PRIMARY COHOMOLOGY OPERATIONS FOR SIMPLICIAL LIE ALGEBRAS

BY

STEWART B. PRIDDY[1]

1. Introduction

Since Lie algebras play an important role in the theory of groups, it is not surprising that simplicial[2] Lie algebras have been successfully applied to the homotopy theory of simplicial groups (which is equivalent to the ordinary homotopy theory of CW-complexes [12]). The most notable such application is the recent computation [2] of a new E^1 term for the Adams spectral sequence.

The purpose of this paper is to study some of the cohomological aspects of simplicial Lie algebras, in particular, the primary cohomology operations. Briefly, we

A. *compute all primary cohomology operations,*

B. *define the analogue of the Steenrod algebra and determine its Hopf algebra structure,*

C. *compute the cohomology algebra of this Steenrod algebra.*

As an application we show that the cohomology algebra of result C is just the new E^1 term (mentioned above) of the Adams spectral sequence for the sphere spectrum.

Throughout the paper we work mod 2, i.e., with the category $\mathcal{L}$ of restricted Lie algebras over Z_2.

We begin (§2) by defining the cohomology groups, $H^*\mathfrak{g}$, of a simplicial restricted Lie algebra $\mathfrak{g}$ and showing that these groups are (Lie) homotopy invariants. We then show that $H^*\mathfrak{g}$ has the structure of a commutative algebra. In §3 we show that the cohomology groups are representable by means of simplicial Eilenberg-MacLane Lie algebras $\mathrm{K}(Z_2, n)$. Thus, the elements of the cohomology groups $H^*K(Z_2, n)$ are in one-to-one correspondence with the primary cohomology operations. A similar discussion is given for stable operations. The Hopf algebra structure of $H^*K(Z_2, n)$ is then investigated in §4, where we show that $H^*K(Z_2, n)$ is isomorphic, as a Hopf algebra, to the ordinary cohomology algebra of $K(Z_2, n)$. The "Steenrod operations", Sq^i, are introduced in §5 and shown to enjoy most of the properties of the Steenrod operations for ordinary cohomology, the only exception being that Sq^0 is identically zero. In §6 we use these Steenrod operations to describe the polynomial generators for the algebra structure of $H^*K(Z_2, n)$ and thus we obtain result A.

Received July 11, 1968.

[1] National Science Foundation Graduate Fellow. This paper is part of the author's doctoral thesis written at the Massachusetts Institute of Technology.

[2] Formerly called semi-simplicial.

Section 7 is devoted to result B. We define the "Steenrod algebra" $\mathfrak{A}(\mathfrak{L})$ for simplicial restricted Lie algebras. The Hopf algebra structure of $\mathfrak{A}(\mathfrak{L})$ is very different from that of the ordinary Steenrod algebra, in fact $\mathfrak{A}(\mathfrak{L})$ is bigraded. The dual algebra $\mathfrak{A}^*(\mathfrak{L}) = \text{Hom}\ (\mathfrak{A}(\mathfrak{L}), Z_2)$ is not a polynomial algebra and contains divisors of zero.

In §8 an explicit calculation of the algebra $\text{Ext}_{\mathfrak{A}(\mathfrak{L})}(Z_2, Z_2)$ is given. The bigraded structure of $\mathfrak{A}(\mathfrak{L})$ induces a trigradation on $\text{Ext}_{\mathfrak{A}(\mathfrak{L})}(Z_2, Z_2)$. Using this extra gradation and the dual of the normalized bar construction we are able to find a set of generators and relations for $\text{Ext}_{\mathfrak{A}(\mathfrak{L})}(Z_2, Z_2)$, hence result C.

Our application is given in §9 where we show that $\text{Ext}_{\mathfrak{A}(\mathfrak{L})}(Z_2, Z_2)$ is the new E^1 term of the Adams spectral sequence for the sphere spectrum. To do this an Adams spectral sequence $\{E^r\}$, with products, is constructed for which

$$E^2_{*,*} = \text{Ext}^{*,*}_{\mathfrak{A}(\mathfrak{L})}(Z_2, Z_2) \Rightarrow \pi_* LAS.$$

This spectral sequence is easily shown to be degenerate ($E^2 = E^\infty$) and our result follows.

I would like to sincerely thank my thesis advisor Daniel M. Kan for his constant encouragement and for his many valuable suggestions which have materially improved the exposition and content of this paper. Also I wish to thank Daniel G. Quillen, who first suggested to me the problem of investigating the Steenrod algebra for simplicial Lie algebras, and who already knew several of the results of this paper (notably Theorem 6.1) before I proved them. His tract *Homotopical algebra* [17] has greatly influenced my point of view. Finally, I thank Edward B. Curtis who served as my advisor until his leave of absence.

2. Cohomology groups

Our aim in this section is to define (co-)homology groups for simplicial restricted Lie algebras ($s\mathfrak{L}$), show them to be homotopy invariant functors and then introduce cup products. We also establish the long exact (co-)homology sequence of an inclusion $\mathfrak{h} \subset \mathfrak{g}$ and define the suspension homomorphism. First, we recall the following.

2.1. *Cohomology groups for simplicial groups.* The (mod 2) *homology groups* of a simplicial group G are given by

$$H_* G = \pi_* \bar{W} Z_2 G$$

where $\bar{W}$ is the Eilenberg-MacLane functor (defined below in 2.3) and Z_2 is the prolongation (i.e., dimension-wise application) of the group algebra functor. The (mod 2) *cohomology groups* of G are then given by

$$H^*G = \text{Hom}\ (H_* G, Z_2)$$

We follow an analogous procedure for simplicial restricted Lie algebras $\mathfrak{g}$ in $s\mathfrak{L}$.

2.2. *The universal enveloping algebra.* Let U be the prolongation of the universal enveloping algebra functor, i.e. $(U\mathfrak{g})_n = \operatorname{Tens} \mathfrak{g}_n / I_n$ where I_n is the ideal of the tensor algebra generated by elements of the form

$$[g, g'] - (g \otimes g' - g' \otimes g) \quad \text{and} \quad g^2 - g \otimes g$$

for $g, g' \in \mathfrak{g}_n$. The natural augmentation of the tensor algebra provides $U\mathfrak{g}$ with an augmentation $\varepsilon : U\mathfrak{g} \to K(Z_2, 0)$.

Note that $U\mathfrak{g}$ *is a simplicial primitively generated Hopf algebra* [14; §6]. The diagonal map $\mathfrak{g} \to \mathfrak{g} \times \mathfrak{g}$, given by $g \to (g, g)$, induces a diagonal map

$$\Delta : U\mathfrak{g} \to U(\mathfrak{g} \times \mathfrak{g}) \simeq U\mathfrak{g} \otimes U\mathfrak{g}$$

which is characterized by (i) Δ is a simplicial Z_2-algebra homomorphism and (ii) $\Delta(g) = 1 \otimes g + g \otimes 1$ for $g \in \mathfrak{g}$.

2.3. *The Eilenberg-MacLane functor* $\bar{W}$. Recall [15; p. 12–05] that if R is an augmented simplicial algebra over Z_2 then

$$(\bar{W}R)_0 = Z_2, \quad (\bar{W}R)_{q+1} = R_q \otimes R_{q-1} \otimes \cdots \otimes R_0, \qquad q \geq 0,$$

with face and degeneracy operators given by

$$\partial_0(r_0) = \varepsilon(r_0) = \partial_1(r_0),$$

$$\partial_0(r_q \otimes \cdots \otimes r_0) = \varepsilon(r_q) \cdot r_{q-1} \otimes \cdots \otimes r_0, \qquad q > 0,$$

$$\partial_{i+1}(r_q \otimes \cdots \otimes r_0)$$
$$= \partial_i r_q \otimes \cdots \otimes \partial_1 r_{q-i+1} \otimes (\partial_0 r_{q-i}) \cdot r_{q-i-1} \otimes \cdots \otimes r_0 \qquad i \geq 0, q > 0,$$

$$s_0(r_q \otimes \cdots \otimes r_0) = 1_{q+1} \otimes r_q \otimes \cdots \otimes r_0, \qquad q \geq 0,$$

$$s_{i+1}(r_q \otimes \cdots \otimes r_0)$$
$$= s_i r_q \otimes \cdots \otimes s_0 r_{q-i} \otimes 1_{q-i} \otimes r_{q-i-1} \otimes \cdots \otimes r_0, \qquad i, q \geq 0.$$

If R is an augmented simplicial cocommutative Hopf algebra over Z_2 (e.g. $R = U\mathfrak{g}$) then $\bar{W}R$ *is a simplicial commutative coalgebra over* Z_2. The diagonal map of $\bar{W}R$, $\Delta : \bar{W}R \to \bar{W}R \otimes \bar{W}R$ is induced by the diagonal map of R.

2.4. *Cohomology groups for simplicial Lie algebras.* If $\mathfrak{g}$ is a simplicial Lie algebra in $s\mathfrak{L}$ define the *homology* and *cohomology groups* of $\mathfrak{g}$ to be

$$H_0 \mathfrak{g} = 0, \quad H_q \mathfrak{g} = \pi_q \bar{W} U\mathfrak{g}, \qquad q > 0, \quad H^*\mathfrak{g} = \operatorname{Hom}(H_* \mathfrak{g}, Z_2).$$

Clearly a map $f : \mathfrak{g} \to \mathfrak{h}$ in $s\mathfrak{L}$ induces a map $H^*f : H^*\mathfrak{h} \to H^*\mathfrak{g}$ and so H^* *is a functor*. A corresponding statement holds for H_*. We use reduced (co-)homology groups ($H_0 \mathfrak{g} = H^0\mathfrak{g} = 0$) because (i) they are well behaved with respect to the suspension homomorphism (2.12) and (ii) the functor H^* is representable (3.4).

2.5. *Lie homotopy.* The maps $f, g : \mathfrak{g} \to \mathfrak{h}$ of $s\mathfrak{L}$ are called *Lie homotopic* if there is a map $h : \mathfrak{g} \otimes \Delta_1 \to \mathfrak{h}$ in $s\mathfrak{L}$ such that $h \circ \varepsilon_0 = f$ and $h \circ \varepsilon_1 = g$ for the canonical maps

$$\Delta_0 \overset{\varepsilon_0}{\underset{\varepsilon_1}{\rightrightarrows}} \Delta_1$$

of the standard complexes Δ_0, Δ_1 (see [12]).

To see that H^*f (or $H_* f$) *depends only on the Lie homotopy class of* f, it suffices to prove

2.6. *If the maps* $f, g : \mathfrak{g} \to \mathfrak{h}$ *of* $s\mathfrak{L}$ *are Lie homotopic then the maps* $\bar{W}Uf, \bar{W}Ug : \bar{W}U\mathfrak{g} \to \bar{W}U\mathfrak{h}$ *are homotopic as maps of simplicial commutative coalgebras over* Z_2.

Proof. The functor U is defined by prolongation and hence by a theorem of Kan [10; Theorem 5.3] preserves homotopies. Thus it suffices to show that $\bar{W}$ preserves homotopies. Suppose given a homotopy $\gamma : A \otimes \Delta_1 \to B$ of the simplicial cocommutative Hopf algebra maps $\alpha, \beta : A \to B$. Now for $n \geq 0$,

$$(\bar{W}A \otimes \Delta_1)_n = V_{\sigma \epsilon (\Delta_1)_n} (\bar{W}A)_n(\sigma)$$

that is, a sum of copies of $(\bar{W}A)_n$ one for each simplex $\sigma \epsilon (\Delta_1)_n$. Define a map $\delta_n(\sigma)$ on each summand $(\bar{W}A)_n(\sigma)$ by

$$a_{n-1} \otimes \cdots \otimes a_0 \to a_{n-1}(\partial_0 \sigma) \otimes \cdots \otimes a_0(\partial_0^n \sigma).$$

Clearly $\delta_n(\sigma)$ is a homomorphism of commutative coalgebras. Let $\delta_n = V_{\sigma \epsilon (\Delta_1)_n} \delta_n(\sigma)$. It is easily verified that (i) δ is a simplicial map and (ii) $\bar{W}\gamma \circ \delta$ is a homotopy of $\bar{W}\alpha$ and $\bar{W}\beta$.

2.7. *Weak equivalence.* A map $f : \mathfrak{g} \to \mathfrak{h}$ of $s\mathfrak{L}$ is called a *weak equivalence* if $\pi_* f : \pi_* \mathfrak{g} \to \pi_* \mathfrak{h}$ is an isomorphism.

The following proposition shows that *the functors* H^* *and* H_* *are weak homotopy invariants.*

2.8. *Let* $f : \mathfrak{g} \to \mathfrak{h}$ *be a weak equivalence then*

$$\pi_* \bar{W}Uf \quad \pi_* \bar{W}U\mathfrak{g} \to \pi_* \bar{W}U\mathfrak{h}$$

is an isomorphism.

