{\perp}(M)$ by $S(X, Y) = T_X Y$. We now list some well-known properties of this operator.

PROPOSITION 3.1. (i) $T_X Y = T_Y X$.
(ii) $\langle T_X Y, N\rangle^{-*} = -\langle T_X N, Y\rangle^{-*}$
(iii) *T is bilinear over* $\mathfrak{F}(M)$.
(iv) $T_X Y \in \mathfrak{X}^{\perp}(M)$; $T_X N \in \mathfrak{X}(M)$

Proof. (i) follows from the generalized second structural equation over I, using $T_X Y = \bar{\nabla}_X Y - \nabla_X Y$. (ii) expresses the compatibility of $\bar{\nabla}$ with the metric on $\bar{M}$. (iii) follows from (i) and (ii). (iv) is trivial.

We will need formulae for these operators, in terms of local base fields for $\mathfrak{X}(M)$ and $\mathfrak{X}^{\perp}(M)$. To obtain these, let $(Z_1, \cdots, Z_{d+k})$ denote a local orthonormal frame field on an $\bar{M}$-neighborhood U of a point $I(p)$, where $p \in M$. Assume further that $Z \circ I \in \mathfrak{X}^{\perp}(M)$ for $d + 1 \leq \alpha, \beta \leq d + k$, while $Z_i \circ I \in \mathfrak{X}(M)$ for $1 \leq i, j, k \leq d$, as can always be arranged. To simplify the notation, we further restrict attention to a sufficiently small M-neighborhood V of p, on which I is injective, so we may identify V with $I(V)$, $\mathfrak{X}(V)$ with $\mathfrak{X}_I(V)$, etc. Now let $\bar{\nabla}_{Z_J} Z_K = \Gamma_{JK}^L Z_L$ by definition, for unrestricted indices J, K, L. Now $P\bar{\nabla} = \nabla$ on $\mathfrak{X}(V)$, or $\nabla_{Z_i} Z_j = \Gamma_{ij}^k Z_k$. Similarly, $P^{\perp}\bar{\nabla} = T$ on $\mathfrak{X}(V)$, or $T_{Z_i} Z_j = \Gamma_{ij}^{\alpha} Z_{\alpha}$. On $\mathfrak{X}^{\perp}(V)$, $\nabla_{Z_i}^{\perp} Z_{\alpha} = \Gamma_{i\alpha}^{\beta} Z_{\beta}$, while $T_{Z_i} Z_{\alpha} = \Gamma_{i\alpha}^{j} Z_j$.

Now let $F : K \to \bar{M}$ be a smooth mapping of a manifold K into $\bar{M}$ which factors through I, i.e. $F = I \circ G$ for some $G : K \to M$. Then we have the $\mathfrak{F}(K)$-module $\mathfrak{X}_F(\bar{M})$ and the orthogonal projection tensors over F, $P \circ F$

and $P^{\perp} \circ F$, which according to our conventions we may abbreviate to P and $P^{\perp}$. Then

$$\mathfrak{X}_F(\bar{M}) = \mathfrak{X}_F(M) \oplus \mathfrak{X}_F^{\perp}(M),$$

where

$$\mathfrak{X}_F(M) = \{X \in \mathfrak{X}_F(\bar{M}) : PX = X\} \quad \text{and} \quad \mathfrak{X}_F^{\perp}(M) = \{X \in \mathfrak{X}_F(\bar{M}) : X = P^{\perp}X\}.$$

The idea now is to start with the generalized structural equations for $\bar{M}$ (over F):

(i) $\bar{R}^*_{A_* B_*} = \bar{\nabla}_{[A,B]} - [\bar{\nabla}_A, \bar{\nabla}_B]$
(ii) $\bar{\nabla}_A B_* - \bar{\nabla}_B A_* = [A, B]_*$

(where $\bar{R}$ denotes the curvature tensor of $\bar{M}$, and $A, B \in \mathfrak{X}(K)$).

We decompose all operators appearing into tangent and normal parts relative to M, and then apply them to vector fields over F which are respectively tangent and normal to M. Carrying out this process for $\bar{R}_{A_* B_*}$ gives four equations.

*F*1. *Operators on* $\mathfrak{X}_F(M)$. We assert that (a) $P\bar{\nabla}_A Y = \nabla_A Y$, for $Y \in \mathfrak{X}_F(M)$. To see this, let $\{Z_1, \cdots, Z_{d+k}\}$ denote the local adapted orthonormal frame field considered above. Then $\nabla_A(Z_j \circ F) = a^i \tilde{\Gamma}^k_{ij} \tilde{Z}_k = P(a^i \tilde{\Gamma}^K_{ij} \tilde{Z}_K) = P\bar{\nabla}_A \tilde{Z}_j$, where $\tilde{\Gamma}^k_{ij} = \Gamma^k_{ij} \circ F$, $A_* = a^i \tilde{Z}_i$, etc., as in Section 2.

