H_2 OF SUBGROUPS OF KNOT GROUPS

BY

R. H. CROWELL[1]

1. Summary of results

For any group G, we mean by $H_i(G)$ the i^{th} homology group of G with integer coefficients. Essential to this paper is the fact that if X is a $K(G, 1)$ space, then $H_i(G) = H_i(X)$ for every i. A group Π will be said to be a *knot group* if there exists a tame (polygonal) knot $k \subset S^3$ such that $\Pi = \pi_1(S^3 - k)$.

Consider a subgroup G of a knot group $\Pi = \pi_1(S^3 - k)$. The asphericity of knots states that $\pi_2(S^3 - k) = 0$. This famous theorem [8] together with the fact that there exists a finite 2-dimensional complex K which is a deformation retract of $S^3 - k$ implies that $S^3 - k$ is a $K(\Pi, 1)$ space. Let X be any covering space of a space of the same homotopy type as $S^3 - k$ with the property that $\pi_1(X) = G$. Then X is a $K(G, 1)$ space, and so $H_i(G) = H_i(X)$ for every i.

(1.1) PROPOSITION. *If G is a subgroup of a knot group Π, then $H_i(G) = 0$, for $i \geq 3$, and $H_2(G)$ is free abelian.*

The proof is very simple. Following the above paragraph, we take for the covering space X with $\pi_1(X) = G$ a complex covering the 2-dimensional complex K. Then X is also 2-dimensional. Hence, if $C_i(X)$ is the group of i-chains, then $C_i(X) = 0$ for $i \geq 3$ and, consequently, $H_i(G) = H_i(X) = 0$ for $i \geq 3$. The group $C_2(X)$ is free abelian (although generally not finitely generated), and, since every subgroup of a free abelian group is free [6, p. 45], we conclude that $H_2(G) = H_2(X)$ is free.

The next problem is the determination of the rank of $H_2(G)$. A simple solution in terms of $H_1(G)$ can be given provided G is a subgroup of finite index.

(1.2) PROPOSITION. *If G is a subgroup of a knot group Π and if Π/G (the set of right cosets) is finite, then the homology groups of G are finitely generated and*

$$\operatorname{rank} H_2(G) = \operatorname{rank} H_1(G) - 1.$$

To prove (1.2), let $\Pi = \pi_1(S^3 - k)$, let K be a finite 2-complex which is a deformation retract of $S^3 - k$, and let X be a covering complex of K such that $\pi_1(X) = G$. Since Π/G is finite, the complex X is also finite and its

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homology groups are therefore finitely generated. From Alexander duality it follows that $H_1(K) \cong H_1(S^3 - k)$ is infinite cyclic and that $H_2(K) \cong H_2(S^3 - k) = 0$. The Euler-Poincaré formula therefore implies that

$$\chi(K) = 1 - 1 + 0 = 0.$$

If cardinality$(\Pi/G) = n$, then X is an n-sheeted covering and so

$$\chi(X) = n\chi(K) = 0.$$

Thus, a second application of the Euler-Poincaré formula gives

$$0 = \chi(X) = 1 - \text{rank } H_1(X) + \text{rank } H_2(X).$$

Since $H_i(X) = H_i(G)$, the proof is complete.

Observe that the above proof contains the known results that

$$H_1(\Pi) = \Pi/\Pi' = H_1(S^3 - k)$$

is infinite cyclic and that $H_2(\Pi) = H_2(S^3 - k) = 0$.