Proof. First we show that $Uf : U\mathfrak{g} \to U\mathfrak{h}$ is a weak equivalence. Filter $U\mathfrak{g}$ and $U\mathfrak{h}$ by their Lie filtrations [14; Definition 6.9]. Since f preserves filtrations it induces a map of the associated spectral sequences

$$\begin{array}{ccc} E^1U\mathfrak{g} & \xrightarrow{E^1Uf} & E^1U\mathfrak{h} \\ \Downarrow & & \Downarrow \\ \pi_* U\mathfrak{g} & \xrightarrow{\pi_* Uf} & \pi_* U\mathfrak{h}. \end{array}$$

According to the Poincaré-Birkhoff-Witt theorem [14, Theorem 5.15], $E^0U\mathfrak{g} = \operatorname{Sym}\mathfrak{g}$ and $E^0U\mathfrak{h} = \operatorname{Sym}\mathfrak{h}$, where Sym is the prolongation of the mod 2 restricted symmetric algebra functor. Now f is a homotopy equivalence of simplicial Z_2-modules, hence by a theorem of Dold, $E^1 Uf$ is an isomorphism. The Lie filtrations of $U\mathfrak{g}$ and $U\mathfrak{h}$ are complete and bounded below, hence the spectral sequences converge and the induced map $E^\infty Uf$ at E^∞ is also an isomorphism. Thus Uf is a weak equivalence.

Now applying a theorem of Cartan, later corrected by Moore ([3; Theorem 2], then see [16; p. 13–01]) it follows that

$$\pi_* \bar{W} Uf : \pi_* \bar{W} U\mathfrak{g} \to \pi_* \bar{W} U\mathfrak{h}$$

is an isomorphism.

2.9. *The algebra structure of* $H^*\mathfrak{g}$. The simplicial commutative coalgebra structure of $\bar{W}U\mathfrak{g}$ *induces a natural commutative algebra structure on* $H^*\mathfrak{g}$. We note that this algebra *does not have a unit* since our cohomology groups are reduced. The *cohomology product* (*cup product*) is given by

$$\begin{aligned} \cup : H^i\mathfrak{g} \otimes H^j\mathfrak{g} &= \operatorname{Hom}(\pi_i \bar{W}U\mathfrak{g}, Z_2) \otimes \operatorname{Hom}(\pi_j \bar{W}U\mathfrak{g}, Z_2) \\ &\to \operatorname{Hom}(\pi_{i+j}(\bar{W}U\mathfrak{g} \otimes \bar{W}U\mathfrak{g}), Z_2) \\ &\xrightarrow{\operatorname{Hom}(\pi_{i+j}\Delta, Z_2)} \operatorname{Hom}(\pi_{i+j} \bar{W}U\mathfrak{g}, Z_2) = H^{i+j}\mathfrak{g} \end{aligned}$$

2.10. *The Hopf algebra structure of* $H^*\mathfrak{a}$, $\mathfrak{a}$ *abelian.* If $\mathfrak{a}$ is a simplicial *abelian* Lie algebra in $s\mathfrak{L}$ then $U\mathfrak{a}$ is a simplicial commutative, cocommutative Hopf algebra and so therefore is $\bar{W}U\mathfrak{a}$. Thus, *for* $\mathfrak{a}$ *abelian,* $H^*\mathfrak{a}$ *has the structure of a commutative, cocommutative Hopf algebra* (*without unit or counit*) *over* Z_2.

2.11. *The long exact* (*co-*)*homology sequence of an inclusion.* If $\mathfrak{h} \subset \mathfrak{g}$ are simplicial Lie algebras in $s\mathfrak{L}$ then $U\mathfrak{h} \subset U\mathfrak{g}$ and

$$0 \to \bar{W}U\mathfrak{h} \to \bar{W}U\mathfrak{g} \to \bar{W}U\mathfrak{g}/\bar{W}U\mathfrak{h} \to 0$$

is exact. Hence there is a long exact sequence in homology (also cohomology)

$$\cdots \to \pi_{i+1} \frac{\bar{W}U\mathfrak{g}}{\bar{W}U\mathfrak{h}} \xrightarrow{\partial_{i+1}} \pi_i \bar{W}U\mathfrak{h} \to \pi_i \bar{W}U\mathfrak{g} \to \pi_i \frac{\bar{W}U\mathfrak{g}}{\bar{W}U\mathfrak{h}} \xrightarrow{\partial_i} \pi_{i-1} \bar{W}U\mathfrak{h} \to \cdots$$

where ∂_* is the connecting homomorphism (boundary).

2.12. *The suspension homomorphism.* If

$$\mathfrak{h} \xrightarrow{i} \mathfrak{c} \xrightarrow{j} \mathfrak{g}$$

is a fibration in $s\mathfrak{L}$ and $\mathfrak{c}$ is contractible then the *suspension homomorphism* (for cohomology) $\sigma : H^{q+1}\mathfrak{g} \to H^q\mathfrak{h}$ for $q \geq 0$ is given by

$$\sigma = \delta^{-1} j^* : (\pi_{q+1} \bar{W}U\mathfrak{g})^* \to \left(\pi_{q+1} \frac{\bar{W}U\mathfrak{c}}{\bar{W}U\mathfrak{h}}\right)^* \to (\pi_q \bar{W}U\mathfrak{h})^*$$

where

$$\bar{\jmath} : \pi_{q+1} \frac{\bar{W}U\mathfrak{c}}{\bar{W}U\mathfrak{h}} \to \pi_{q+1} \bar{W}U\mathfrak{g}$$

is induced by $\bar{W}Uj$ and δ is the transpose of ∂, which is an isomorphism since $\mathfrak{c}$ is contractible. There is a dual notion for homology but our concern in later sections will be stable cohomology operations and so we shall only consider the suspension homomorphisms for cohomology.

3. Representability and primary cohomology operations

In this section we introduce and classify primary cohomology operations for $s\mathfrak{L}$. First, we show that the cohomology functor H^* is representable by means of Eilenberg-MacLane complexes $K(Z_2, n)$ for $s\mathfrak{L}$; then, using this fact, show that the primary cohomology operations are in one-to-one correspondence with $H^*K(Z_2, n)$.

A similar discussion is given for stable primary cohomology operations.

3.1. *Eilenberg-MacLane complexes for* $s\mathfrak{L}$. Recall that for each $n \geq 0$, the simplicial abelian group $K(Z_2, n) = \bar{Z}_2 S_n$ (where $\bar{Z}_2$ is the prolongation of the free Z_2-module functor with the basepoint set equal to zero) is an Eilenberg-MacLane complex for the category of simplicial groups, $s\mathcal{G}$. The *Eilenberg-MacLane complex* $K(Z_2, n)$ *of* $s\mathfrak{L}$ is the simplicial abelian group $K(Z_2, n)$ endowed with the structure of a simplicial abelian restricted Lie algebra (i.e., trivial Lie bracket and square operators).

In the next section (4.1) we show that $H^{n+1}K(Z_2, n) = Z_2$. The non-zero class (or "fundamental" class) ι_{n+1} is represented by

$$\dot{\imath}_n \otimes 1_{n-1} \otimes \cdots \otimes 1_0$$

where i_n is the the nondegenerate n-simplex of S_n.

In order to state our representability theorem we need the concept of

3.2. *Free simplicial Lie algebras. A free simplicial Lie algebra* $\mathfrak{g}$ of $s\mathfrak{L}$ is an object of $s\mathfrak{L}$ such that

(i) For $n \geq 0$, there is a Z_2-module M_n with $\mathfrak{g}_n = LM_n$, where L is the prolongation of the free restricted Lie algebra functor.

(ii) For $n \geq i \geq 0$, $y \in M_n$ implies $s_i y \in M_{n+1}$.

If $\mathfrak{g}$ is a simplicial Lie algebra, then there is a free simplicial Lie algebra $\hat{\mathfrak{g}}$ and a weak equivalence $f : \hat{\mathfrak{g}} \to \mathfrak{g}$ which is also a fibration, see [17; Chpt. II, p. 4,11]. If $\mathfrak{h}$ is a free simplicial Lie algebra, then f induces a bijection $[\mathfrak{h} \to \hat{\mathfrak{g}}]^*_{s\mathfrak{L}} \to [\mathfrak{h} \to \mathfrak{g}]_{s\mathfrak{L}}$; see [17; Chpt. I, p. 1.10].

A free simplicial Lie algebra $\hat{\mathfrak{g}}$ together with a weak equivalence $\hat{\mathfrak{g}} \to \mathfrak{g}$ which is a fibration is called a *free resolution of* $\mathfrak{g}$.

3.3. *Representability*. The following proposition shows that H^* is a representable functor.

3.4. *Let* $\mathfrak{g}$ *be a simplicial Lie algebra in* $s\mathfrak{L}$ *and let* $\hat{\mathfrak{g}} \to \mathfrak{g}$ *be a free resolution of* $\mathfrak{g}$. *Then there is a one-to-one correspondence*

$$H^{n+1}\mathfrak{g} \leftrightarrow [\hat{\mathfrak{g}} \to K(Z_2, n)]_{s\mathfrak{L}}, \qquad n \geq 0.$$

This correspondence is such that if $\gamma \leftrightarrow f_\gamma$ *then* $f_\gamma^*(\iota_{n+1}) = \gamma$.

Proposition 3.4 is an immediate consequence of 2.8 and 3.5 and 3.6 below.

3.5. *If* $\mathfrak{g}$ *is a free simplicial Lie algebra in* $s\mathfrak{L}$ *then there is a natural isomorphism*

$$H_* \mathfrak{g} \simeq \pi_{*-1} \operatorname{Ab} \mathfrak{g}, \qquad * \geq 1,$$

where Ab *is the abelianization functor.*

3.6. *Let* $\mathfrak{g}$ *be a free simplicial Lie algebra in* $s\mathfrak{L}$. *Then there is a one-to-one correspondence*

$$\operatorname{Hom}(\pi_n \operatorname{Ab} \mathfrak{g}, Z_2) \leftrightarrow [\mathfrak{g} \to K(Z_2, n)]_{s\mathfrak{L}}, \qquad n \geq 0,$$

such that if $\gamma \leftrightarrow f_\gamma$ *then* $(\pi_n \operatorname{Ab} f_\gamma)^*(i_n^*) = \gamma$.

Proof of 3.5. The proof depends on the following spectral sequence of Quillen [17; Chpt. II, Theorem 6(a)]:

$$E^2_{p,q} = \pi_p(\operatorname{Tor}_q^{U\mathfrak{g}}(K(Z_2, 0), K(Z_2, 0)) \Rightarrow \pi_{p+q}(K(Z_2, 0) \otimes^L_{U\mathfrak{g}} K(Z_2, 0)) \qquad \text{for } p, q \geq 0$$

We will need the following information about the E^2 term:

$$(*) \qquad \begin{aligned} \operatorname{Tor}_q^{U\mathfrak{g}_n}(Z_2, Z_2) &= Z_2 && \text{if } q = 0 \\ &= \operatorname{Ab} \mathfrak{g}_n && \text{if } q = 1 \\ &= 0 && \text{if } q > 1. \end{aligned}$$

(See [4; p. 271].)

Using this we will derive the proposition. The term

$$K(Z_2, 0) \otimes^L_{U\mathfrak{g}} K(Z_2, 0)$$

is by definition equal to $P \otimes_{U\mathfrak{g}} Q$ where P (resp. Q) is a projective resolution of $K(Z_2, 0)$ as a right (resp. left) $U\mathfrak{g}$-module. However,

$$\pi_*(P \otimes_{U\mathfrak{g}} Q) \xrightarrow{\sim} \pi_*(K(Z_2, 0) \otimes_{U\mathfrak{g}} Q)$$

and so we may dispense with P. Let $Q = WU\mathfrak{g}$ the acyclic free left $U\mathfrak{g}$-module which is the total space of the Eilenberg-MacLane construction ($U\mathfrak{g}$, $WU\mathfrak{g}$, $\bar{W}U\mathfrak{g}$) [15; p. 12–05].

By $(*)$, the spectral sequence collapses and we have for $p \geq 0$,

$$\begin{aligned} \pi_p(\operatorname{Ab} \mathfrak{g}) &= \pi_p(\operatorname{Tor}_1^{U\mathfrak{g}}(K(Z_2, 0), K(Z_2, 0)) \\ &\simeq \pi_{p+1}(K(Z_2, 0) \otimes_{U\mathfrak{g}} WU\mathfrak{g}) = \pi_{p+1} \bar{W}U\mathfrak{g} \end{aligned}$$

The naturality of this isomorphism follows from the naturality of the spectral sequence.