(b) $P^{\perp}\bar{\nabla}_A Y = T^*_{A_*} Y$. This follows from $P^{\perp}\bar{\nabla}_A \tilde{Z}_j = P(a^i \tilde{\Gamma}^K_{ij} \tilde{Z}_K) = a^i \tilde{\Gamma}^{\alpha}_{ij} Z_{\alpha}$. But

$$(T^*_{A_*} \tilde{Z}_j)_p = S(a^i \tilde{Z}_i, \tilde{Z}_j)_p = (a^i \tilde{\Gamma}^{\alpha}_{ij} \tilde{Z}_{\alpha})_p .$$

Hence as an operator on $\mathfrak{X}_F(M)$, (c) $\bar{\nabla}_A = \nabla_A + T_{A_*}$, since we may delete the star over a tensor.

*F*2. *Operators on* $\mathfrak{X}_F^{\perp}(M)$. (a) $P^{\perp}\bar{\nabla}_A = \nabla^{\perp}_A$. Here $\nabla^{\perp}_A$ denotes the pullback $F^*\nabla^{\perp}_A$ of the normal connection operator $\nabla^{\perp}_A$. This identity is verified as follows:

$$P^{\perp}\bar{\nabla}_A \tilde{Z}_{\alpha} = P^{\perp} a^i \tilde{\Gamma}^K_{i\alpha} \tilde{Z}_K = a^i \tilde{\Gamma}^{\beta}_{i\alpha} \tilde{Z}_{\beta} .$$

But $F^*\nabla^{\perp}_{A_p} \tilde{Z}_{\alpha} = \nabla^{\perp}_{F_* A_p} Z_{\alpha} = (a^i(\nabla^{\perp}_{Z_i} Z_{\alpha} \circ F))_p = (a^i \tilde{\Gamma}^{\beta}_{i\alpha} \tilde{Z}_{\beta})_p$.

(b) $P\bar{\nabla}_A = T^*_{A_*}$, since $P\bar{\nabla}_A \tilde{Z}_{\alpha} = Pa^i \tilde{\Gamma}^K_{i\alpha} \tilde{Z}_K = a^i \tilde{\Gamma}^j_{i\alpha} \tilde{Z}_j = T^*_{A_*}$.
Hence as an operator on $\mathfrak{X}_F^{\perp}(M)$, (c) $\bar{\nabla}_A = \nabla^{\perp}_A + T_{A_*}$. We can now prove the following well-known result.

PROPOSITION 3.2. *M-parallel and $\bar{M}$-parallel translation along a curve α in M of a vector tangent to M at $\alpha(t_1)$ agree if and only if $T_{\alpha'} = 0$ along α.*

Proof. $\bar{\nabla}_{d/dt} Y = \nabla_{d/dt} Y + T_{\alpha'} Y$, if Y is a vector field in $\mathfrak{X}_{\alpha}(M)$.

COROLLARY. *M is totally geodesic in $\bar{M}$ (i.e. I maps M-geodesics γ into $\bar{M}$-geodesics $I \circ \gamma$) if and only if T vanishes identically over I.*

With these preliminaries, we are ready to decompose the generalized structural equations of $\bar{M}$. Here $P\cdot\bar{\nabla}_A$ denotes the composition of the operators P and $\bar{\nabla}_A$, etc.

I. $P\cdot\bar{R}_{A_*B_*}$ on $\mathfrak{X}_F(M)$ yields the *Gauss Equation:*

$$P\cdot\bar{R}_{A_*B_*} = P\cdot\bar{\nabla}_{[A,B]} - P\cdot[\bar{\nabla}_A, \bar{\nabla}_B];$$

using $\bar{\nabla}_A = \nabla_A + T_{A_*}$ and the generalized structural equation $R_{A_*B_*} = \nabla_{[A,B]} - [\nabla_A, \nabla_B]$ of M we have

$$P\cdot\bar{R}_{A_*B_*} = R_{A_*B_*} - [T_{A_*}, T_{B_*}].$$

II. $P^\perp\cdot\bar{R}_{A_*B_*}$ on $\mathfrak{X}_F(M)$ gives the *Codazzi-Mainardi Equation*

$$P^\perp\cdot\bar{R}_{A_*B_*} = T_{[A,B]_*} - \nabla_A^\perp\circ T_{B_*} - T_{A_*}\cdot\nabla_B + \nabla_B^\perp\circ T_{A_*} + T_{B_*}\cdot\nabla_A.$$