We shall give an explicit computation of $H_2(G)$ for the subgroups G corresponding to the cyclic coverings of knots. Consider a knot group $\Pi = \pi_1(S^3 - k)$. The fact that the commutator quotient group Π/Π' is infinite cyclic implies that, for every nonnegative integer n, there exists a normal subgroup Π_n of Π and an exact sequence

$$(1) \qquad 1 \to \Pi_n \to \Pi \to Z/nZ \to 0$$

and Π_n is uniquely determined by this sequence. In particular, Π_0 is the commutator subgroup Π', and $\Pi_1 = \Pi$. Denote by $Z[t, t^{-1}]$ the ring of polynomials in t and t^{-1} with integer coefficients, and consider in this ring the knot polynomials $\Delta_j(t)$ of the knot k, as defined in [3] and normalized so that $\Delta_j(1) = 1$. We recall that $\Delta_{j+1}(t) \mid \Delta_j(t)$ in $Z[t, t^{-1}]$ and that, for all i sufficiently large, $\Delta_j(t)$ is the constant 1. We shall prove

(1.3) THEOREM. *If Π is a knot group and if Π_n is the subgroup defined by the sequence* (1), *then*

$$\begin{aligned} \text{rank } H_2(\Pi_n) &= 0, && \textit{if } n = 0, \\ &= \textstyle\sum_{j=1}^{\infty} b_j, && \textit{if } n > 0, \end{aligned}$$

where b_j is the number of distinct complex n^{th} roots of 1 which are zeros of $\Delta_j(t)/\Delta_{j+1}(t)$.

The case $n = 0$ will be proved in Section 2. Actually, the fact that $H_2(\Pi') = 0$ for every knot group Π has been shown by R. G. Swan in [9, p. 198]. However, the present proof is geometric and very different from his. The 1-dimensional group $H_1(\Pi')$ is of fundamental importance in knot theory. From the fact that

$$H_1(\Pi') = H_1(\Pi; Z(\Pi/\Pi')) = H_1(\Pi; Z[t, t^{-1}])$$

it follows that $H_1(\Pi')$, which as an abelian group is equal to Π'/Π'', is also a $Z[t, t^{-1}]$-module. Specifically, it is the module having the Alexander polynomial $\Delta_1(t)$ of the knot as generator of its 0^{th} elementary ideal and having the matrix $tV - V'$ as a relation matrix (V is the Seifert matrix, and V' is its transpose). It is known [1, p. 349] that rank $H_1(\Pi') =$ degree $\Delta_1(t)$. Since the latter is an even integer, we see that the conclusion of Proposition (1.2) is always false if $G = \Pi_0 = \Pi'$.

For $n > 0$, the group $H_1(\Pi_n)$ is the first homology group of the n-fold cyclic (unbranched) covering space of $S^3 - k$. This group has been studied by many knot theorists, most notably by H. Seifert and R. H. Fox. Let X_n be the unbranched, and X_n^b the branched, n-fold cyclic covering space of $S^3 - k$. In Section 3 we have given a new proof of Fox's theorem that

$$H_1(X_n) = H_1(X_n^b) \oplus Z. \tag{2}$$

Since $H_1(X_n) = H_1(\Pi_n)$, it follows from (1.2) that

$$\text{rank } H_2(\Pi_n) = \text{rank } H_1(X_n) - 1 = \text{rank } H_1(X_n^b). \tag{3}$$

The expression of $\sum_{j=1}^{\infty} b_j$ which appears in (1.3) is then easily shown to be the same as in Fox's formula [4, p. 417] for the rank of $H_1(X_n^b)$.

It is an immediate corollary of (1.1) and (1.3) that

(1.4) *If n is a positive integer, then $H_2(\Pi_n) \neq 0$ if and only if there exists a complex n^{th} root of* 1 *which is a zero of the Alexander polynomial $\Delta_1(t)$.*

For every knot, we have $\Delta_1(1) = 1$ and $\Delta_1(-1) \equiv 1 \pmod 2$. Hence, we obtain $H_2(\Pi) = H_2(\Pi_1) = 0$ and also $H_2(\Pi_2) = 0$. For the trefoil knot, however, it is a consequence of (1.1), (3), and [5, p. 156] that

$$\begin{aligned} H_2(\Pi_n) &= Z \oplus Z, \quad \text{if} \quad n > 0 \text{ and } n \equiv 0 \pmod 6, \\ &= 0, \qquad\qquad \text{otherwise.} \end{aligned}$$

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2. Proof of (1.3) for $n = 0$