Proof of 3.6. Let $s\mathfrak{M}$ be the category of simplicial Z_2-modules. Now

$$\mathrm{Hom}\ (\pi_* \mathrm{Ab}\ \mathfrak{g}, Z_2) \leftrightarrow [\mathrm{Ab}\ \mathfrak{g} \to K(Z_2, n)]_{s\mathfrak{M}}$$

and this correspondence is such that if $\alpha \leftrightarrow f_\alpha$ then $(\pi_n \mathrm{Ab} f_\alpha)^* (i_n^*) = \alpha$.

Since $K(Z_2, n)$ is an abelian object of $s\mathfrak{L}$,

$$\mathrm{Hom}_{s\mathfrak{L}}\ (\mathfrak{g}, K(Z_2, n)) = \mathrm{Hom}_{s\mathfrak{M}}\ (\mathrm{Ab}\ \mathfrak{g}, K(Z_2, n))$$

and

$$\mathrm{Hom}_{s\mathfrak{L}}\ (\mathfrak{g} \otimes \Delta_1, K(Z_2, n)) = \mathrm{Hom}_{s\mathfrak{M}}\ (\mathrm{Ab}\ (\mathfrak{g} \otimes \Delta_1), K(Z_2, n))$$

$$= \mathrm{Hom}_{s\mathfrak{M}}\ (\mathrm{Ab}\ \mathfrak{g} \otimes \Delta_1, K(Z_2, n)).$$

Hence $\mathrm{Hom}\ (\pi_n \mathrm{Ab}\ \mathfrak{g}, Z_2) = [\mathrm{Ab}\ \mathfrak{g} \to K(Z_2, n)]_{s\mathfrak{M}} = [\mathfrak{g} \to K(Z_2, n)]_{s\mathfrak{L}}$.

3.7. *Primary cohomology operations.* A *primary cohomology operation* T *of type* $(n; q)$ is a natural transformation of functors

$$T : H^n \to H^q.$$

The standard argument using 3.4 yields

3.8. *The primary cohomology operations for* $s\mathfrak{L}$ *of type* $(n; q)$ *are in one-to-one correspondence with* $H^q K(Z_2, n - 1), n > 0$.

Denote this correspondence by $T_u \leftrightarrow u$.

3.9. *Stable cohomology operations. A stable primary cohomology operation of type* $i \geq 0$ is a sequence $\{T_n\}_{n \geq 0}$ of primary cohomology operations of type $(n, n + i)$ with the property that if $\mathfrak{h} \to \mathfrak{c} \to \mathfrak{g}$ is a fibration in $s\mathfrak{L}$ and $\mathfrak{c}$ is contractible then

$$\begin{array}{ccc} H^{n+1}\mathfrak{g} & \xrightarrow{T_{n+1}} & H^{n+1+i}\mathfrak{g} \\ \downarrow \sigma & & \downarrow \sigma \\ H^n\mathfrak{h} & \xrightarrow{T_n} & H^{n+i}\mathfrak{h} \end{array}$$

commutes, where σ is the suspension homomorphism of 2.12.
Stable operations may be classified in the following manner.

3.10. *The stable primary cohomology operations for* $s\mathfrak{L}$ *of type* $i \geq 0$ *are in one-to-one correspondence with sequences* $\{u_0 = 0, u_1, u_2, \cdots\}$ *of elements* $u_j \in H^{j+i}K(Z_2, j - 1)$ *such that* $\sigma u_{j+1} = u_j$, *where* σ *is the suspension homomorphism associated with the fibration*

$$K(Z_2, j - 1) \to WK(Z_2, j - 1) \to K(Z_2, j).$$

Proof. Given a stable primary operation $\{T_j\}$ of type $i \geq 0$, consider the sequence

$$u_0 = 0, \quad u_j = T_j(\iota_j) \in H^{j+i}K(Z_2, j-1), \qquad j > 0,$$

where ι_j is the "fundamental class" of $H^jK(Z_2, j-1)$. Since the Eilenberg-MacLane Lie algebras $K(Z_2, j)$ are connected by fibrations

$$K(Z_2, j) \to WK(Z_2, j) \to K(Z_2, j+1)$$

with $WK(Z_2, j)$ contractible, the stability of $\{T_j\}$ implies $\sigma u_{j+1} = u_j$. By 3.4, this correspondence is clearly one-to-one.

Now let $\{u_0 = 0, u_1, u_2, \cdots\}$ be a sequence as described in the hypothesis and let $\mathfrak{h} \to \mathfrak{c} \to \mathfrak{g}$ be a fibration with $\mathfrak{c}$ contractible (in which we may assume $\mathfrak{h}$, $\mathfrak{c}$, and $\mathfrak{g}$ are free). Let $f_z : \mathfrak{g} \to K(Z_2, j+1)$ represent $z \in H^{j+2}\mathfrak{g}$. By a lifting argument there exists a map $g : \mathfrak{c} \to WK(Z_2, j)$ such that

$$\begin{array}{ccccc} \mathfrak{h} & \longrightarrow & \mathfrak{c} & \longrightarrow & \mathfrak{g} \\ \downarrow & & \downarrow g & & \downarrow f_z \\ K(Z_2, j) & \to & WK(Z_2, j) & \to & K(Z_2, j+1) \end{array}$$

commutes. Denote by $f_{\sigma z}$ the map $\mathfrak{h} \to K(Z_2, j)$ which is the restriction of g. Now by naturality the following diagram commutes

$$\begin{array}{ccc} H^*K(Z_2, j) & \xrightarrow{f_{\sigma z}^*} & H^*\mathfrak{h} \\ \uparrow \sigma & & \uparrow \sigma \\ H^*K(Z_2, j+1) & \xrightarrow{f_z^*} & H^*\mathfrak{g} \end{array}$$

Hence $f_{\sigma z}$ represents $\sigma z \in H^{j+1}\mathfrak{h}$ and so $T_u\, \sigma(z) = \sigma T_v(z)$.

4. The Hopf algebra structure of $H^*K(Z_2, n)$

We now relate the (mod 2) primary cohomology operations for $s\mathfrak{L}$ to the more familiar (mod 2) primary cohomology operations for $s\mathcal{G}$ (simplicial groups). For convenience, in this section we shall adjoin a unit (in dimension zero) to the cohomology algebra of the Eilenberg-MacLane complex $K(Z_2, n)$ for $s\mathfrak{L}$ thus making it a Hopf algebra (see 2.10). We shall show that this Hopf algebra is isomorphic to the cohomology algebra of the Eilenberg-MacLane complex $K(Z_2, n)$ of $s\mathcal{G}$.

Recall (2.1) that as a simplicial group the (mod 2) cohomology groups of $K(Z_2, n)$ are given by

$$H_{\mathcal{G}}^*\, K(Z_2, n) = \operatorname{Hom}\,(\pi_*\, \bar{W}Z_2\, K(Z_2, n), Z_2)$$

while (2.4) as a simplicial abelian Lie algebra the (mod 2) cohomology groups (with unit) of $K(Z_2, n)$ are given by

$$H_{\mathfrak{L}}^*\, K(Z_2, n) = \operatorname{Hom}\,(\pi_*\, \bar{W}UK(Z_2, n), Z_2).$$

Here we temporarily use the subscripts $\mathcal{G}$ and $\mathcal{L}$ to distinguish the group and Lie algebra cases. The main result of this section is

4.1. *There is an isomorphism of Hopf algebras over* Z_2

$$H^*_{\mathcal{L}}\, K(Z_2\,,n) \approx H^*_{\mathcal{G}}\, K(Z_2\,,n).$$

This result is an immediate consequence of 4.3 and 4.4 below and the fact that $K(Z_2\,,n) = \bar{Z}_2\, S_n$ (see 3.1)

4.2. Proposition 4.1 is somewhat unexpected. The "algebraic" diagonal (2.2, 2.3) $\Delta_{\bar{W}UK}$ for $\bar{W}UK(Z_2\,,n)$ is fundamentally different from the "geometric" diagonal $\Delta_{\bar{W}Z_2K}$ for $\bar{W}Z_2\,K(Z_2\,,n)$. The diagonal $\Delta_{\bar{W}UK}$ is induced by $k \to 1 \otimes k + k \otimes 1$ while $\Delta_{\bar{W}Z_2K}$ is induced by $k \to k \otimes k$. Proposition 4.1 states that $\Delta_{\bar{W}UK}$ and $\Delta_{\bar{W}Z_2K}$ induce the same cup product. In the next section (§5) we shall see that they do *not* yield the same Steenrod operations.

4.3. *If* X, $*$ *is a simplicial set with basepoint then there is an isomorphism of simplicial augmented* Z_2-*algebras*

$$\phi : Z_2\, \bar{Z}_2\, X \simeq U\bar{Z}_2\, X$$

where $\bar{Z}_2$ *is the prolongation of the free* Z_2-*module functor with the basepoint set equal to zero.*

4.4. *If* X, $*$ *is a simplicial set with basepoint then the geometric and algebraic diagonals for* $\bar{Z}_2\, X$ *induce the same cup product structure, i.e., the diagram*

$$\begin{array}{ccc}
N\bar{W}Z_2\bar{Z}_2X & \xrightarrow{\quad N\bar{W}\phi \quad} & N\bar{W}U\bar{Z}_2X \\
\Big\downarrow N\Delta_{\bar{W}Z_2\bar{Z}_2X} & & \Big\downarrow N\Delta_{\bar{W}U\bar{Z}_2X} \\
N(\bar{W}Z_2\bar{Z}_2X \otimes \bar{W}Z_2\bar{Z}_2X) & & N(\bar{W}U\bar{Z}_2X \otimes \bar{W}U\bar{Z}_2X) \\
\Big\downarrow f & & \Big\downarrow f \\
N\bar{W}Z_2\bar{Z}_2X \otimes N\bar{W}Z_2\bar{Z}_2X & \xrightarrow{\quad N\bar{W}\phi \otimes N\bar{W}\phi \quad} & N\bar{W}UZ_2X \otimes N\bar{W}UZ_2X
\end{array}$$

commutes, where N *is the normalization functor* [15; p. 12–01] *and* f *is the Eilenberg-Zilber map* [15; p. 12–02]. *Moreover, all maps in the diagram are differential graded* Z_2-*algebra homomorphisms.*

Proof of 4.3. It suffices to show that if X is a set with basepoint $*$ then there is a natural isomorphism

$$\phi : Z_2\, \bar{Z}_2\, X \simeq U\bar{Z}_2\, X$$

of augmented Z_2-algebras. Let $X - \{*\} = \{x_\alpha\}$. The underlying set of $Z_2\, X$ is a Z_2-basis for the group algebra $Z_2\, \bar{Z}_2\, X$. The enveloping algebra $U\bar{Z}_2\, X$ is isomorphic to $Z_2[x_\alpha]/(x_\alpha^2)$, the truncated polynomial algebra in the variables x_α. Both algebras have natural augmentations:

$$\varepsilon_{Z_2\bar{Z}_2X} : Z_2\, \bar{Z}_2\, X \to Z_2$$

given by

$$\varepsilon_{Z_2 \bar{Z}_2 X}(1 \cdot z) = 1 \quad \text{for} \quad z \in \bar{Z}_2 X$$

and

$$\varepsilon_{U\bar{Z}_2 X} : U\bar{Z}_2 X \to Z_2$$

given by

$$\varepsilon_{U\bar{Z}_2 X}(1) = 1, \quad \varepsilon_{U\bar{Z}_2 X}(x_{i_1} \cdots x_{i_n}) = 0 \text{ for } x_{i_1} \cdots x_{i_n} \text{ a monomial.}$$

Define $\phi : Z_2 \bar{Z}_2 X \to U\bar{Z}_2 X$ by

$$1 \cdot (x_{i_1} + \cdots + x_{i_n}) \to \textstyle\prod_{j=1}^{n} (1 + x_{i_j}).$$

This defines ϕ on the basis for $Z_2 \bar{Z}_2 X$ and hence on $Z_2 \bar{Z}_2 X$. The map ϕ is easily verified to be an isomorphism. It is clear from the definition of ϕ that $\varepsilon_{Z_2 \bar{Z}_2 X} = \varepsilon_{U\bar{Z}_2 X} \phi$.

