INTEGRAL BASES IN LINEAR TOPOLOGICAL SPACES

BY

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1. Introduction

In [2], Dean has shown that m does not admit a Schauder basis of subspaces. More generally he has shown that no P_λ space; cf. [1, p. 94], for a definition of P_λ space; admits a Schauder basis of subspaces. The purpose of this paper is to consider a basis structure, based upon integration rather than summation, which exists in every P_1 space, and in fact in every P_λ space which is isomorphic to a P_1 space. The existence of these integral bases is not limited to P_λ spaces. It is shown for example that any linear topological space which admits a summation basis also admits an integral basis.

The integral bases considered here differ from the integral bases constructed by Edwards in [3] in two major ways. First, the vector-valued set functions used in the construction of integrals in this paper need not be countably additive. Secondly, the basis structure in this paper is based upon an integral of Hellinger type. There is one big advantage to using a Hellinger type integral in constructing integral bases. The approximating sums for such an integral allow a net of projection operators to be associated with the basis in a natural way which is analogous to the sequence of partial sum operators associated with a summation basis or basis of subspaces. Because of this it is possible to show very simply that there are theorems for the integral basis which are analogous to the Bessaga-Pelcynski weak basis theorem and the theorem of Banach-Newns-Arsove which asserts that a summation basis for a complete metric space is a Schauder basis.

In addition to the topics previously mentioned, this paper also considers relations between integral bases and a type of biorthogonality condition introduced by Kaplan in [6].

2. Preliminary results

Throughout this paper all linear vector spaces are assumed, unless otherwise stated, to be infinite dimensional spaces over the real or complex number field. If V is a linear vector space N will denote the null vector for V. For the purposes of this paper, linear Hausdorff space means a linear topological space which is a Hausdorff space in its vector topology. Finally, if V is a linear Hausdorff space, a sequence $\{b_i\}_{i=1}^{\infty}$ of elements of V will be said to be a summation basis for V if there exists a unique sequence $\{l_i\}_{i=1}^{\infty}$ of linear functionals on V such that if y is in V then $y = \sum_{i=1}^{\infty} l_i(y)b_i$. If each l_i is continuous $\{b_i\}_{i=1}^{\infty}$ will be said to be a Schauder basis for V.

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In order to define the integral used in this paper a few preliminary results are needed.

Definition 1. Suppose X is a non-void set. The statement that a collection of subsets of X, $\mathcal{P}$, is a proto-ring means that if A and B are in $\mathcal{P}$ then there exist finite disjoint collections, $\{E_i\}_{i=1}^{p}$ and $\{F_j\}_{j=1}^{q}$, in $\mathcal{P}$ such that $A \cap B = \cup_{i=1}^{p} E_i$ and $A - B = \cup_{j=1}^{q} F_j$. If $\mathcal{P}$ contains a finite disjoint collection $\{G_k\}_{k=1}^{r}$ such that $X = \cup_{k=1}^{r} G_k$ then $\mathcal{P}$ will be said to be a proto-algebra.

It is an immediate consequence of this definition that $\emptyset$ belongs to every proto-ring.

If $\mathcal{P}$ is a non-void collection of subsets of a set X and $D = \{D_i\}_{i=1}^{p}$ is a finite disjoint collection of non-void elements of $\mathcal{P}$ such that $X = \bigcup_{i=1}^{p} D_i$ then D will be said to be a $\mathcal{P}$-subdivision of X. If E is a $\mathcal{P}$-subdivision of X, E will be said to refine D if each element of E is contained in some element of D. If $\mathcal{P}$ is a proto-algebra, the collection of all $\mathcal{P}$-subdivisions of X will be denoted by $\mathfrak{D}$. It is an immediate consequence of Definition 1 that $\mathfrak{D}$ is directed by refinement. It also follows from Definition 1 that if $\mathcal{P}$ is a proto-algebra and E is in $\mathcal{P}$ then there exists a $\mathcal{P}$-subdivision of X to which E belongs. These two properties in fact characterize proto-algebras.