In this section we give a new proof of Swan's theorem that $H_2(\Pi') = 0$ for an arbitrary knot group $\Pi = \pi_1(S^3 - k)$. Let S be an orientable spanning surface for the knot k. Specifically, S is semi-linearly embedded in S^3, and $\partial(S) = k$. The genus of S, which we denote by h, need not be minimal. We construct an embedding $f: S \times [-1, 1] \to S^3$ such that $f(s, 0) = s$, for all $s \in S$, and set

$$A = S^3 - f(\mathrm{Int}(S) \times (-1, 1)).$$

(2.1) $H_2(A) = 0$.

Proof. Since A and $S^3 - S$ are of the same homotopy type, $H_2(A) \cong H_2(S^3 - S)$. By Alexander duality we have $H_2(S^3 - S) \cong \tilde{H}^0(S) = 0$.

Let $\sharp, \flat : S \to A$ be the two mappings defined, for every $s \in S$, by $\sharp(s) = f(s, 1)$ and $\flat(s) = f(s, -1)$. Denoting the homomorphisms induced by $\sharp$ and $\flat$ by the same symbols respectively, we have

$$H_1(S) \underset{\flat}{\overset{\sharp}{\rightrightarrows}} H_1(A).$$

It can be shown [10] that there exist bases for $H_1(S)$ and $H_1(A)$ with respect to which the matrices of $\sharp$ and $\flat$ are the Seifert matrix V and its transpose V' respectively. If $\Delta_1(t)$ is the Alexander polynomial of k, then $\Delta_1(t) = \det(tV - V')$. Since $\Delta_1(1) = 1$, we have $\det(V - V') = 1$ and, therefore,

(2.2) *The homomorphism* $\sharp - \flat : H_1(S) \to H_1(A)$ *is an isomorphism.*

Let $\{h_j : S^3 \to S_j^3\}$ be a family, indexed by the integers, of homeomorphisms onto disjoint copies of S^3. For each integer $j \in Z$, consider the embedding $f_j : S \times [-1, 1] \to S_j^3$ defined by $f_j = h_j f$, and set $A_j = h_j(A)$. Let $\sim$ be the equivalence relation on the disjoint union $\bigcup_{j \in Z} A_j$ which identifies $f_j(s, -1)$ with $f_{j+1}(s, 1)$, for every $s \in S$ and $j \in Z$. The identification is indicated schematically in Figure 1. We denote the identification space $(\bigcup_{j \in Z} A_j)/\sim$ by X, and henceforth shall regard the spaces A_j as closed subspaces of X. We define

$$S_j = A_j \cap A_{j+1},$$

and inclusion mappings

$$A_j \xleftarrow{\flat_j} S_j \xrightarrow{\sharp_j} A_{j+1}.$$

The mappings $\theta_j : S \to S_j$ and $\eta_j : A \to A_j$ defined by $\theta_j(s) = f_j(s, -1) = f_{j+1}(s, 1)$ and $\eta_j(a) = h_j(a)$ are homeomorphisms, and for every $j \in Z$, the following diagram is commutative.

$$\begin{array}{ccccccc} & & A & \xleftarrow{\flat} & S & \xrightarrow{\sharp} & A & \\ & & \downarrow{\scriptstyle \eta_j} & & \downarrow{\scriptstyle \theta_j} & & \downarrow{\scriptstyle \eta_{j+1}} & \\ \cdots & \to & A_j & \xleftarrow{\flat_j} & S_j & \xrightarrow{\sharp_j} & A_{j+1} & \leftarrow \cdots \end{array} \tag{4}$$

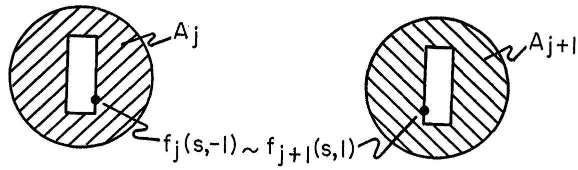

FIGURE 1

It is obvious that X is an infinite cyclic covering space of $S^3 - \text{nbd}(k)$, where $\text{nbd}(k)$ is an open regular neighborhood of the knot k. Since Π/Π' is infinite cyclic, it follows that $\pi_1(X) = \Pi'$. Hence, $H_i(\Pi') = H_i(X)$ for every i. This construction of the covering space X was used by L. Neuwirth [7] in his study of the structure of the group Π'. The proof that $H_2(\Pi') = 0$ is completed by proving that $H_2(X) = 0$.