Proof of 4.4. To simplify the notation let $\Delta_{Z_2} = fN\Delta_{\bar{W}Z_2\bar{Z}_2 X}$ and $\Delta_U = fN\Delta_{\bar{W}U\bar{Z}_2 X}$. Recall that Δ_{Z_2} is just the familiar Alexander-Whitney diagonal map

$$\Delta_{Z_2}(\langle z_{n-1}, \cdots, z_0 \rangle) = \textstyle\sum_{i=0}^{n} \langle \tilde{\partial}^{n-i} z_{n-1}, \cdots, \tilde{\partial}^{n-i} z_{n-1} \rangle \otimes \langle z_{n-i-1}, \cdots, z_0 \rangle$$

where

$$\langle z_{n-1}, \cdots, z_0 \rangle \in (\bar{W}Z_2 \bar{Z}_2 X)_n \quad \text{and} \quad \tilde{\partial}^{n-i} z_j = \partial_{j-n+i+1} \cdots \partial_j z_j .$$

Now it may be verified that Δ_U is also given by the Alexander-Whitney formula. The commutativity of the diagram is then a trivial verification using the map ϕ of 4.3. With the exception of the map f all maps in the diagram are obviously multiplicative. A proof of the multiplicativity of f may be found in [8; Theorem 3.2].

5. The Steenrod operations for $s\mathfrak{L}$

The purpose of this section is to define (mod 2) Steenrod cohomology operations for $s\mathfrak{L}$ and to describe their properties.

5.1. *Dold's construction.* In [5] Dold defines a family of cohomology operations, Sq^i, $i \geq 0$, called *Steenrod operations*, for the category of simplicial commutative coalgebras over Z_2. These operations are defined with respect to the cohomology functor $H^* = \text{Hom}(\pi_*, Z_2)$ and they generalize Steenrod's original construction in that *they satisfy all the properties of the usual Steenrod operations except* Sq^0 *is not necessarily the identity.*

5.2. *The Steenrod operations for* $s\mathfrak{L}$. Now consider the category $\bar{W}Us\mathfrak{L}$ with objects $\bar{W}U\mathfrak{g}$ for $\mathfrak{g}$ in $s\mathfrak{L}$ and maps

$$\bar{W}U\mathfrak{g} \xrightarrow{\bar{W}Uf} \bar{W}U\mathfrak{h} \quad \text{for} \quad \mathfrak{g} \xrightarrow{f} \mathfrak{h}$$

in $s\mathfrak{L}$. For every object $\bar{W}U\mathfrak{g}$ there is a diagonal

$$\Delta : \bar{W}U\mathfrak{g} \to \bar{W}U\mathfrak{g} \otimes \bar{W}U\mathfrak{g}$$

induced by the natural diagonal of $U\mathfrak{g}$ (see 2.2, 2.3). The commutativity of Δ follows immediately from the commutativity of the diagonal for $U\mathfrak{g}$. Thus $\bar{W}Us\mathfrak{L}$ is a subcategory of the category of simplicial commutative coalgebras over Z_2 and hence inherits Steenrod operations. Since the cohomology functor (2.4) for $s\mathfrak{L}$ is precisely $H^* = \mathrm{Hom}\,(\pi_*, Z_2)$ for $\bar{W}Us\mathfrak{L}$, with $H^0 = 0$, we shall call the resulting operations, Sq^i, *the* (mod 2) *Steenrod operations for* $s\mathfrak{L}$.

5.3. *The* (mod 2) *Steenrod operations*, Sq^i, *for* $s\mathfrak{L}$ *satisfy*

(1) $Sq^i : H^* \to H^{*+1}$ *is a homomorphism and is natural* $(*, i \geq 0)$,

(2) $Sq^0 \equiv 0 : H^* \to H^*$,

(3) $Sq^n u = u^2$ *if* $\dim u = n$,

(4) $Sq^i u = 0$ *if* $\dim u < i$,

(5) (*Cartan formula*) $Sq^k(u \cdot v) = \sum_{i=0}^{k} Sq^i u \cdot Sq^{k-i} v$,

(6) (*Adem relations*) $Sq^a Sq^b = \sum_{i=0}^{[a/2]} \binom{b-1-j}{a-2j} Sq^{a+b-j} Sq^j \quad (0 < a < 2b)$,

(7) (*Stability*) $Sq^i \sigma = \sigma Sq^i$.

Parts (1) and (3)–(6) of 5.3 follow from Dold. The proofs of (7) and (2) are given in 5.4–6 and 5.7–11 respectively.

5.4. *Steenrod operations for quotient objects.* In certain circumstances Dold shows that the definition of the Sq^i can be extended to include quotient objects: Suppose K, L are simplicial commutative coalgebras over Z_2 and $L \subset K$ is a subcoalgebra; then Δ_K induces a diagonal for K/L:

$$\Delta_{K/L} : K/L \to (K \otimes K)/(L \otimes L) \to K/L \otimes K/L.$$

5.5 [5; p. 273, §5.8–9]. *If* L, K *and* Δ_K *satisfy the conditions of the preceding paragraph then the* Sq^i *are defined for* K/L *and satisfy*

$$Sq^i \delta = \delta\, Sq^i$$

where $\delta : H^*L \to H^{*+1}K/L$ *is the coboundary map.*

If $\mathfrak{h} \subset \mathfrak{g}$ are simplicial Lie algebras in $s\mathfrak{L}$ then $\bar{W}U\mathfrak{g} \subset \bar{W}U\mathfrak{h}$ and

$$\Delta_{\bar{W}U\mathfrak{g}}\, \bar{W}U\mathfrak{h} \subset \bar{W}U\mathfrak{h} \otimes \bar{W}U\mathfrak{h} \subset \bar{W}U\mathfrak{g} \otimes \bar{W}U\mathfrak{g}$$

hence

5.6. *The Steenrod operations for* $s\mathfrak{L}$ *are stable*, i.e.,

$$Sq^i \sigma = \sigma\, Sq^i.$$

5.7. *The operation* Sq^0. Dold gives an example [5; p. 282] of a category for which Sq^0 is identically zero, namely the category of simplicial symmetric algebras SM generated by simplicial modules over Z_2, with diagonal given by

$$SM \to SM \otimes SM, \qquad m \to 1 \otimes m + m \otimes 1, \qquad m \in M.$$

This example is closely related to the present work and can be used to prove a similar result about Sq^0 for $s\mathfrak{L}$. However, a more elementary description of

Sq^0 is obtained by working directly from the definition. Before proceding to this description we will need the following facts which are easily verified by straightforward calculations.

5.8. *Let* $\mathfrak{g}$ *be a free Lie algebra in* $\mathcal{L}$ *and let*

$$\Delta : U\mathfrak{g} \to U\mathfrak{g} \otimes U\mathfrak{g}$$

denote the diagonal of $U\mathfrak{g}$. *If* $u \in (U\mathfrak{g})^*$ *and* $u(1) = 0$, *then* $\Delta^*(u) = 0$ *where* Δ^* *is the transpose of* Δ.

5.9. *Let* $\mathfrak{g}$ *and* $\mathfrak{h}$ *be free Lie algebras in* $\mathcal{L}$ *and let*

$$\Delta : U\mathfrak{g} \otimes U\mathfrak{h} \to (U\mathfrak{g} \otimes U\mathfrak{g}) \otimes (U\mathfrak{g} \otimes U\mathfrak{h})$$

be the diagonal map of $U\mathfrak{g} \otimes U\mathfrak{h}$. *If* $w \in (U\mathfrak{g} \otimes U\mathfrak{h})^*$ and $w(1) = 0$ *then* $\Delta^*(w \otimes w) = 0$.

5.10. *Let* $\mathfrak{g}$ *be a free simplicial Lie algebra in* $s\mathcal{L}$ *and let*

$$\Delta : \bar{W}U\mathfrak{g} \to \bar{W}U\mathfrak{g} \otimes \bar{W}U\mathfrak{g}$$

be the induced diagonal. If $w \in (\bar{W}U\mathfrak{g})^*$ *is an n-simplex,* $n > 0$, *and* $w(1) = 0$ *then* $\Delta^*(w \otimes w) = 0$.

We now prove part 2) of 5.3.

5.11. *If* $\mathfrak{g}$ *is a free simplicial Lie algebra in* $s\mathcal{L}$ *then*

$$Sq^0 \equiv 0 : H^q\mathfrak{g} \to H^q\mathfrak{g}, \qquad q \geq 0.$$

Proof. We may assume $q > 0$ since $H^0\mathfrak{g} = 0$. For each element $z \in H^q\mathfrak{g}$ there corresponds a unique (up to homotopy) map

$$\bar{W}U\mathfrak{g} \xrightarrow{\alpha} K(Z_2, q).$$

The value of Sq^0 on the class z is defined to be the image of $e_0 \otimes (i_q \otimes i_q)$ under the map (see [5; p. 264–7])

$$W \otimes_{\mathcal{S}(2)} (\text{ch } (K(Z_2, q) \otimes \text{ch } K(Z_2, q))^* \xrightarrow{\Phi/\mathcal{S}(2)} \text{ch } ((K(Z_2, q) \otimes K(Z_2, q))^*/\mathcal{S}(2) \xrightarrow{\text{ch } (\alpha \otimes \alpha)^*/\mathcal{S}(2)} \cdot \text{ch}(\bar{W}U\mathfrak{g} \otimes \bar{W}U\mathfrak{g})^*/\mathcal{S}(2) \xrightarrow{\Delta^*/\mathcal{S}(2)} \text{ch } (\bar{W}U\mathfrak{g})^*$$

where $\mathcal{S}(2)$ is the symmetric group acting on two letters and ch denotes the functor which assigns to each simplicial module X the unique chain complex ch X with $(\text{ch } X)_n = X_n$ and $d_n = \sum_{i=0}^n (-1)^i d_i$.

Now $\Phi/\mathcal{S}(2)(e_0 \otimes (i_q \otimes i_q)) = i_q \otimes i_q$ (see [9; 9. 217]). Hence

$$\text{ch } (\alpha \otimes \alpha)^*/\mathcal{S}(2) \circ \Phi/\mathcal{S}(2)(e_0 \otimes (i_q \otimes i_q)) = z \otimes z.$$

By 5.10, $\Delta^*(z \otimes z) = 0$, hence $Sq^0 z = 0$.

5.12. *Elementary properties of the operations Sq^i for $s\mathfrak{L}$.* Several elementary properties follow immediately from 5.3. We list them at this point in order that the reader may contrast them with the corresponding properties for the ordinary Steenrod operations (for topological spaces).

5.13. *The operation Sq^i is indecomposable for all $i > 0$.*

Proof. Parts (2) and (6) of 5.3.

5.14. *If u is a one-dimensional class then*

$$Sq^i u^k = u^{2k} \quad \text{if } i = k > 0$$
$$= 0 \quad \text{otherwise.}$$

Proof. Obvious for $k = 1$. The result follows by induction using (2) and (5) of 5.3.

5.15. *If u is a one-dimensional class then $Sq^i(u^{2^k}) = u^{2^{k+1}}$ if $i = 2^k$ and zero otherwise.*

5.16. *The Steenrod operations do not act freely on $H^*K(Z_2, 0)^n$ in dimension $\leq n$, contrary to the simplicial group case.*

6. Polynomial generators for $H^*K(Z_2, n)$

According to 4.1, $H^*_{\mathfrak{L}}K(Z_2, n)$ is isomorphic as a Hopf algebra to $H^*_{\mathfrak{G}}K(Z_2, n)$. Serre's computation [18] of the latter algebra shows it to be a polynomial algebra with generators $Sq^I\iota_{n+1}$, for the fundamental class $\iota_{n+1} \in H^{n+1}_{\mathfrak{G}}K(Z_2, n)$ and admissible sequences I of non-negative integers of excess $e(I) \leq n$. *The main result of this section gives a similar set of generators for $H^*_{\mathfrak{L}}K(Z_2, n)$.* The proof compares $H^*_{\mathfrak{G}}K(Z_2, n)$ and $H^*_{\mathfrak{L}}K(Z_2, n)$ and so we assume (6.1–6.8) that the cohomology algebras have units.

6.1. *For $n \geq 0$, $H^*_{\mathfrak{L}}K(Z_2, n) = Z_2[\iota_{n+1}, Sq^I\iota_{n+1}]$ where $\iota_{n+1} \in H^{n+1}_{\mathfrak{L}}K(Z_2, n)$ is the fundamental class and Sq^I are compositions of the Steenrod operations for $s\mathfrak{L}$ defined in §5 with I any admissible sequence of positive integers of excess $e(I) \leq n$.*

It is possible to prove 6.1 by adapting Serre's argument to the Moore spectral sequence in cohomology of the construction

$$(\bar{W}UK(Z_2, n), \bar{W}U\bar{W}K(Z_2, n), \bar{W}UWK(Z_2, n))$$

(see [15]). However, one must show that cup products behave properly in this spectral sequence and this involves a lengthy (although completely straightforward) technical digression. An elementary proof is obtained by considering the following definition and theorem of Borel [1; Theorem 6.1].

6.2. Let A be a commutative graded algebra with algebra generators $\{x_1, x_2, x_3, \cdots\}$ such that $0 < \dim x_i \leq \dim x_{i+1}$. If x_i is not a polynomial in the generators $\{x_j\}_{j\neq i}$ then the set $\{x_1, x_2, x_3, \cdots\}$ is called a *minimal system of generators for* A.