Lemma 1. *Suppose X is a non-void set and $\mathcal{P}$ is a collection of subsets of X such that*

(1) some subcollection of $\mathcal{P}$ subdivides X;

(2) the collection of all $\mathcal{P}$-subdivisions of X is directed by refinement;

(3) if $E \in \mathcal{P}$ there exists a $\mathcal{P}$-subdivision of X to which E belongs.

Then $\mathcal{P} \cup \{\emptyset\}$ is a proto-algebra.

The proof of this lemma is omitted since it is an immediate consequence of the definitions of subdivision and of refinement. A more detailed discussion of the properties of proto-rings including a complete proof of Lemma 1 will be found in [8].

If $\mathcal{P}$ is a proto-ring, a finitely-additive function μ on $\mathcal{P}$ into a linear Hausdorff space will be said to be a p-volume.

Definition 2. Suppose X is a non-void set, $\mathcal{P}$ is a proto-algebra of subsets of X, φ is a choice function on $\mathcal{P} - \{\emptyset\}$ and μ is a p-volume on $\mathcal{P}$ into a linear Hausdorff space (V, τ). A function f on X into the scalar field for V is said to be φ-integrable with respect to μ if and only if the net

$$\{\textstyle\sum_{i=1}^{p} f(\varphi(D_i))\mu(D_i) : \{D_i\}_{i=1}^{p} \in \mathfrak{D}\}$$

τ-converges. The limit of this net will be denoted by $\varphi \int_X f\, d\mu$.

For the most part only very simple properties of the integral introduced in Definition 2 will be used in the remainder of this paper. These properties will be introduced as needed. A less obvious property of the φ integral which will be needed later is given in

LEMMA 2. *Suppose X is a non-void set, $\mathcal{P}$ a proto-algebra of subsets of X, φ a choice function on $\mathcal{P} - \{\emptyset\}$ and μ a p-volume on $\mathcal{P}$ into a linear Hausdorff space. Let X_1 denote the range of φ. Then $\mathcal{P} \cap X_1$ is a proto-algebra of subsets of X_1 and there exists a p-volume μ_1 on $\mathcal{P} \cap X_1$, with the range of μ_1 equal to the range of μ, and a choice function φ_1 on $\mathcal{P} \cap X_1$, with range X_1, such that $\varphi \int_X f\, d\mu$ exists if and only if $\varphi_1 \int_{X_1} f\, d\mu_1$ exists. Furthermore both integrals have the same value.*

Proof. Let F denote the mapping of $\mathcal{P}$ onto $\mathcal{P} \cap X_1$ defined by $F(E) = E \cap X_1$, $\forall E \in \mathcal{P}$. Since no non-void subset of $\mathcal{P}$ is contained in $X_1^{\sim}$ it follows that F is one to one from which it follows that $\mathcal{P} \cap X_1$ is a proto-algebra, that $\mu_1 = \mu \circ F^{-1}$ is a p-volume on $\mathcal{P} \cap X_1$, and that $\varphi_1 = \varphi \circ F^{-1}$ is a choice function on $\mathcal{P} \cap X_1 - \{\emptyset\}$. F also defines in a natural way a one to one isotone correspondence between the $\mathcal{P}$-subdivisions of X and the $\mathcal{P} \cap X_1$-subdivisions of X_1. The conclusion of the lemma follows from this observation and Definition 2.

3. Integral bases

The work of Fichtenholtz and Kantorovich [4] and of Hildebrandt [5] on integral representations of continuous linear functionals on spaces of bounded measurable functions strongly suggests that it should be possible to define a type of basis in spaces such as m if one is willing to use some sort of integral-like process in the construction of the basis. If in addition one wishes to construct a basis which has the property that a summation basis for a space generates an integral basis for that space one is led to a definition such as the following.

DEFINITION 3. Suppose V is a linear Hausdorff space and B is a subset of V which does not contain N. The statement that B is an I-basis for V means that there exists a non-void set X, a proto-algebra $\mathcal{P}$ of subsets of X, a p-volume μ on X with range $B \cup \{N\}$, a choice function φ on $\mathcal{P} - \{\emptyset\}$ and a unique scalar-valued map l on $V \times X$ such that if y is in V then $y = \varphi \int_X l(y, -)\, d\mu$ and the net

$$\{\textstyle\sum_D l(y, \varphi(D_i))\mu(D_i) : D = \{D_i\}_{i=1}^{p} \in \mathfrak{D}\}$$

is bounded.