III. $P\cdot\bar{R}_{A_*B_*}$ on $\mathfrak{X}_F^\perp(M)$ gives the second *Codazzi-Mainardi Equation*

$$P\cdot\bar{R}_{A_*B_*} = T_{[A,B]_*} - \nabla_A\circ T_{B_*} - T_{A_*}\cdot\nabla_B^\perp + \nabla_B\circ T_{A_*} + T_{B_*}\cdot\nabla_A^\perp.$$

IV. $P^\perp\cdot\bar{R}_{A_*B_*}$ on $\mathfrak{X}^\perp(M)$ gives the *Ricci Equation*

$$P^\perp\cdot\bar{R}_{A_*B_*} = R^\perp_{A_*B_*} - [T_{A_*}, T_{B_*}],$$

where $R^\perp_{A_*B_*}$ denotes the generalized curvature operator $\nabla^\perp_{[A,B]} - [\nabla_A^\perp, \nabla_B^\perp]$ of the normal connection $\nabla^\perp$.

Remarks. The two Codazzi equations evidently contain the same information, by the antisymmetry of the curvature operator.

The Codazzi equations can be simplified somewhat, and in fact all of these equations can be combined into one by the following device. Define a connection $\tilde{\nabla}$ in $I^*T(\bar{M})$ as follows:

$$\tilde{\nabla}_X(PY + P^\perp Y) = P\bar{\nabla}_X PY + P^\perp\bar{\nabla}_X P^\perp Y = \nabla_X PY + \nabla_X^\perp P^\perp Y,$$

where $X\in\mathfrak{X}(M)$, $Y\in\Gamma(I^*T(\bar{M})) = \mathfrak{X}_I(\bar{M})$, and $\bar{\nabla}_X$ denotes $I^*\bar{\nabla}_X$ here. Then we can redefine T by $T = \bar{\nabla} - \tilde{\nabla}$, or $T_XY = P^\perp\bar{\nabla}_X PY + P\bar{\nabla}_X P^\perp Y$. Now these operators pull back over $G: K \to M$ to give operators on $\Gamma(F^*T(\bar{M}))$, which we denote by the same symbols. With this terminology, the Codazzi Equation II is easily shown to be equivalent to

$$P\bar{R}_{A_*B_*} = (\tilde{\nabla}_A T)_{B_*} - (\tilde{\nabla}_B T)_{A_*} = T_{[A,B]_*} + [T_{B_*}, \tilde{\nabla}_A] - [T_{A_*}, \tilde{\nabla}_B].$$

If $\tilde{R}$ denotes the curvature tensor of $\tilde{\nabla}$, we have the generalized structural equation

$$\tilde{R}_{A_*B_*} = \tilde{\nabla}_{[A,B]} - [\tilde{\nabla}_A, \tilde{\nabla}_B].$$

A simple calculation using $T = \bar{\nabla} - \tilde{\nabla}$ shows that

$$\bar{R}_{A_*B_*} - \tilde{R}_{A_*B_*} - R(T)_{A_*B_*} = [T_{B_*}, \tilde{\nabla}_A] - [T_{A_*}, \tilde{\nabla}_B]$$

where $R(T)_{A_*B_*}$ denotes the "curvature operator" $T_{[A,B]_*} - [T_{A_*}, T_{B_*}]$ of T. This equation contains all the information of equations I–IV above.

4. An application

In this section we prove the theorem given in the introduction. All of the nomenclature and conventions of previous sections are assumed in force unless otherwise noted. The following definition and lemmas are due essentially to Chern and Kuiper [3]. Here M is assumed immersed in $\bar{M}$ by I; restrictions on $\bar{M}$ will be added as needed.

Let $\mathcal{R}(p) = \{x \in M_p : T_x = 0\}$ the *space of relative nullity* at $p \in M$. Set $\dim \mathcal{R}(p) = \nu(p)$, the *index of relative nullity* at p. ν is upper-semicontinuous, so the set G on which it assumes its minimal value n is open.

We write $X \in \mathcal{R}$ if X is a relative nullity vector field, i.e. if $X_p \in \mathcal{R}(p)$ for all p in question.