6.3 (Borel). *Let* H *be a connected commutative Hopf algebra of finite type over* Z_2 *in which no element is nilpotent. If* $\{x_1, x_2, \cdots\}$ *is a minimal system of generators for* H *then* $H = Z_2[x_1, x_2, \cdots]$.

Proof of 6.1. In 6.7 and 6.8 we shall show that the set $\{\iota_{n+1}, Sq^I\iota_{n+1}\}$ defined in 6.1 is a minimal system of generators for $H^*_{\mathfrak{L}}K(Z_2, n)$. Since by 4.1, $H^*_{\mathfrak{L}}K(Z_2, n)$ is isomorphic as a Hopf algebra to a polynomial algebra, it is certainly a connected commutative Hopf algebra of finite type over Z_2 in which no element is nilpotent. Hence by 6.3,

$$H^*_{\mathfrak{L}}K(Z_2, n) = Z_2[\iota_{n+1}, Sq^I\iota_{n+1}].$$

The remainder of this section is devoted to establishing 6.7 and 6.8.

Conventions. In the following lemmas, we shall always assume

$$Sq^I = Sq^{a_k} \cdots Sq^{a_1}$$

is admissible with $a_i > 0$, $k > 0$. The notations ${}_{\mathfrak{G}}Sq^I$ and ${}_{\mathfrak{L}}Sq^I$ will be used when necessary to distinguish the group and Lie algebra Steenrod operations. Let

$$n(I) = n(a_k, \cdots, a_1) = a_k + \cdots + a_1.$$

6.4. *Suppose* $Sq^{a_k} \cdots Sq^{a_1}$ *has excess* $e(a_k, \cdots, a_1) \leq m$. *If*

$$q = m + 1 - e(a_k, \cdots, a_1)$$

then there is a positive integer j *such that*

$$(1)\quad \sigma^q(Sq^{a_k} \cdots (Sq^{a_1}\iota_{m+1}) = \sigma \cdots \sigma(Sq^{a_k} \cdots Sq^{a_1}\iota_{m+1}) \qquad (q\ \sigma\text{-factors})$$
$$= (Sq^{a_{k-j}} \cdots Sq^{a_1}\iota_{m-q+1})^{2^j}$$

and

$$(2)\qquad e(a_{k-j}, \cdots, a_1) \leq m - q.$$

Proof. Since $a_k = \sum_{i=1}^{k-1} a_i + m - q + 1$ we have

$$\sigma^q(Sq^{a_k} \cdots Sq^{a_1}\iota_{m+1}) = (Sq^{a_{k-1}} \cdots Sq^{a_1}\iota_{m-q+1})^2.$$

Let $j > 0$ be the largest integer such that

$$\sigma^q(Sq^{a_k} \cdots Sq^{a_1}\iota_{m+1}) = (Sq^{a_{k-j}} \cdots Sq^{a_1}\iota_{m-q+1})^{2^j}.$$

Now we claim

$$e(a_{k-j}, \cdots, a_1) \leq m - q = e(a_k, \cdots, a_1) - 1.$$

To see this note that

$$e(a_k, \cdots, a_1) = (2a_k - a_{k-1}) + \cdots + (2a_{k-j+1} - a_{k-j}) + e(a_{k-} , \cdots, a_1).$$

Hence

$$e(a_k, \cdots, a_1) \geq e(a_{k-j}, \cdots, a_1).$$

If $e(a_{k-j}, \cdots, a_1) = e(a_k, \cdots, a_1)$ then

$$a_k = 2a_{k-1}, \cdots, a_{k-j+1} = 2a_{k-j}.$$

So

$$a_k = a_{k-1} + \cdots + a_{k-j} + \sum_{i=1}^{k-j-1} a_i + (m - q + 1)$$

or

$$2^j a_{k-j} = 2^{j-1} a_{k-j} + \cdots + a_{k-j} + \sum_{i=1}^{k-j-1} a_i + (m - q + 1)$$

or

$$a_{k-j} = \sum_{i=1}^{k-j-1} a_i + (m - q + 1).$$

But this implies that

$$\sigma^q (Sq^{a_k} \cdots Sq^{a_1} \iota_{m+1}) = (Sq^{a_{k-j-1}} \cdots Sq^{a_1} \iota_{m+1})^{2^{j+1}}$$

contradicting the maximality of j. Hence

$$e(a_{k-j}, \cdots, a_1) \leq m - q.$$

6.5. *If $P = P(\iota_{m+1}, Sq^I \iota_{m+1}; e(I) \leq m)$ is a homogeneous polynomial with at least one linear term then $P \neq 0$ in $H_{\mathfrak{L}}^* K(Z_2, m)$.*

Proof. The proof is by induction on the maximum length of the linear terms, where length $(\iota_{m+1}) = 0$, and length $(Sq^I \iota_{m+1}) = k$ for $I = (a_k, \cdots, a_1)$.

If the maximum length is zero then $P = \iota_{m+1} \neq 0$ in $H_{\mathfrak{L}}^* K(Z_2, m)$. Assume that for all $m > 0$ the lemma is true for homogeneous polynomials with linear terms of maximum length $< l$.

Let $P = P(\iota_{m+1}, Sq^I \iota_{m+1}; e(I) \leq m)$ be a homogeneous polynomial with linear terms $\sum Sq^J \iota_{m+1}$ such that the maximum length of these terms is l. Suppose $P = 0$ in $H_{\mathfrak{L}}^* K(Z_2, m)$. Apply iterations of the suspension homomorphism σ until the remaining terms are squares (Lemma 6.4), i.e.,

$$0 = \sigma^r(0) = \sigma^r(P) = \sigma^r(P' + \sum Sq^J \iota_{m+1}) = \sigma^r(\sum Sq^J \iota_{m+1})$$
$$= \sum (Sq^{J'} \iota_{m-r+1})^{2^j}$$

where $e(J') \leq m - r$. Since we are working over Z_2 we may remove the largest common exponent of 2 from each term. Hence at least one of the remaining terms has exponent $2^0 = 1$, and the remaining terms form a homogeneous polynomial

$$Q = Q(\iota_{m-r+1}, Sq^I \iota_{m-r+1}; e(I) \leq m - r)$$

with a linear term such that $Q = 0$ in $H^* K(Z_2, m - r)$. However the length of each linear term of Q is less than l and hence by induction, $Q \neq 0$ in $H_{\mathfrak{L}}^* K(Z_2, m - r)$. This contradiction shows that $P \neq 0$ in $H_{\mathfrak{L}}^* K(Z_2, m)$ and establishes the lemma.

6.6. *If $0 < n \leq m$ then $\{Sq^I \iota_{m+1}; n(I) = n\}$ forms a Z_2-module basis for $H_{\mathfrak{L}}^{m+1+n} K(Z_2, m)$.*

Proof. Any I with $n(I) = n$ has excess $e(I) \leq m$ since $e(\mathrm{I}) \leq n(I) = n \leq m$. Hence by 6.5, $\{Sq^I \iota_{m+1}, n(I) = n\}$ is linearly independent. However

$$\dim_{Z_2} H_{\mathcal{L}}^{m+1+n} K(Z_2, m) = \dim_{Z_2} H_{\mathcal{G}}^{m+1+n} K(Z_2, m) = \mathrm{Card}\ \{I, n(I) = n\}$$

and so $\{Sq^I \iota_{m+1}; n(I) = n\}$ also generates.

6.7. *The set* $\{\iota_{m+1}, Sq^I \iota_{m+1}; e(I) \leq m\}$ *generates* $H_{\mathcal{L}}^* K(Z_2, m)$ *as a graded algebra over* Z_2.

Proof. Suppose $x \in H_{\mathcal{L}}^* K(Z_2, m)$ is a homogeneous element. By 4.1, there is a Hopf algebra isomorphism

$$\phi : H_{\mathcal{G}}^* K(Z_2, m) \simeq H_{\mathcal{L}}^* K(Z_2, m)$$

hence there is a homogeneous polynomial

$$P = P(\iota_{m+1}, {}_{\mathcal{G}}Sq^I \iota_{m+1}; e(I) \leq m)$$

such that $\phi(P) = x$.

Thus to show that $\{\iota_{m+1}, {}_{\mathcal{L}}Sq^I \iota_{m+1}; e(I) \leq m\}$ generates we are reduced to showing that for each ${}_{\mathcal{G}}Sq^J \iota_{m+1}$ with $e(J) \leq m$ there is a polynomial

$$Q = Q(\iota_{m+1}, {}_{\mathcal{L}}Sq^I \iota_{m+1}; e(I) \leq m),$$

depending on J, such that $\phi({}_{\mathcal{G}}Sq^J \iota_{m+1}) = Q$.

By naturality the following diagram commutes

$$\begin{array}{ccccc} \bar{W}Z_2 \bar{Z}_2 S_i & \rightarrow & \bar{W}Z_2 \bar{Z}_2 CS_i & \rightarrow & \bar{W}Z_2 \bar{Z}_2 S_{i+1} \\ \downarrow \phi & & \downarrow \phi & & \downarrow \phi \\ \bar{W}U\bar{Z}_2 S_i & \rightarrow & \bar{W}U\bar{Z}_2 CS_i & \rightarrow & \bar{W}U\bar{Z}_2 S_{i+1} \end{array}$$

where CS_i is the cone on S_i. By the naturality of the connecting homomorphism we have commutativity in the diagram

$$\begin{array}{ccc} H_{\mathcal{G}}^{*+1} K(Z_2, i+1) & \xrightarrow{\sigma} & H_{\mathcal{G}}^* K(Z_2, i) \\ \downarrow \phi & & \downarrow \phi \\ H_{\mathcal{L}}^{*+1} K(Z_2, i+1) & \xrightarrow{\sigma} & H_{\mathcal{L}}^* K(Z_2, i). \end{array}$$

Let q be a positive integer such that $q \geq n(J) - m$. By 6.6,

$$\phi({}_{\mathcal{G}}Sq^J \iota_{m+q+1}) = \text{some sum} \sum {}_{\mathcal{L}}Sq^I \iota_{m+q+1}$$

with $e(I) \leq m + q$. Hence

$$\begin{array}{ccc} {}_{\mathcal{G}}Sq^J \iota_{m+q+1} & \xrightarrow{\phi} & \sum {}_{\mathcal{L}}Sq^I \iota_{m+q+1} \\ \downarrow \sigma^q & & \downarrow \sigma^q \\ {}_{\mathcal{G}}Sq^J \iota_{m+1} & \xrightarrow{\phi} & \sum ({}_{\mathcal{L}}Sq^{I'} \iota_{m+1})^2 \end{array}$$

commutes and we may choose $Q = \sum ({}_{\mathcal{L}}Sq^{I'}\iota_{m+1})^{2^i}$ since by 6.4, $e(I) \leq m$.

6.8. *If $e(I) \leq m$ then $Sq^I\iota_{m+1} \neq P(Sq^J\iota_{m+1}\,;\, J \neq I,\, e(J) \leq m)$ that is, $Sq^I\iota_{m+1}$ is not equal to a homogeneous polynomial in the variables $Sq^J\iota_{m+1}, e(J) \leq m$ excluding $J = I$.*

Proof. Apply 6.5.

7. The Steenrod algebra and its dual

In this section we define the analogue of the Steenrod algebra for $s\mathcal{L}$ and determine its structure as a Hopf algebra. In particular, we shall show that *this Steenrod algebra is generated by the stable operations, Sq^i, $i \geq 0$, subject to the Adem relations* of 5.3 (6).

7.1. *The Steenrod algebra $\mathfrak{A}(\mathcal{L})$ for $s\mathcal{L}$.* As in the topological case we *define the Steenrod algebra $\mathfrak{A}(\mathcal{L})$ for $s\mathcal{L}$ to be the algebra of all stable primary cohomology operations of $s\mathcal{L}$* (3.9). The product of two operations is given by composition. The unit is the identity operation, denoted by 1.

The following proposition gives generators and relations for $\mathfrak{A}(\mathcal{L})$.

7.2. *The Steenrod operations, Sq^i, $i \geq 0$, generate $\mathfrak{A}(\mathcal{L})$ as a Z_2-algebra and the Adem relations generate the ideal of relations in $\mathfrak{A}(\mathcal{L})$.*

This result is an immediate corollary of

7.3. (1) *As a Z_2-module, $\mathfrak{A}(\mathcal{L}^*) \simeq \lim_{m\to\infty} H^{m+1+*}K(Z_2, m)$*

(2) *The admissible monomials, Sq^I, form a Z_2-basis for $\mathfrak{A}(\mathcal{L})$.*

Proof. To establish (1) recall (3.10) that each stable primary operation of type i corresponds to a unique sequence $\{u_0 = 0, u_1, u_2, \cdots\}$ of elements $u_j \in H^{j+i}K(Z_2, j-1)$ such that $\sigma u_{j+1} = u_j$. However, using 6.3 we see that such a sequence stabilizes for $j \geq i + 1$. Hence

$$\mathfrak{A}(\mathcal{L}^*) \simeq \lim_{n\to\infty} H^{m+1+*}K(Z_2, m)$$

as a Z_2-module.