Suppose now that V is a linear Hausdorff space and that B is an I-basis for V. It is an immediate consequence of Definition 2 that for each x in X, $l(-, x)$ is a linear functional on V. This functional will be said to be the x-coordinate functional. The uniqueness requirement of Definition 3 together with Definition 2 implies that the range of φ is X and that for each x in X and each $\mathcal{P}$-subdivision D there exists a refinement E of D such that for some E_i in E, $\varphi(E_i)$ is x. This in turn implies that if E is in $\mathcal{P}$ then $l(\mu(E), -)$ is χ_E and that μ is one to one.

If $D = \{D_i\}_{i=1}^{p}$ is a $\mathcal{P}$-subdivision of X then

$$\textstyle\sum_{i=1}^{p} l(-, \varphi(D_i))\mu(D_i)$$

defines a linear transformation of V into itself. This transformation will be denoted by $\mathfrak{J}_D$. These $\mathfrak{J}_D$ transformations are a direct analogue to the partial sum operators of summation basis theory. There are however some differences in behavior between the $\mathfrak{J}_D$ operators and partial sum operators. One of these differences which is of some consequence is that if D and E are $\mathcal{P}$-subdivisions of X then $\mathfrak{J}_D \mathfrak{J}_E$ is not in general equal to $\mathfrak{J}_D$ even if E refines D. It is however easily shown to be true that $\mathfrak{J}_E \mathfrak{J}_D = \mathfrak{J}_D$ if E refines D. Because of this last relation several of the major theorems in summation basis theory also hold for I-bases with only small changes in the proof.

LEMMA 3. *Suppose (V, τ) is a locally convex Hausdorff space, B is a $\sigma(V, V^*)$ I-basis for V, and $\mathfrak{U} = \{U_i\}_{i \in \mathfrak{s}}$ is a local base for τ consisting of closed convex balanced neighborhoods of N. Then*

(a) $\mathfrak{U}' = \{U_i' = \bigcap_{D \in \mathfrak{D}} \mathfrak{J}_D^{-1}[U_i] : i \in \mathfrak{s}\}$ *is a local base for a separated locally convex topology τ' for V, and $\tau \subset \tau'$;*

(b) $\{\mathfrak{J}_D : D \in \mathfrak{D}\}$ *is an equi-continuous collection of maps from (V, τ') to (V, τ);*

(c) *if (V, τ) is metrizable so is (V, τ');*

(d) *if (V, τ) is complete so is (V, τ').*

Proof. This lemma is an analogue of Lemma 2 in [7]. The proofs of parts (a) and (c) are almost identical to the proofs given by McArthur in [7]. Part (b) is a trivial consequence of part (a) and is somewhat weaker than the corresponding part of McArthur's lemma. The stronger form would hold in the I-basis setting if B were a basis with the property that $\mathfrak{J}_D \mathfrak{J}_E = \mathfrak{J}_D$ when E refines D.

Suppose now that $\{y_j : j \in \mathfrak{J}\}$ is a τ'-Cauchy net in V. It follows from part (b) that for each D in $\mathfrak{D}$, $\{\mathfrak{J}_D y_j : j \in \mathfrak{J}\}$ is a τ-Cauchy net and that in fact this is true uniformly in D. Suppose that x is an element of X. Let E be an element of $\mathcal{P}$ such that $\varphi(E) = x$ and let $D(E) = \{E_i\}_{i=1}^{p}$ be a $\mathcal{P}$-subdivision of X to which E belongs. It is easily shown that $\{\mu(E_i)\}_{i=1}^{p}$ is a Hamel basis for the range of $\mathfrak{J}_{D(E)}$ and therefore since $\{\mathfrak{J}_{D(E)} y_j : j \in \mathfrak{J}\}$ is a Cauchy net it follows that $\{l(y_j, x) : j \in \mathfrak{J}\}$ is a Cauchy net of scalars. Let θ denote the scalar valued function on X which is the pointwise limit of the net $\{l(y_j, -) : j \in \mathfrak{J}\}$ on X. Suppose now that $D = \{D_i\}_{i=1}^{q}$ is a $\mathcal{P}$-subdivision of X and that U is a τ-neighborhood of N. Let V be a τ-neighborhood of N such that $\sum_{i=1}^{q} V$ is contained in U. There exists an element $\jmath$ of $\mathfrak{J}$ such that if j follows $\jmath$ then