For every positive integer n, we set $B_n = A_1 \cup \cdots \cup A_n$. The basic lemma is the following:

(2.3) $H_2(B_n) = 0, \quad n = 1, 2, 3, \cdots.$

Proof. If $n = 1$, the conclusion is a direct corollary of (2.1), since $B_1 = A_1 \cong A$. So we assume that $n \geq 2$. Define

$$B_n' = B_n \cap \textstyle\bigcup_{j \in Z} A_{2j+1} \quad \text{and} \quad B_n'' = B_n \cap \textstyle\bigcup_{j \in Z} A_{2j}.$$

Then, $B_n = B_n' \cup B_n''$ and $B_n' \cap B_n'' = S_1 \cup \cdots \cup S_{n-1}$. Moreover,

$$H_i(B_n') \oplus H_i(B_n'') = H_i(A_1) \oplus \cdots \oplus H_i(A_n),$$

$$H_i(B_n' \cap B_n'') = H_i(S_1) \oplus \cdots \oplus H_i(S_{n-1}).$$

Thus, part of the Mayer-Vietoris sequence of the pair consisting of B_n' and B_n'' is

$$H_2(A_1) \oplus \cdots \oplus H_2(A_n) \xrightarrow{j_*} H_2(B_n) \xrightarrow{\partial_*}$$

$$H_1(S_1) \oplus \cdots \oplus H_1(S_{n-1}) \xrightarrow{i_*} H_1(A_1) \oplus \cdots \oplus H_1(A_n).$$

Since $A_j \cong A$, we have $H_2(A_j) = 0$, from which it follows that ∂_* is a monomorphism. We conclude from the exactness of the above sequence that

$$H_2(B_n) \cong \text{Image }(\partial_*) = \text{Kernel }(i_*).$$

It therefore only remains to prove that i_* is a monomorphism. We have

$$\begin{aligned} i_*(u_1 \oplus \cdots \oplus u_{n-1}) = \flat_1(u_1) &- \sharp_1(u_1) \\ &- \flat_2(u_2) + \sharp_2(u_2) \\ &\qquad + \flat_3(u_3) - \sharp_3(u_3) \\ &\qquad\qquad \text{etc.} \end{aligned}$$

The groups $H_1(S)$, $H_1(S_j)$, $H_1(A)$, and $H_1(A_j)$ are all free with rank $2h$. With respect to some choice of bases for $H_1(S)$ and $H_1(A)$, let V and W be the matrices defining the homomorphisms

$$\sharp : H_1(S) \to H_1(A) \quad \text{and} \quad \flat : H_1(S) \to H_1(A),$$

respectively. As a result of the commutative diagram (4), it follows that (up to sign) the homomorphism i_* is defined by the matrix

$$M_n = \begin{array}{c|cccccc|}
 & 1 & 2 & 3 & 4 & \cdots & n \\
\hline
1 & -W & V & 0 & 0 & \cdots & 0 \\
2 & 0 & -W & V & 0 & & 0 \\
3 & 0 & 0 & -W & V & & 0 \\
\vdots & \vdots & & & & & \\
n-1 & 0 & 0 & 0 & & & V
\end{array}$$