Part (2) is merely a restatement of 6.6 using part (1).

There is a natural augmentation $\varepsilon : \mathfrak{A}(\mathcal{L}) \to Z_2$ for $\mathfrak{A}(\mathcal{L})$ given by $\varepsilon(1) = 1$ and $\varepsilon(Sq^I) = 0$.

7.4. *The bigradation of $\mathfrak{A}(\mathcal{L})$.* If $I = (i_1, \cdots, i_k)$ is a finite sequence of positive integers then I will be called a *label*. The *length* of I or $l(I)$ is k and the *degree* of I or deg (I) is $i_1 + \cdots + i_k$. If two labels $I = (i_1, \cdots, i_k)$ and $J = (j_1, \cdots, j_k)$ are of the same length then their *sum* is $(i_1 + j_1, \cdots, i_k + j_k)$ and they are *ordered*, $I \leq J$, by the lexicographical ordering from the left.

Observe that since $Sq^iSq^0 = 0$, $i \geq 0$, the Adem relations show that $\mathfrak{A}(\mathcal{L})$

is bigraded. More precisely a monomial Sq^I is bigraded by the bigradation of its label, first by degree and second by length.

7.5. *The Hopf algebra structure of* $\mathfrak{A}(\mathfrak{L})$. We now show that the Steenrod algebra $\mathfrak{A}(\mathfrak{L})$ has a diagonal map. However there is no counit and hence we say that $\mathfrak{A}(\mathfrak{L})$ is a Hopf algebra without counit.

7.6. *There is a graded Z_2-algebra homomorphism*

$$\Delta : \mathfrak{A}(\mathfrak{L}) \to \mathfrak{A}(\mathfrak{L}) \otimes \mathfrak{A}(\mathfrak{L})$$

which is induced by the Cartan formula. Moreover, Δ is associative and commutative and gives $\mathfrak{A}(\mathfrak{L})$ the structure of a cocommutative Hopf algebra without counit.

Proof. The proof of the existence of the homomorphism Δ is identical with that given by Milnor [13] for the topological Steenrod algebra. Since Δ is associative and commutative on the generators of $\mathfrak{A}(\mathfrak{L})$ it is associative and commutative on $\mathfrak{A}(\mathfrak{L})$.

Note that *the diagonal Δ does not preserve the second gradation* or length of a monomial Sq^I, i.e., if I is an admissible label then

$$\Delta(Sq^I) = \sum_{J+J'=I,\, l(J)=l(J')=l(I)} Sq^J \otimes Sq^{J'}$$

where J and J' run over all pairs of admissible labels of length $l(I)$ and sum I.

The dual $\mathfrak{A}^*(\mathfrak{L}) = \text{Hom}\ (\mathfrak{A}(\mathfrak{L}), Z_2)$ is a commutative Hopf algebra without unit but, unlike the dual of the topological Steenrod algebra, it is not a polynomial algebra.

Let $\{\xi_I\}$ be the basis dual to the basis $\{Sq^I\}$ where I runs over the set of admissible labels.

7.7. *The dual algebra $\mathfrak{A}^*(\mathfrak{L})$ is a commutative Hopf algebra without unit generated by* $\{1^*, \xi_I, I$ an admissible label$\}$ *subject to the following relations*:

(a) $1^* \cdot 1^* = 1^*$

(b) $1^* \cdot \xi_I = 0$

(c) $\xi_I \cdot \xi_J = \xi_{I+J}$ *if* $l(I) = l(J)$
$\quad = 0$ *otherwise.*

Proof. Commutativity follows from the commutativity of Δ. The first two relations are clear. Now suppose I, J, K are admissible labels, then

$$\langle \xi_I \cdot \xi_J , Sq^K \rangle$$
$$= \langle \xi_I \otimes \xi_J , \Delta Sq^K \rangle = \sum_{K'+K''=K,\, l(K')=l(K'')=l(K)} \langle \xi_I , Sq^{K'} \rangle \langle \xi_J , Sq^{K''} \rangle = 1$$

if and only if $K = I + J$ and $l(I) = l(J)$.

Let χ_l be the submodule of $\mathfrak{A}^*(\mathfrak{L})$ generated by ξ_I for $l(I) = l > 0$ and by

1^* *for* $l = 0$. By 7.7 we have

7.8. *The algebra* $\mathfrak{A}^*(\mathfrak{L})$ *is the direct product of the ideals* χ_l:

$$\mathfrak{A}^*(\mathfrak{L}) = \prod_{l=0}^{\infty} \chi_l$$

The diagonal map δ for $\mathfrak{A}^*(\mathfrak{L})$ is the dual of the multiplication map for $\mathfrak{A}(\mathfrak{L})$.

7.9. *Let* I *be an admissible label. If* $l(I) = n > 0$ *then*

$$\delta\xi_I = \sum_{j=1}^{n} \sum_{(i_1,\cdots,i_n)} e_I(i_1, \cdots, i_n)\xi_{(i_1,\cdots,i_j)} \otimes \xi_{(i_{j+1},\cdots,i_n)}$$

where the second sum is taken over all labels $(i_1, \cdots, i_j, i_{j+1}, \cdots, i_n)$ *such that* $(i_1, \cdots, i_j)$ *and* $(i_{j+1}, \cdots, i_n)$ *are admissible labels. The function* e_I *is defined on all labels of length* $l(I)$ *and has values in* $Z_2 : e_I(J) = 1$ *if* Sq^I *has coefficient one in the admissible expansion of* Sq^J *and* $e_I(J) = 0$ *otherwise.*

Proof. Let $I = (a_1, \cdots, a_n)$ be an admissible label. Fix k, $k \in \{1, \cdots, n\}$ and let

$$J_1 = (b_1, \cdots, b_k) \quad \text{and} \quad J_2 = (b_{k+1}, \cdots, b_n)$$

be admissible labels. Then

$$\langle \delta\xi_I, Sq^{J_1} \otimes Sq^{J_2} \rangle = \langle \xi_I, Sq^{J_1} \cdot Sq^{J_2} \rangle = e_I(b_1, \cdots, b_n).$$

Also

$$\begin{aligned}\langle \textstyle\sum_{j=1}^{n} \sum_{(i_1,\cdots i_n)} & e_I(i, \cdots, i_n)\xi_{(i_1,\cdots,i_j)} \otimes \xi_{(i_{j+1},\cdots,i_n)}, Sq^{J_2} \otimes Sq^{J_2} \rangle \\ &= \textstyle\sum_{j=1}^{n} \sum_{(i_1,\cdots,i_n)} e_I(i_1, \cdots, i_n)\langle \xi_{(i_1,\cdots,i_j)}, Sq^{J_1}\rangle \cdot \langle \xi_{(i_{j+1},\cdots,i_n)}, Sq^{J_2}\rangle \\ &= \textstyle\sum_{(i_1,\cdots,i_n)} e_I(i_1, \cdots, i_n)\langle \xi_{(i_1,\cdots,i_k)}, Sq^{J_1}\rangle \cdot \langle \xi_{(i_{k+1},\cdots,i_n)}, Sq^{J_2}\rangle \\ &= e_I(b_1, \cdots, b_n).\end{aligned}$$

7.10. An easy calculation using 7.9 shows that if (a, b) is an admissible label, i.e., $a \geq 2b > 0$, then

$$\delta\xi_{(a,b)} = \xi_a \otimes \xi_b + \sum_{j=2b}^{\alpha} \binom{a-j-1}{j-2b} \xi_j \otimes \xi_{a+b-j}$$

where $\alpha = \{\frac{2}{3}(a + b)\}$ is the greatest integer less than $\frac{2}{3}(a + b)$. This fact will be used in §8.

8. The calculation of $\mathrm{Ext}_{\mathfrak{A}(\mathfrak{L})}(Z_2, Z_2)$

Using the dual of the normalized bar construction, we explicitly compute the cohomology algebra, $\mathrm{Ext}_{\mathfrak{A}(\mathfrak{L})}(Z_2, Z_2)$ of the Steenrod algebra $\mathfrak{A}(\mathfrak{L})$ of §7. In the case of the ordinary Steenrod algebra, this method is too complicated to be practical; however, since $\mathfrak{A}(\mathfrak{L})$ is a bigraded algebra, $\mathrm{Ext}_{\mathfrak{A}(\mathfrak{L})}(Z_2, Z_2)$ is a trigraded algebra with the Yoneda product and using this extra structure we are able to carry out the calculation. We denote the trigradation by $\mathrm{Ext}_{\mathfrak{A}(\mathfrak{L})}^{m,n,l}(Z_2, Z_2)$ where

$$m = \text{(co)homological degree}$$

n = degree by which maps are lowered

l = length.

We shall prove that off certain "diagonals" $m = l$, $\mathrm{Ext}_{\mathfrak{a}(\mathfrak{L})}^{m,n,l}(Z_2, Z_2)$ vanishes and so the trigradation reduces to a *bigradation*. Let $\xi_i \in \mathrm{Ext}_{\mathfrak{a}(\mathfrak{L})}^{1,i,1}(Z_2, Z_2)$ be the class of the dual of Sq^i, $i > 0$. The main result of this section is

8.1. *The bigraded algebra* $\mathrm{Ext}_{\mathfrak{a}(\mathfrak{L})}^{*,*}(Z_2, Z_2)$ *is generated by the elements* ξ_i, $i > 0$ *subject to the following relations*: *if* $a \geq 2b > 0$,

$$\xi_a \cdot \xi_b = \sum_{j=2b}^{\alpha} \binom{a-j-1}{j-2b} \xi_j \cdot \xi_{a+b-j}$$

where $\alpha = \{\frac{2}{3}(a+b)\}$ *is the greatest integer less than* $\frac{2}{3}(a+b)$.

This result follows from 8.2–4 below.

Recall that the cohomology algebra of a graded Z_2-algebra B_*, of finite type, with unit $\eta : Z_2 \to B_*$ and augmentation $\varepsilon : B_* \to Z_2$ is given by the cohomology of the following cochain complex (Y, δ), (dual of the normalized bar construction of B_*):

$$Y^0 = Z_2, \quad Y^n = I(B^*) \otimes \cdots \otimes I(B^*) \quad (n\text{-factors}),$$

$$\delta(b_1 \otimes \cdots \otimes b_n) = \sum_{i=1}^{n} b_1 \otimes \cdots \otimes b_{i-1} \otimes \delta b_i \otimes b_{i+1} \otimes \cdots \otimes b_n$$

where $B^* = \mathrm{Hom}(B_*, Z_2)$, $I(B^*)$ denotes the augmentation ideal $= \ker \eta^*$ and δ is the dual of the multiplication of B_*. The cohomology product or Yoneda product is easily shown to be induced by the tensor algebra product.

8.2. *The following relations (and no more) hold in* $\mathrm{Ext}_{\mathfrak{a}(\mathfrak{L})}^{2,*,2}(Z_2, Z_2)$ *for* $a \geq 2b > 0$

$$\xi_a \cdot \xi_b = \sum_{j=2b}^{\{(2/3)(a+b)\}} \binom{a-j-1}{j-2b} \xi_j \cdot \xi_{a+b-j}$$

Hence $\{\xi_c \cdot \xi_d : 0 < c < 2d\}$ *is a* Z_2*-module basis for* $\mathrm{Ext}_{\mathfrak{a}(\mathfrak{L})}^{2,*,2}(Z_2, Z_2)$.

Proof. Given any positive integers a, b, $\xi_a \otimes \xi_b$ is a cocycle since $\delta(\xi_a \otimes \xi_b) = \delta\xi_a \otimes \xi_b + \xi_a \otimes \delta\xi_b = 0$.

Now $Y^{1,*,2} = \{\xi_{(a,b)} : a \geq 2b > 0)$, hence the only relations in $\mathrm{Ext}_{\mathfrak{a}(\mathfrak{L})}^{2,*,2}$ are $\delta\xi_{(a,b)} = 0$ or by 7.10,

$$(*) \qquad \xi_a \otimes \xi_b + \sum_{j=2b}^{\{(2/3)(a+b)\}} \binom{a-j-1}{j-2b} \xi_j \otimes \xi_{a+b-j} = 0.$$

Notice that in the range $2b \leq j \leq \{\frac{2}{3}(a \quad b)\}$ we have $j < 2(a+b-j)$. Hence, using the relations $(*)$, we can express any cocycle as a unique sum of $\xi_c \otimes \xi_d$, $0 < c < 2d$.