$$[l(y_j, \varphi(D_i)) - \theta(\varphi(D_i))]\mu(D_i) \in V, \qquad \forall D_i \in D,$$

and therefore

$$\mathfrak{J}_D y_j - \sum_{i=1}^{q} \theta(\varphi(D_i))\mu(D_i) \in U.$$

Hence the net $\{\mathfrak{J}_D y_j : j \in \mathfrak{J}\}$ τ-converges to $\sum_{i=1}^{q} \theta(\varphi(D_i))\mu(D_i)$. A straightforward argument now shows that θ is $l(y, -)$, where y is the τ-limit of the net $\{y_j : j \in \mathcal{J}\}$. A similar argument then shows that this net also τ'-converges to y. This completes the proof.

LEMMA 4. *Suppose (V, τ) is a barrelled space and B is a $\sigma(V, V^*)$ I-basis for V with continuous coordinate functionals. Then B is a τI-basis for V.*

Proof. Since the net $\{\mathfrak{J}_D : D \in \mathfrak{D}\}$ is by hypothesis weakly pointwise bounded and since $\mathfrak{J}_E \mathfrak{J}_D = \mathfrak{J}_D$ if E refines D a standard proof (cf. [7, Lemma 1]) from Schauder basis theory may be used to prove the lemma.

Lemmas 3 and 4 then yield an analogue for I-bases to the Bessaga-Pelcynski weak basis theorem for summation bases.

THEOREM 1. *Suppose (V, τ) is a Frechet space and B is a $\sigma(V, V^*)$ I-basis for V. Then B is a τ I-basis for V with continuous coordinate functionals.*

Lemma 2 remains valid if the words "locally convex", "$\sigma(V, V^*)$", and "convex" are deleted from its statement. This observation yields

THEOREM 2. *Suppose V is a complete linear metric space and B is an I-basis for V. Then the coordinate functionals for B are continuous.*

The preceding group of theorems show that the theory of I-bases parallels very closely the theory of summation bases. There is however an even closer relation; every summation basis generates an I-basis.

THEOREM 3. *Suppose V is a linear Hausdorff space and that $B' = \{b_i\}_{i=1}^{\infty}$ is a summation basis for V. Then $B = \{b_i\}_{i=1}^{\infty} \cup \{b_i - b_{i+1}\}_{i=1}^{\infty}$ is an I-basis for V.*

Proof. Let X be the set of all positive integers and let $\mathcal{P}$ be

$$\{\emptyset\} \cup \{\{p\}\}_{p=1}^{\infty} \cup \{F_q = \{r \in X : r \geq q\}\}_{q=1}^{\infty}.$$

Define μ by

$$\begin{aligned} \mu(E) &= N, & E &= \emptyset, \\ &= b_q, & E &= F_q, q = 1, 2, 3, \cdots \\ &= b_p - b_{p+1}, & E &= \{p\}, p = 1, 2, 3, \cdots \end{aligned}$$

and φ by

$$\begin{aligned} \varphi(E) &= p, & E &= \{p\}, p = 1, 2, 3, \cdots \\ &= q, & E &= F_q, q = 1, 2, 3, \cdots. \end{aligned}$$

Finally, let l be the scalar-valued function defined by

$$l(y, j) = \sum_{i=1}^{\infty} l_i(y) \chi_{F_i}(j), \qquad \forall y \in V, \ \forall j \in X.$$

The above function is well defined since a given j in X belongs to only finitely many of the F_i sets. If D is a $\mathcal{P}$-subdivision of X then it follows from the definition of $\mathcal{P}$ that there exists an integer q such that D is $\{\{1\}, \cdots, \{q-1\},$

$F_q\}$. Therefore there is a one to one mapping of X onto $\mathfrak{D}$, and the mapping is in fact order-preserving. If $D = \{\{1\}, \cdots, \{n-1\}, F_n\}$ is an arbitrary $\mathcal{P}$-subdivision of X then a straightforward computation yields,

$$\mathfrak{J}_D y = \sum_{i=1}^{n} l_i(y) b_i, \qquad \forall y \in V.$$

The conclusion of the theorem now follows.