LEMMA 4.1. *The distribution $p \to \mathcal{R}(p)$ is autoparallel (i.e. if $X, Y \in \mathcal{R}$ then $\nabla_X Y \in \mathcal{R}$) and involutive if $P^\perp \bar{R}_{I_* X, I_* Z} = 0$ for $X \in \mathcal{R}$, $Z \in \mathfrak{X}(M)$, in particular if $\bar{M}$ has constant curvature.*

Proof. We apply the Codazzi Equation II with F the identity map on M. We may identify X with $I_* X$, etc., as usual. We have (by assumption on $\bar{R}$)

$$0 = T_{[X,Z]} Y - \nabla_X^\perp \circ T_Z Y - T_X \cdot \nabla_Z Y + \nabla_Z^\perp \circ T_X Y + T_Z \cdot \nabla_X Y,$$

where $X, Y \in \mathcal{R}$, $Z \in \mathfrak{X}(M)$. The first four terms on the right vanish by relative nullity of X and Y, leaving $T_Z \cdot \nabla_X Y = 0$. But this means $\nabla_X Y \in \mathcal{R}$. Now $[X, Y] = \nabla_X Y - \nabla_Y X \in \mathcal{R}$ also, or $\mathcal{R}$ is involutive. If

$$\bar{R}_{XY} = K(\langle X, \ \rangle Y - \langle Y, \ \rangle X) \quad \text{for} \quad X, Y \in \mathfrak{X}(\bar{M})$$

then $P^\perp \bar{R}^* = 0$.

The distribution induced by $\mathcal{R}$ on G has constant dimension n, so it is integrable. Let L denote a leaf of the resulting foliation on G, and let $J : L \to M$ denote the immersion of L in M. Let ∇^L, T^L, P_L, etc. denote the induced connection on L and the operators associated with the immersion J.

LEMMA 4.2. *The leaf L is totally geodesically immersed in M and (by $I \circ J$) in $\bar{M}$, and has curvature operator $P \cdot \bar{R}_{X_* Y_*}$ in the induced metric, for $X, Y \in \mathfrak{X}(L)$.*

Proof. Locally, since $J : L \to M$ is an imbedding, X and Y are of form $X = \tilde{X} \circ J$, $Y = \tilde{Y} \circ J$ for $\tilde{X}, \tilde{Y} \in \mathcal{R}$. By a remark at the end of Section 2, we have $\nabla_{\tilde{X}} \tilde{Y}_{|L} = (J^* \nabla)_X Y$. It follows that

$$P_L^\perp J^* \nabla_X Y = P_L^\perp \nabla \tilde{Y} \circ J = 0,$$

by Lemma 4.1. Hence $T_X^L Y = P_L^\perp J^* \nabla_X Y = 0$, so L is totally geodesic in M.

Now if $\tilde{\gamma}$ is a geodesic of L, $\gamma = J \circ \tilde{\gamma}$ is geodesic in M as well. Now $\bar{\nabla}_{\gamma'} \gamma' = \nabla_{\gamma'} \gamma' + T_{\gamma'} \gamma'$. But $T_{\gamma'} \gamma' = 0$ by relative nullity of γ', and $\nabla_{\gamma'} \gamma' = 0$ since γ is a geodesic of M. Hence $\bar{\nabla}_{\gamma'} \gamma' = 0$ and γ is a geodesic of $\bar{M}$ also. Hence L is totally geodesic in $\bar{M}$ also.

Now the Gauss Equation I gives $P \cdot \bar{R}_{X_* Y_*} = R_{XY}$ where $*$ denotes $(I \circ J)_*$.

The following lemma is not needed later, but it gives a condition guaranteeing positive relative nullity.

Let $\mathcal{R}(m)^{\perp}$ denote the orthogonal complement (in M_m) of $\mathcal{R}(m)$.

LEMMA 4.3. *There is a vector x in $\mathcal{R}(m)^{\perp}$ such that $T_x : \mathcal{R}(m)^{\perp} \to M_m^{\perp}$ is one-to-one if $R_{XY} = P \cdot \bar{R}_{XY}$. This implies $\nu(m) \geq d - k$.*

For proof see Stiel [8], or [3].

We now prove the main theorem. We assume M is complete and satisfies the condition of Lemma 2. Then if G is as before, we must show that a leaf L of the relative nullity foliation on G is complete.

Let $\gamma : [0, c) \to L$ be a geodesic segment in L, and let $\gamma^* : [0, \infty) \to M$ be the extension of γ in M guaranteed by completeness. It suffices to show that $p^* = \gamma^*(c)$ lies in G also, or that $\nu(p^*) = n$ (we have $\nu(p^*) \geq n$ by upper-semicontinuity).