8.3. *The classes* $\xi_{a_1} \otimes \cdots \otimes \xi_{a_n}$, $0 < a_i < 2a_{i+1}$ *form a* Z_2*-module basis for* $\mathrm{Ext}_{\mathfrak{a}(\mathfrak{L})}^{n,*,n}(Z_2, Z_2)$. *These monomials will be called coadmissible.*

Proof. It follows from 8.2 that the coadmissible monomials generate $\mathrm{Ext}_{\mathfrak{a}(\mathfrak{L})}^{n,*,n}(Z_2, Z_2)$. However, $Y^{n+1,*,n}$ has a basis consisting of elements of the form

$$\xi_{b_1} \otimes \cdots \otimes \xi_{b_{i-1}} \otimes \xi_{(b_i,b_{i+1})} \otimes \xi_{b_{i+2}} \otimes \cdots \otimes \xi_{b_n}$$

and

$$\delta(\xi_{b_1} \otimes \cdots \otimes \xi_{b_{i-1}} \otimes \xi_{(b_i,b_{i+1})} \otimes \xi_{b_{i+2}} \otimes \cdots \otimes \xi_{b_n})$$
$$= \xi_{b_1} \otimes \cdots \otimes \xi_{b_{i-1}} \otimes \delta\xi_{(b_i,b_{i+1})} \otimes \xi_{b_{i+2}} \otimes \cdots \otimes \xi_{b_n},$$

hence the only relations introduced are precisely those required to express any monomial in coadmissible form.

8.4. $\mathrm{Ext}^{m,*,l}_{\mathfrak{a}(\mathfrak{L})}(Z_2, Z_2) = 0$ *if* $m \neq l$.

The proof of 8.4 is given below. We will filter the complex Y and show that off the diagonals, $m = l$, the quotients of this filtration are cohomologous to zero. The result then follows from a standard exact sequence argument.

The *cochain complex* Y *is given the following filtration*: If I is a label of length $l(I) = l$ then *define* $F_I Y^{u,*,l} = Z_2$-module generated by the tensors

$$\xi_{(a_1,\cdots,a_{k_1})} \otimes \xi_{(a_{k_1+1},\cdots,a_{k_2})} \otimes \cdots \otimes \xi_{(a_{k_{u-1}+1},\cdots,a_l)}$$

where $(a_1, \cdots, a_{k_1}), \cdots, (a_{k_{u-1}+1}, \cdots, a_l)$ are admissible labels and $I \geq (a_1, \cdots, a_l)$.

It is clear that *the filtration is increasing*, i.e., $I \geq J$ and $l(I) = l(J)$ implies $F_I Y^{*,*,l} \supset F_J Y^{*,*,l}$.

Of course we must show that $F_I Y^{*,*,l}$ *is a subcomplex* of $Y^{*,*,l}$, that is, closed under δ. This follows from 7.9 and the following lemma.

8.5. *Any inadmissible monomial* Sq^I *can be expressed as a sum of monomials each of whose labels is strictly greater than* I.

Proof. Suppose $I = (i_1, \cdots, i_k)$ with $n = i_r < 2i_{r+1} = 2m$. Then

$$Sq^I = Sq^N Sq^n Sq^m Sq^M = \sum_{j=1}^{[n/2]} \binom{m-j-1}{n-2j} Sq^N Sq^{n+m-j} Sq^j Sq^M$$

Now $m - j > 0$ so $n + m - j > n$ hence

$$(N, n + m - j, j, M) > (N, n, m, M).$$

Notation. If I is a label let $F_{I-1} Y$ denote $\bigcup_{J<I,\, l(J)=l(I)} F_J Y$. Note that if $I = (i_1, \cdots, i_k)$ and $i_k > 1$ then

$$F_{I-1} Y = F_{(i_1,\cdots,i_{k-1},i_k-1)} Y$$

Again using 7.9 and 8.5,

8.6. *The induced map* $\delta : F_I Y/F_{I-1} Y \to F_I Y/F_{I-1} Y$ *is given by the following formula*: (write I for ξ_I for simplicity)

$$\delta(a_1 \cdots a_{k_1} \otimes a_{k_1+1} \cdots a_{k_2} \otimes \cdots \otimes a_{k_{u-1}+1} \cdots a_l)$$
$$= \sum_{j=2}^{k_1-1} a_1 \cdots a_{j-1} \otimes a_j \cdots a_{k_1} \otimes \cdots \otimes a_{k_{u-1}+1} \cdots a_l$$
$$+ \sum_{j=k_1+2}^{k_2-1} a_1 \cdots a_{k_1} \otimes a_{k_1+1} \cdots a_{j-1} \otimes a_j \cdots a_{k_2} \otimes \cdots \otimes a_{k_{u-1}+1} \cdots a_l$$

$$+ \cdots + \sum_{j=k_{u-1}+2}^{l-1} a_1 \cdots a_{k_1} \otimes \cdots \otimes a_{k_{u-1}+1} \cdots a_{j-1} \otimes a_j \cdots a_l \pmod{F_{I-1} Y}$$

Proof of 8.4. Fix $l > 0$. If $m > l$ then there are no non-zero elements of length l in $Y^{m,*,l}$; this is because each tensor factor has length at least one.

For $m < l$, we will examine the quotients of the filtration and show that they have trivial cohomology.

To each tensor

$$x = a_1 \cdots a_{k_1} \otimes a_{k+1} \cdots a_{k_2} \otimes \cdots \otimes a_{k_{u-1}+1} \cdots a_l$$

of length l we assign an integer ai (x), called the *admissibility index*, defined as follows: Let ai (x) be the smallest integer i, $1 \leq i \leq l-1$ such that $a_i \geq 2a_{i+1}$. If no such i exists then set ai $(x) = l$.

We now define a contracting homotopy

$$\Phi : F_I Y^{u,*,l}/F_{I-1} Y^{u,*,l} \to F_I Y^{u-1,*,l}/F_{I-1} Y^{u-1,*,l} \qquad \text{for } 1 \leq u \leq l$$

given by

$$\Phi(x) = a_1 \otimes a_2 \otimes \cdots \otimes a_{i-1} \otimes a_i a_{i+1} \cdots a_{k_{i+1}} \otimes \cdots \otimes a_{k_{u-1}+1} \cdots a_l$$

if $x = a_1 \otimes a_2 \otimes \cdots \otimes a_i \otimes a_{i+1} \cdots a_{k_{i+1}} \otimes \cdots \otimes a_{k_{u-1}+1} \cdots a_l$ where ai $(x) = i$ and $k_i = i$ and

$$\Phi(x) = 0 \quad \text{otherwise.}$$

Notice that $a_i a_{i+1} \cdots a_{k_{i+1}}$ is admissible since $a_i \geq 2a_{i+1}$.

It is now a routine verification to show that if

$$x \in F_I Y^{m,*,l}/F_{I-1} Y^{m,*,l}, \quad m < l,$$

then $\delta\Phi(x) + \Phi\delta(x) = x$. One considers the two cases $k_i > i$ and $k_i = i$.

9. The E^1 term of the Adams spectral sequence

Recall from [2] that $\pi_* LAS$ is the E^1 term of the Adams spectral sequence, derived from the lower 2-central series filtration, of the free group sphere spectrum FS. As in [2; §5.2] let $\lambda_n \in \pi_n L_2 AS$ denote the class $i \otimes i$. Then we can state the main result of this section.

9.1. *There is an isomorphism of bigraded algebras over* Z_2,

$$\Phi : \mathrm{Ext}_{\mathfrak{A}(\mathfrak{L})}^{*,*} (Z_2, Z_2) \simeq \pi_* LAS$$

given by $\xi_{i+1} \to \lambda_1$, $i \geq 0$. *For bigradation* $(m, m+1)$ $m, i \geq 0$,

$$\Phi : \mathrm{Ext}_{\mathfrak{A}(\mathfrak{L})}^{m,m+i} (Z_2, Z_2) \simeq \pi_i L_{2m} AS.$$

Proof. It follows from 9.3–11 below that there is a convergent Adams spectral sequence.

$$E^2_{s,t} = \mathrm{Ext}_{\mathfrak{A}(\mathfrak{L})}^{s,s+t} (Z_2, Z_2) \Rightarrow \pi_* LAS$$

with a product structure induced by composition. On E^2 this product agrees with the Yoneda product and on E^∞ it is induced by the composition product of $\pi_* LAS$. Since $\pi_* LAS = \sum_{r\geq 1} \pi_* L_r AS$ is a bigraded algebra with respect to the composition product it suffices to show $E^2 = E^\infty$. Now $\lambda_n \neq 0 \in \pi_n L_2 AS$ for all $n \geq 0$ (every $2n + 1$ simplex of $L_2 AS_n$ is degenerate), hence ξ_{i+1} persists to E^∞ and $\xi_{i+1} \Rightarrow \lambda_i$. Thus since the differentials d^r are derivations (9.10 (ii)) and the ξ_{n+1}, $n \geq 0$ generate E^2 we have $E^2 = E^\infty$.

9.2 *Remark.* This result gives a cohomological determination of the algebra structure of the E^1 term. In effect it "explains" the origin of the relations among the λ_i's (compare [2, §2.4 (iii)] with 8.2).

9.3. *The Adams spectral sequence for* $s\mathfrak{L}$. We will derive the Adams spectral sequence called for in the preceding proof.

Let

$$\cdots \subset D_{r+1}\mathfrak{g} \subset D_r\mathfrak{g} \subset \cdots \subset D_0\mathfrak{g} = \mathfrak{g}$$

denote the 2-*derived series filtration* of $\mathfrak{g}$ (where $D_r = \Gamma_2 \cdots \Gamma_2$, r factors).

Denote by $\{E^r\mathfrak{g}\}$ the spectral sequence derived from the homotopy exact couple of this filtration.

9.4. *The* E^2 *term.* The following result describes the E^2 term of $\{E^r\mathfrak{g}\}$

9.5. *If* $\mathfrak{g}$ *is an m-connected* ($m \geq 1$) *free simplicial Lie algebra in* $s\mathfrak{L}$ *then*

$$E^2_{s,t}\mathfrak{g} = \operatorname{Ext}^{s,s+t}_{\mathfrak{A}(\mathfrak{L})} (\operatorname{Hom} (\pi_* \operatorname{Ab}\mathfrak{g}, Z_2), Z_2), \qquad * < 2m + s$$

Proof. It suffices to show that (in the stable range)

$$i_* \equiv 0 : H_* D_{r+1}\mathfrak{g} \to H_* D_r\mathfrak{g}$$

for all $r \geq 0$. Applying the homology functor to the fibration

$$D_{r+1}\mathfrak{g} \xrightarrow{\ i\ } D_r\mathfrak{g} \xrightarrow{\ j\ } \operatorname{Ab} D_r\mathfrak{g} = K(\pi_* \operatorname{Ab} D_r \mathfrak{g}, *)$$

we have (in the stable range) an exact triangle

$$\begin{array}{ccc} H_* D_{r+1}\mathfrak{g} & \xrightarrow{\ i_*\ } & H_* D_r\mathfrak{g} \approx \pi_{*-1} \operatorname{Ab} D_r\mathfrak{g} \\ {\scriptstyle \partial_*}\nwarrow & & \swarrow{\scriptstyle j_*} \\ & H_* K(\pi_* \operatorname{Ab} D_r\mathfrak{g}, *) \approx \otimes_{i+j=*-1} \pi_i \operatorname{Ab} D_r\mathfrak{g} \otimes \mathfrak{A}^j(\mathfrak{L}). & \end{array}$$

Thus $i_* \equiv 0$ since j_* is the injection given by $[g] \to [g] \otimes 1$.

9.6. *Convergence.*

9.7. *Let* $\mathfrak{g}$ *be an m-connected free simplicial Lie algebra in* $s\mathfrak{L}$, *Then* $\{E^r\mathfrak{g}\}$ *is convergent and* $E^\infty\mathfrak{g}$ *is the graded group associated with the induced* 2-*derived series filtration of* $\pi_*\mathfrak{g}$, *in the stable range* $* < 2m$.