Since the proof of Theorem 3 gives a one to one correspondence between the collection of operators $\{\mathfrak{J}_D\}$ generated by B and the partial sum operators for B' it follows that B' is a Schauder basis if and only if the coordinate functionals for B are continuous.

There exist many Banach spaces whose elements are functions on a set X which have the property that their norm is equivalent to the sup norm and which contain a fundamental set whose elements are the characteristic functions of the sets of a proto-algebra of subsets of X. The most obvious example of a space of this type is the space of all functions on a set X which are bounded and measurable with respect to a given σ-algebra of subsets of X. A second example of such a space, and one which is of some analytical interest, is the space of quasi-continuous functions on a number interval $[a, b]$. Here the collection of characteristic functions of open subintervals of $[a, b]$ and singleton sets is fundamental. All such Banach spaces admit I-bases.

Example 1. Suppose X is a non-void set and $\mathcal{P}$ is a proto-algebra of subsets of X. Let $Q(X, \mathcal{P})$ denote the space of scalar (i.e. real or complex) valued functions on X which are uniformly approximatable by linear combinations of characteristic functions of sets in $\mathcal{P}$. Suppose further that $Q(X, \mathcal{P})$ is given the sup norm topology. $Q(X, \mathcal{P})$ is clearly a Banach space with this topology. It follows from Definition 1 by a straightforward argument that a function f is in $Q(X, \mathcal{P})$ if and only if for each positive number ε there exists a $\mathcal{P}$-subdivision D of X such that if $\{E_i\}_{i=1}^{r}$ refines D and p and q are in E_i, $i = 1, 2, \cdots, r$, then $|f(p) - f(q)| < \varepsilon$. This observation together with Definition 2 implies that if μ is the p-volume defined by $\mu(E) = \chi_E$, $\forall E \in \mathcal{P}$, and φ is an arbitrary choice function on $\mathcal{P} - \{\emptyset\}$ then $f = \varphi \int_X f\, d\mu$, $\forall f \in Q(X, \mathcal{P})$. It may be shown by an elementary transfinite induction argument that there exists a choice function φ on $\mathcal{P} - \{\emptyset\}$ having the property that if E and F are in $\mathcal{P} - \{\emptyset\}$, $E \subset F$, and $\varphi(F)$ is in $E \cap F$ then $\varphi(F) = \varphi(E)$. It will be assumed in the remainder of this discussion that such a choice function has been selected. Suppose now that g is a scalar valued function on X such that $\varphi \int_X g\, d\mu = N$. Then if ε is a positive number, there exists a $\mathcal{P}$-subdivision D of X such that if $E = \{E_i\}_{i=1}^{p}$ refines D then $\| \sum_{i=1}^{p} g(\varphi(E_i))\chi_{E_i} \| < \varepsilon$. If x is in the range of φ it follows from the properties of a proto-ring and the properties of φ that there exists a $\mathcal{P}$-subdivision $\{E_i\}_{i=1}^{p}$ which refines D and has the property that for some $\bar{\imath}$, $1 \leq \bar{\imath} \leq p$, $\varphi(E_{\bar{\imath}}) = x$. Thus $|g(x)| < \varepsilon$ and since ε and x are arbitrary, g must be identically zero on the range of φ. Thus it follows from Lemma 2 that if X_1, φ_1, and μ_1 are defined as in Lemma

2 then the function l defined by $l(f, x) = f(x), \forall f \in Q(X, \mathcal{P}), \forall x \in X_1$, is the only scalar valued function on $X_1 \times Q(X, \mathcal{P})$ such that

$$f = \varphi_1 \int_{X_1} l(f, -)\, d\mu_1, \qquad \forall f \in Q(X, \mathcal{P}).$$

Since the $\mathfrak{I}_D$ operators associated with this integral each have norm one it follows that $B = \{\chi_E : E \in \mathcal{P} - \{\emptyset\}\}$ is an I-basis for $Q(X, \mathcal{P})$. Because of the special character of the choice function φ_1, the I-basis just constructed has one additional property of interest, namely $\mathfrak{I}_D \mathfrak{I}_E = \mathfrak{I}_D$ if E refines D.