Since $\sharp - \flat$ is an isomorphism, the matrix $V - W$ is invertible. We contend that

$$\text{rank } M_n = (n-1)(2h). \tag{5}$$

Since

$$\text{rank Kernel } (i_*) = (n-1)(2h) - \text{rank } M_n,$$

proving (5) will finish the proof of (2.3). The argument is inductive. For $n = 2$, we have

$$M_2 = (-W \quad V) \sim (-W \quad V - W),$$

and the rank of the equivalent righthand matrix is obviously $2h$. We shall give in detail the reduction from $n = 5$ to $n = 4$, and this will convincingly illustrate the general inductive step from $n \geq 3$ to $n - 1$.

$$M_5 = \begin{bmatrix} -W & V & 0 & 0 & 0 \\ 0 & -W & V & 0 & 0 \\ 0 & 0 & -W & V & 0 \\ 0 & 0 & 0 & -W & V \end{bmatrix}$$

Add the 1st column block to the 2nd, the new 2nd to the third, the new 3rd to the 4th, etc., to obtain the equivalent matrix

$$\begin{bmatrix} -W & V-W & V-W & V-W & V-W \\ 0 & -W & V-W & V-W & V-W \\ 0 & 0 & -W & V-W & V-W \\ 0 & 0 & 0 & -W & V-W \end{bmatrix}.$$

Subtract the 2nd row block from the 1st, the 3rd from the 2nd, and the 4th from the 3rd, to get the equivalent matrix

$$M_5' = \begin{bmatrix} -W & V & 0 & 0 & 0 \\ 0 & -W & V & 0 & 0 \\ 0 & 0 & -W & V & 0 \\ 0 & 0 & 0 & -W & V-W \end{bmatrix} = \left[\begin{array}{c|c} M_4 & 0 \\ \hline 0 \quad -W & V - W \end{array}\right]$$

Since rank $M_4 = 3(2h)$ by induction and since rank $(V - W) = 2h$, it follows that rank $M_5 = 4(2h)$. This completes the proof of equation (5), and also of Proposition (2.3).

For every nonnegative integer n, we now define

$$B_n^* = A_{-n} \cup \cdots \cup A_0 \cup \cdots \cup A_n .$$

Since $B_n^* \cong B_{2n+1}$, it is a corollary of (2.3) that $H_2(B_n^*) = 0$, for $n = 0, 1, 2, \cdots$. But the covering space X is the union of the infinite chain of subspaces $B_0^* \subset B_1^* \subset B_2^* \subset \cdots$. Since the homology functor commutes with direct limits, it follows at once that $H_2(X) = 0$, and, as observed above, this proves that $H_2(\Pi') = 0$.

3. Finite cyclic covering spaces

For $n > 0$, the unbranched n-fold cyclic covering space X_n of $S^3 - \text{nbd}(k)$ is obtained from B_n by identifying S_0 and S_n. Specifically, we consider the equivalence relation $\sim$ on B_n which identifies $f_1(s, 1)$ and $f_n(s, -1)$, for every $s \in S$, and we form the identification space $X_n = B_n/\sim$. Our primary objective is to give a proof of equation (2) in Section 1, which relates the 1st homology of the branched and unbranched covering spaces. The equation is obviously true for $n = 1$, and we shall therefore assume that $n \geq 2$. As a result, the spaces $A_1, \cdots, A_n$ and $B_1, \cdots, B_{n-1}$ are embedded in X_n and henceforth will be regarded as subspaces. Thus, we have

$$B_{n-1} \cup A_n = X_n, \qquad B_{n-1} \cap A_n = S_{n-1} \cup S_n \quad (\text{and } S_n = S_0).$$

The space B_{n-1} is a 3-dimensional manifold with a boundary consisting of the union of an annulus and the two homeomorphic surfaces S_0 and S_{n-1}. The same is true of A_n. The union $B_{n-1} \cup A_n = X_n$, indicated schematically in Figure 2, is a 3-dimensional manifold whose boundary is a torus formed by the union of the two annuli. Let T be a solid torus with interior disjoint from X_n and such that $\partial(T) = \partial(X_n)$. The union $X_n \cup T$ is the branched covering space X_n^b. In the following mapping diagram the two rows are corresponding parts of reduced Mayer-Vietoris sequences: one for B_{n-1} and A_n, and the other for $B_{n-1} \cup T$ and A_n. The homomorphism φ_1 is induced

