A REMARK ON THE BIRKHOFF ERGODIC THEOREM

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In this note we will prove the following theorem:

THEOREM. *Let T be a 1–1, invertable, measure-preserving, ergodic transformation of a measure space X onto itself. Let*

$$f^*(x) = \sup_n (1/n) \sum_{i=1}^n f(T^i(x)).$$

(a) *Assume X has finite measure. Then for $f \geq 0$, $f^*(x)$ is integrable if and only if $[f(x) \log (x)]^+$ is integrable (g^+ is the positive part of g).*

(b) *Assume Z has infinite measure. Then for $f \geq 0$, $f^*(x)$ is not integrable.*

The "if" part of (a) is well known and is only stated here for the sake of completeness.

This paper has as its starting point the following theorem of Burkholder: Let X_i be a sequence of independent identically distributed, non-negative random variables. Then $\sup_n (1/n) \sum_{i=1}^n X_i(\omega)$ is integrable if and *only if* $[X_i(\omega) \log (X_i(\omega))]^+$ has finite expectation. Gundy, in an unpublished paper, proves a reverse maximal inequality from which he deduces the above theorem. (This is generalized in Proposition 1.) Gundy also suggested that his theorem holds in the more general case of an ergodic transformation, and that is what we prove here. This seems to be the natural setting for the theorem, since it does not hold for the identity transformation, $T(x) = x$. Furthermore, the theorem does not seem to generalize in a natural way to the operator case, since it does not hold for the linear operator that sends every function into a constant.

LEMMA 1. *Given a set D, of non-zero measure, we can find disjoint sets A_i^j, $1 \leq i < M_j < \infty$, $1 \leq j < \infty$, such that*

$$T(A_i^j) = A_{i+1}^j, \quad \textit{unless } i = M_j - 1,$$

$$\bigcup_{j=1}^\infty A_1^j = D \quad \textit{and} \quad \bigcup_{j=1}^\infty \bigcup_{i=1}^{M_j-1} A_i^j = X.$$

Proof. For each $x \in D$ let $N(x)$ be the first integer ≥ 1 such that $T^{N(x)}(x) \in D$. Let A_1^j be the set of x, in D, such that $N(x) = j$. Let $M_j = j$ and let $A_i^j = T^{i-1}(A_1^j)$ for $i \leq j$. The A_i^j are disjoint because T is 1–1 and their union is X because T is ergodic.

PROPOSITION 1. *Fix $\alpha > 0$. Let E be the set where $f^* \geq \alpha$. Let F be the set where $f \geq \alpha$. Assume that $m(X - E) \neq 0$ ($m(C)$ is the measure of C). Then $\alpha \cdot m(E) \geq \frac{1}{2} \int_F f$.*

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Proof. (1) It is easy to see that we may assume without loss of generality that $f = 0$ outside of F.

We can see (1) as follows: Let ψ_F be the function that is 1 on F and 0 elsewhere. If Proposition 1 holds for $\psi_F \cdot f$, then it holds for f since $\int_F f = \int_F \psi_F \cdot f$ and $f^* \geq (\psi_F \cdot f)^*$.

(2) Let $D = X - E$ and apply Lemma 1.

(3) Now fix j and $x \in A_1^j$. Define a block to be a sequence of consecutive integers, say from l to k, where $k < M_j$ and if $l \leq i \leq k$, then

$$(1/(k - i + 1)) \sum_{s=i}^{k} f(T^s x) \geq \alpha.$$

By a maximal block we mean a block that is not included in any larger block. It is easy to see that

(a) any two maximal blocks are disjoint;

(b) no block starts with 1 (because $A_1^j \subset D = X - E$);

(c) if the integers from k to l form a maximal block, then

$$(1/(k - l + 1)) \sum_{s=l}^{k} f(T^s x) < 2\alpha$$

(otherwise we could extend the block by adding $l - 1$).

(4) Let C be the union of all the points y such that we can find integers i and j (depending on y) and $T^{-i}y \in A_1^j$, $i < M_j$ and i is in a maximal block (for $T^{-i}y$ and A_1^j). We then have $m(C)\cdot\alpha \geq \frac{1}{2}\int_C f$.

We get (4) as follows: We can write A_1^j as the disjoint union of sets ${}^rA_1^j$ where any two points in ${}^rA_1^j$ have the same maximal blocks. If the integers from l to k form a maximal block for all points in ${}^rA_1^j$, then 3(c) gives us

$$\int_{{}^rA_1^j} \sum_{i=l}^{k} f(T^i x) \leq 2\alpha(k - l + 1)m({}^rA_1^j).$$

Since T is measure preserving, this gives

$$\bigcup_{i=l}^{k} \int_{T^i({}^rA_1^j)} f(x) \leq 2\alpha m\left(\bigcup_{i=l}^{k} T^i({}^rA_1^j)\right).$$

Now sum over all the maximal blocks for ${}^rA_1^j$, and then over all the ${}^rA_1^j$ to get (4).

(5) Since $C \subset E$, (4) gives $M(E)\cdot\alpha \geq \frac{1}{2}\int_C f$. Since we assumed that $f = 0$ on $X - F$ and since $F \subset C$ we get $\int_C f = \int_F f$.

Proof of Theorem (a). (1) Let g_i be the function that is 2^i on the set where $g \geq 2^i$ and 0 elsewhere.

(2) $g \leq \sup\ (\sum_{i=0}^{\infty} 2g_i),\ 1]$.

(3) $g \geq \sum_{i=0}^{\infty} \frac{1}{2} g_i$.

(4) The standard maximal inequality shows that for all α large enough $m(X - E) \neq 0$ and hence, Proposition 1 holds. Applying it, we get that there is an N such that for $i \geq N$, $\int f_i^* \geq \sum_{l>i} r_l$ where

$$r_l = \int_{E_l} [\inf (f, 2^{l+1}) - 2^l]$$

and E_l is the set where $f > 2^l$.

(5) If we sum over i in (4) we get

$$\int \sum_{i=N}^{\infty} f_i^* \geq \sum_{i=N}^{\infty} (i - N) r_i.$$

(6) We next note the following simple fact:

$$\sum_{i=0}^{\infty} (i + 1) r_i \geq \int [f(x) \cdot \log f(x)]^+$$

(g^+ is the positive part of g).

We now put (3), (5) and (6) together to get

$$\begin{aligned} 2 \int f^* &\geq \int \sum_{i=N}^{\infty} f_i^* \\ &\geq \sum_{i=0}^{\infty} (i + 1) r_i - (N + 1) \sum_{i=N}^{\infty} r_i - \sum_{i=0}^{N-1} (i + 1) r_i \\ &\geq \int [f(x) \log f(x)]^+ - (N + 1) \int f. \end{aligned}$$

Proof of Theorem (b). (1) Define g_{-i} as the function that is 2^{-i} on the set where $g \geq 2^{-i}$ and 0 elsewhere.

(2) The standard maximal inequality now tells us that $m(X - E) \neq 0$ for any α and hence we can apply proposition 1 for any α. We therefore get that there exists an N and $\beta > 0$ such that

$$\int f_{-i}^* > \beta \quad \text{for} \quad i > N.$$

(3) We use (2) and the fact that $f^* > \frac{1}{2} \sum_{i=1}^{\infty} f_{-i}^*$ to finish the proof.

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