Proof. Inductively construct a sequence

$$\cdots \longrightarrow F_{n+1}\mathfrak{g} \longrightarrow F_n\mathfrak{g} \longrightarrow \cdots \longrightarrow F_0\mathfrak{g} = \mathfrak{g}$$

of induced fibrations (a Postnikov tower for $\mathfrak{g}$) with fibre

$$K_n = K(\pi_{i_n} F_n \mathfrak{g}, i_n - 1),$$

where $\pi_{i_n} F_n \mathfrak{g}$ is the first non-zero homotopy group of $F_n\mathfrak{g}$:

$$\begin{array}{ccc} K_n & \rightarrow & K_n \\ \downarrow & & \downarrow \\ F_{n+1}\mathfrak{g} & \rightarrow & WK_n \\ \downarrow & & \downarrow \\ F_n\mathfrak{g} & \rightarrow & \bar{W}K_n . \end{array}$$

Now the simplicial Lie algebra version of [6, Proposition IB, p. 18–03] (established using the representability of H^*) shows that since

$$i_* \equiv 0 : H_* D_{r+1}\mathfrak{g} \longrightarrow \pi_* D_r \mathfrak{g}.$$

there are maps $g_n : D_n\mathfrak{g} \longrightarrow F_n\mathfrak{g}$ such that

$$\text{(9.5)} \qquad \begin{array}{ccccccc} \cdots \longrightarrow & D_{n+1}\,\mathfrak{g} & \longrightarrow & D_n\,\mathfrak{g} & \longrightarrow & \cdots & \mathfrak{g} \\ & \downarrow g_{n+1} & & \downarrow g_n & & & \downarrow \mathrm{id} \\ \cdots \longrightarrow & F_{n+1}\,\mathfrak{g} & \longrightarrow & F_n\,\mathfrak{g} & \longrightarrow & \cdots & \mathfrak{g} \end{array}$$

commutes. Hence there is an inclusion $D_n \pi_* \mathfrak{g} \longrightarrow F_n \pi_* \mathfrak{g}$ relating the filtration induced by $\{D_n \mathfrak{g}\}$ to the filtration induced by $\{F_n \mathfrak{g}\}$. Since $\{F_n \mathfrak{g}\}$ is a Postnikov tower it becomes increasingly highly connected, hence

$$\bigcap_n D_n \pi_* \mathfrak{g} \subset \bigcap_n F_n \pi_* \mathfrak{g} = 0.$$

This fact also established convergence since $D_1 \mathfrak{g}$ (and hence $D_r \mathfrak{g}$) is free.

9.8. *Products.* The remainder of this section deals with the Lie algebra sphere spectrum LAS. *The spectral sequence* $\{E^r LAS\}$ *for LAS has a natural product*

$$E^r_{s,t} LAS \otimes E^r_{s',t'} LAS \rightarrow E^r_{s+s',t+t'} LAS$$

induced by composition, i.e., the natural transformation $LL \rightarrow L$ which for a Z_2-module M is the unique map of such that $L_1 LM = LM$. To establish this pairing we first prove a preliminary result.

9.9. *There is a composition pairing*

$$\phi_r : \pi_s(D_n/D_{n+r})LAS \otimes \pi_t(D_q/D_{q+r})LAS \rightarrow \pi_{s+t}(D_{n+q}/D_{n+q+r})LAS, \quad r \geq 0,$$

(also denoted by $a \otimes b \to a \cdot b$) such that

(i) ϕ_r *is natural with respect to the maps*

$$\eta_* : \pi_*(D_n/D_{n+q})LAS \to \pi_*(D_{n'}/D_{n'+q'})LAS$$

for all $0 \leq n' \leq n,\ q' \leq q \leq \infty$,

(ii) $\partial_{s+t}(a \cdot b) = \eta_s a \cdot \partial_t b + \partial_s a \cdot \eta_t b$ in the diagram

$$\begin{array}{ccc}
\pi_s\left(\dfrac{D_n}{D_{n+r}}\right)LAS \otimes \pi_t\left(\dfrac{D_q}{D_{q+r}}\right)LAS & \xrightarrow{\phi_r} & \pi_{s+t}\left(\dfrac{D_{n+q}}{D_{n+q+r}}\right)LAS \\
\downarrow{\scriptstyle \partial_s \otimes \eta_t} \quad \searrow{\scriptstyle \eta_s \otimes \partial_t} & & \downarrow{\scriptstyle \partial_{s+t}} \\
\pi_s\left(\dfrac{D_n}{D_{n+1}}\right)LAS \otimes \pi_{t-1}\left(\dfrac{D_{q+r}}{D_{q+r+1}}\right)LAS & \searrow{\scriptstyle \phi_1} & \\
\pi_{s-1}\left(\dfrac{D_{n+r}}{D_{n+r+1}}\right)LAS \otimes \pi_t\left(\dfrac{D_q}{D_{q+1}}\right)LAS & \xrightarrow{\phi_1} & \pi_{s+t-1}\left(\dfrac{D_{n+q+r}}{D_{n+q+r+1}}\right)LAS.
\end{array}$$

Proof. The construction of the pairing ϕ_r is an easy generalization of the construction used for the composition pairing of [2; §3.1–2]. Naturality (i) also follows from this argument and (ii) follows from the argument [2; p. 334] showing that the differential d^1 is a derivation in the Adams spectral sequence for FS.

Using 9.9 we have from [7; Theorèmé IIA, p. 19–06].

9.10. *The composition pairing* ϕ_r *induces a product*

$$\mu^r : E^r_{s,t}\, LAS \otimes E^r_{s',t'}\, LAS \to E^r_{s+s',t+t'}\, LAS$$

(also denoted by $a \otimes b \to a \cdot b$) such that

(i) $\mu' = \phi_1$ *on* E^1,

(ii) d^r *is a derivation,* $d^r(a \cdot b) = d^r a \cdot b + a \cdot d^r b$,

(iii) μ^{r+1} *is induced by* μ^r *and* μ^∞ *is induced by the* μ^r,

(iv) μ^∞ *is also induced (upon passage to the associated graded group)* by $\phi_0 : \pi_s\, LAS \otimes \pi_t\, LAS \to \pi_{s+t}\, LAS$.

9.11. *The induced product*

$$\mu^2 : E^2_{s,t}\, LAS \otimes E^2_{s',t'}\, LAS \to E^2_{s+s',t+t'}\, LAS$$

of 9.10 *agrees with the Yoneda product*

$$\cup : \mathrm{Ext}^{s,s+t}_{\mathcal{A}(\mathcal{L})}(Z_2, Z_2) \otimes \mathrm{Ext}^{s',s'+t'}_{\mathcal{A}(\mathcal{L})}(Z_2, Z_2) \to \mathrm{Ext}^{s+s',s+s'+t+t'}_{\mathcal{A}(\mathcal{L})}(Z_2, Z_2)$$

under the isomorphism of 9.5.

Proof. Let

$$\omega \in a \in \pi_t(D_s/D_{s+1})LAS = E^1_{s,t}\, LAS,$$

$$\lambda \in b \in \pi_{t'}(D_{s'}/D_{s'+1})LAS = E^1_{s',t'}\, LAS.$$

Let $\gamma \in D_{s'}\, LAS$ be such that proj $\gamma \in b$. Write γ in the form $B(Ai)$ where B is a formula involving only degeneracy operators and the operations tensor product of sum. The composition product $a \cdot b$ is represented by $\mathrm{B}(a)$. Using the element γ and composition $LL \rightarrow L$ construct natural maps

$$\lambda_q : (D_q/D_{q+1})LAS \rightarrow (D_{q+s'}/D_{q+s'+1})LAS.$$

The map λ_0 has the property that $\lambda_0(Ai) = \text{proj}\ \gamma$. Now using the maps λ_q and the $\mathcal{A}(\mathcal{L})$-resolution of H^*LAS constructed from the short exact sequences

$$0 \rightarrow D_{q+1}\, LAS \rightarrow D_q\, LAS \rightarrow (D_q/D_{q+1})LAS \rightarrow 0,$$

construct the following commutative ladder (rows exact)

$$\begin{array}{ccccccccc}
0 \leftarrow H^*LAS \leftarrow & H^*\left(\frac{D_0}{D_1}\right)LAS & \leftarrow \cdots \leftarrow & H^*\left(\frac{D_s}{D_{s+1}}\right)LAS & \leftarrow & H^*\left(\frac{D_{s+1}}{D_{s+2}}\right)LAS & \leftarrow \cdots \\
\nwarrow \lambda^* & \uparrow \lambda_0^* & & \uparrow \lambda_s^* & & \uparrow \lambda_{s+1}^* & \\
 & H^*\left(\frac{D_{s'}}{D_{s'+1}}\right)LAS & \leftarrow \cdots \leftarrow & H^*\left(\frac{D_{s+s'}}{D_{s+s'+1}}\right)LAS & \leftarrow & H^*\left(\frac{D_{s+s'+1}}{D_{s+s'+2}}\right)LAS & \leftarrow \cdot
\end{array}$$

Using this ladder we may define the Yoneda product of cls (κ^*) and cls (λ^*). In $\mathrm{Ext}_{\mathcal{A}(\mathcal{L})}^{s+s',s+s'+t+t'}(Z_2, Z_2)$,

$$\text{cls}\ (\kappa^*) \cup \text{cls}\ (\lambda^*) = \text{cls}\ (\kappa^* \circ \lambda_s^*) = \text{cls}\ ((\lambda_s \circ \kappa)^*)$$

where

$$(\lambda_s \circ \kappa)^* \in \mathrm{Hom}_{\mathcal{A}(\mathcal{L})}^{t+t'+1}\ (H^*(D_{s+s'}/D_{s+s'+1})LAS, Z_2)$$

In $E^2_{s+s',t+t'}\, LAS$,

$$\mu^2([a] \otimes [b]) = [\lambda_s \circ \kappa]$$

where

$$\lambda_s \circ \kappa \in \pi_{t+t'}(D_{s+s'}/D_{s+s'+1})LAS.$$

This completes the proof since these classes correspond under the isomorphism

$$E^2_{s+s',t+t'}\, LAS \simeq \mathrm{Ext}_{\mathcal{A}(\mathcal{L})}^{s+s',s+s'+t+t'}\ (Z_2, Z_2)$$

of 9.5

Bibliography

1. A. Borel, *Sur la cohomologie des espaces fibrés principaux et des espaces homogènes groupes de Lie compact*, Ann. of Math., vol. 57 (1953), pp. 115–207.

2. A. K. Bousfield, E. B. Curtis, D. M. Kan, D. G. Quillen, D. L. Rector and

J. W. SCHLESINGER, *The* mod-*p* *lower central series and the Adams spectral sequence*, Topology, vol. 5 (1966), pp. 331–342.
3. H. CARTAN, *DGA-algèbres et DGA-modules*, Seminaire H. Cartan (1954/1955), exposé 2.
4. H. CARTAN AND S. EILENBERG, *Homological algebra*, Princeton University Press, Princeton, 1956.
5. A. DOLD, *Uber die Steenrodschen Kohomologie-operationen*, Ann. of Math., vol. 73 (1961), pp. 258–294.
6. A. DOUADY, *Le suite spectrale d'Adams*, Seminaire H. Cartan (1958/59), exposé 18.
7. ———, *Le suite spectrale d'Adams: structure multiplicative*, Seminaire H. Cartan (1958/59), exposé 19.
8. S. EILENBERG AND S. MACLANE, *On the groups* $H(\pi, n)$, *II*, Ann. of Math., vol. 60 (1954), pp. 49–139.
9. D. B. A. EPSTEIN, *Semisimplicial objects and the Eilenberg-Zilber theorem*, Invent. Math., vol. 1 (1966), pp. 209–220.
10. D. M. KAN, *On the homotopy relation for c.s.s. maps*, Bol. Soc. Math. Mexicana, vol. 2 (1957), pp. 75–81.
11. ———, *On c.s.s. categories*, Bol. Soc. Math. Mexicana, vol. 2 (1957), pp. 82–94.
12. ———, *On homotopy theory and c.s.s. groups*, Ann. of Math., vol. 68 (1958), pp. 38–53.
13. J. MILNOR, *The Steenrod algebra and its dual*, Ann. of Math. vol. 67 (1958), pp. 150–171.
14. J. MILNOR AND J. C. MOORE, *On the structure of Hopf algebras*, Ann. of Math., vol. 81 (1965), pp. 211–264.
15. J. C. MOORE, *Constructions sur des complexes d'anneax*, Seminaire H. Cartan (1954/55, exposé 12.
16. ———, *Comparison de la bar construction a la construction W et aux complexes* $K(\pi, n)$, Seminaire H. Cartan (1954/55), exposé 13.
17. D G. QUILLEN, *Homotopical algebra*, Lecture Notes in Mathematics, vol. 43, Springer, Berlin, 1967.
18. J. P. SERRE, *Cohomologie modulo 2 des complexes d'Eilenberg-MacLane*, Comment. Math. Helv., vol. 27 (1953), pp. 198–232.

MASSACHUSETTS INSTITUTE OF TECHNOLOGY
CAMBRIDGE, MASSACHUSETTS
NORTHWESTERN UNIVERSITY
EVANSTON, ILLINOIS