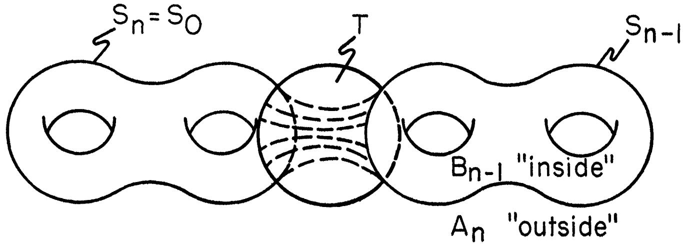

FIGURE 2

by inclusion, and φ_2 is the direct sum of the homomorphisms induced by the inclusion $B_{n-1} \to B_{n-1} \cup T$ and by the identity $A_n \to A_n$.

$$\begin{array}{ccccccc}
H_1(B_{n-1} \cap A_n) & \xrightarrow{i_*} & H_1(B_{n-1}) \oplus H_1(A_n) & \xrightarrow{j_*} & H_1(X_n) & \xrightarrow{\partial_*} & \tilde{H}_0(B_{n-1} \cap A_n) \to 0 \\
\downarrow \varphi_1 & & \downarrow \varphi_2 & & \uparrow \psi & & \\
H_1((B_{n-1} \cup T) \cap A_n) & \xrightarrow{i'_*} & H_1(B_{n-1} \cup T) \oplus H_1(A_n) & \xrightarrow{j'_*} & H_1(X^b_n) & \xrightarrow{\partial'_*} & 0
\end{array}$$

It follows easily from the theory of the homology of orientable 2-manifolds that φ_1 is an isomorphism. Since B_{n-1} is obviously a deformation retract of $B_{n-1} \cup T$, we conclude that φ_2 is also an isomorphism. Since the relevant homomorphisms are induced by inclusion, the first square of the diagram is commutative, i.e., $\varphi_2 i_* = i'_* \varphi_1$. Simple diagram chasing then shows that

$$\text{Kernel } (j'_*) = \text{Kernel } (j_* \varphi_2^{-1}).$$

Since j'_* is an epimorphism, one direction of this equality implies that there exists a homomorphism $\psi : H_1(X^b_n) \to H_1(X_n)$ such that

$$\psi j'_* = j_* \varphi_2^{-1}.$$

The other direction implies that ψ is a monomorphism. Moreover,

$$\text{Image } (\psi) = \text{Image } (\psi j'_*) = \text{Image } (j_* \varphi_2^{-1}) = \text{Image } (j_*).$$

Hence, the sequence

$$0 \to H_1(X^b_n) \xrightarrow{\psi} H_1(X_n) \xrightarrow{\partial_*} \tilde{H}_0(B_{n-1} \cap A_n) \to 0$$

is exact. Since $B_{n-1} \cap A_n$ is the disjoint union of S_{n-1} and S_n, it follows that $\tilde{H}_0(B_{n-1} \cap A_n) = Z$, and we finally obtain the sequence

$$0 \to H_1(X^b_n) \xrightarrow{\psi} H_1(X_n) \xrightarrow{\partial_*} Z \to 0,$$

which is split exact. This proves equation (2) in Section 1.

The proof of Theorem (1.3) for $n > 0$ is finished provided it is assured that the number $\sum_{j=1}^{\infty} b_j$, which appears there, equals the analogous number in Fox's formula [4, p. 417] for the rank of $H_1(X^b_n)$. The only question is whether or not the jth elementary divisor of his matrix $\mathbf{F}(t)$ is equal to the ratio $\Delta_j(t)/\Delta_{j+1}(t)$ of the knot polynomials. An affirmative answer is implied by Fox at the bottom of page 416 in [4], and is also proved on page 698 of [2].

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DARTMOUTH COLLEGE
HANOVER, NEW HAMPSHIRE