Let $\eta = (y^1, \cdots, y^d)$ be a Frobenius coordinate system on a neighborhood U of $p = \gamma(0)$. We make the convention that $1 \leq i, j, \cdots \leq n$; $1 \leq I, J, \cdots \leq d$; $n + 1 \leq \alpha, \beta, \cdots \leq d$, and assume that $\partial/\partial y^i \in \mathcal{R}$, $\partial/\partial y^{\alpha} \notin \mathcal{R}$. We can further assume that $\eta(p) = 0 \in \mathbf{R}^d$. Now let Σ denote the slice $y^i = 0$ of U with induced coordinate $\eta_2 = (y^{n+1}, \cdots, y^d)$, and let $V \subset \mathbf{R}^{d-n}$ denote $\eta_2(\Sigma)$. Let $E(s) = (E_1(s), \cdots, E_d(s))$ be an $\mathcal{R}$-adapted orthogonal frame field on Σ (such that $E_i(s) \in \mathcal{R}(s)$, $E_{\alpha}(s) \in \mathcal{R}^{\perp}(s)$, the orthogonal complement of $\mathcal{R}(s)$, for all $s \in \Sigma$). Assume further that $E_1(p) = \gamma'(0)$. Now define $F : \mathbf{R}^n \times V \to M$ by

$$F(u^1, \cdots, u^n, \eta_2(s)) = \exp_s(u^i E_i(s)).$$

Define vector fields over F $X_I = F_*(\partial/\partial u^I)$, where u^I denote the natural Euclidean coordinates on $\mathbf{R}^n \times V \subset \mathbf{R}^d$.

Let $U_I = \partial/\partial u^I$ in order to simplify notation.

We now apply the Codazzi Equation II to (U_1, U_{α}) to obtain
$0 = P^{\perp}\bar{R}_{U_1,U_{\alpha}} = T_{[U_1,U_{\alpha}]} - T_{X_1} \cdot \nabla_{U_{\alpha}} - \nabla^{\perp}_{U_1} \cdot T_{X_{\alpha}} + T_{X_{\alpha}} \cdot \nabla_{U_1} + \nabla^{\perp}_{U_{\alpha}} \cdot T_{X_1}$.
Most of these terms vanish, as we now show.

By construction, $F(t, 0, \cdots, 0) = \gamma(t)$, and $X_1(t, 0, \cdots, 0) = \gamma'(t) \in \mathcal{R}$ for $0 \leq t < c$. Let G^* denote an open neighborhood of $\gamma[0, c)$ in the open set G, and $W = F^{-1}(G^*)$. For fixed $q \in V$ and all $(u^1, \cdots, u^n)$ such that $(u^1, \cdots, u^n, q) \in W$, F parametrizes the relative nullity leaf through $F(0, \cdots, 0, q)$, so the $X_i = F_*(\partial/\partial u^i)$ are relative nullity vector fields over $F_{|W}$. It follows that T_{X_1} is a vanishing tensor field over $F_{|W}$, and hence $\nabla^{\perp}_{U_{\alpha}} T_{X_1} = P^{\perp}\nabla_{U_{\alpha}} T_{X_1}$ must also vanish over $F_{|W}$. Finally, $F_*([\partial/\partial u^1, \partial/\partial u^{\alpha}]) = 0$ over F.

Hence the only surviving terms (on W) in the Codazzi Equation above are

$$\nabla^{\perp}_{U_1} \cdot T_{X_{\alpha}} = T_{X_{\alpha}} \cdot \nabla_{U_1}.$$

Further restricting the parameters to the domain of γ, and using the fact that

$$\bar{\nabla}_{U_1} = \nabla^{\perp}_{U_1} + T_{X_1} = \nabla^{\perp}_{U_1} \quad \text{on} \quad \mathfrak{X}^{\perp}_F(M),$$

we obtain $\nabla_{\gamma'} \cdot T_{X_\alpha} = T_{X_\alpha} \cdot \nabla_{\gamma'}$. Let Y denote an M- or $\bar{M}$-parallel vector field along γ^*. Then $\nabla_{\gamma'}(T_{X_\alpha} Y) = T_{X_\alpha} \cdot \nabla_{\gamma'} Y = 0$. If $X = x^\alpha X_\alpha$ is a vector field along γ for constants x^α, we have $\nabla_{\gamma'}(T_X Y) = 0$, or $T_X Y$ is parallel along γ. It follows that if $y \in M_p$ is non-nullity, and $T_x y \neq 0$ for $x = x^\alpha X_\alpha(p)$, then $T_X*Y^*(p^*) \neq 0$ where Y^* is the parallel translate of y to p^*, and $X^* = x^\alpha X_\alpha(p^*)$. Hence $\nu(p^*)$ is not greater than $\nu(p)$.

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