To obtain another example of a $Q(X, \mathcal{P})$ space, let X be a compact Hausdorff space which is Boolean; i.e. the family of compact open subsets of X is a base for the topology on X; and let $\mathcal{P}$ be the algebra of open closed subsets of X. Since the characteristic function of an open closed subset of X is continuous it follows that in this case $Q(X, \mathcal{P})$ is contained in $C(X)$. It then follows immediately from the Stone-Weierstrass theorem that $Q(X, \mathcal{P})$ is $C(X)$.

It is a straightforward consequence of Definition 3 that if V is a linear Hausdorff space which admits an I-basis B with continuous co-ordinate functionals and there exists a linear homeomorphism $\mathcal{L}$ of V onto a linear Hausdorff space W then $\mathcal{L}[B]$ is an I-basis for W. Since it is known (cf. [1, theorem 3, p. 94]) that every P_1 space is isometrically isomorphic to a space of continuous functions on an extremely disconnected compact Hausdorff space it follows from Example 1 and the remarks of the previous paragraph that every P_1 space admits an I-basis; in fact every P_λ space which is isomorphic to a P_1 space admits an I-basis.

4. *I*-bases and biorthogonality

In the previous section of this paper the similarities between summation bases and I-bases have been pointed out and exploited. There are however also major differences between the two concepts. For example, it is clear from Theorem 3 that an I-basis need not be a linearly independent set. In fact a necessary and sufficient condition for an I-basis to be linearly independent is that the associated proto-ring have the property that no finite disjoint union of non-void elements of the proto-ring is an element of the proto-ring, and this condition is not satisfied by any of the concrete examples of $Q(X, \mathcal{P})$ spaces considered in Section 2. Related to this lack of linear independence is another property which is well illustrated by $Q(X, \mathcal{P})$ spaces. It is easily seen from definition 1 (cf. [8]) that if $\mathcal{P}$ is a proto-algebra of subsets of a set X then $Q(X, \mathcal{P}) = Q(X, R(\mathcal{P}))$, where $R(\mathcal{P})$ is the algebra generated by $\mathcal{P}$. It therefore follows from Example 1 that one I-basis for a space may be a proper subset of another I-basis for the space.

In view of the preceding remarks one would not expect the concept of biorthogonality to be of much importance in the study of I-bases. There is however an extension of the concept of biorthogonality introduced by Kaplan in [6] which plays much the same role in the study of I-bases as that which biorthogonality plays in the study of summation bases. For convenience some of the definitions in Kaplan's paper are reproduced here.

DEFINITION 4. Suppose V is a linear topological space and $B = \{b_i\}_{i \epsilon I}$ and $\Upsilon = \{f_x\}_{x \epsilon X}$ are subsets of V and V^* respectively. The pair (B, Υ) will be said to be biorthogonal in the wide sense (abbreviated as bows) if it satisfies the following two conditions.

I. (a) If $b_i \epsilon B$ and $f_x \epsilon \Upsilon$ then $f_x(b_i)$ is zero or one;

(b) No f_x is zero at every b_i, and for each b_i in B there is some f_x in Υ such that $f_x(b_i)$ is one.

A finite subset $\{b_j\}_{j=1}^k$ of B will be called a Υ-orthogonal set if no f_x in Υ has value one at more than one b_j, $j = 1, 2, \cdots, k$.

II. The Υ-orthogonal subsets of B form a direct set under the ordering $\prec$ defined by

$$\{b_j\}_{j=1}^k \prec \{b_l\}_{l=1}^q \text{ iff } \{b_j\}_{j=1}^k \subset \text{sp } \{b_l\}_{l=1}^q .$$

In [6], Definition 4 is stated only for real Banach spaces but it is clear that the definition is quite meaningful in a general context. Furthermore, those theorems in [6] which are needed in this paper clearly hold in a general setting and will be used without further comment. In addition to the terminology introduced in [6] the following terminology is useful here.

DEFINITION 5. Suppose V is a linear topological space and $(B \subset V, \Upsilon \subset V^*)$ is bows. A Υ-orthogonal subset $\{b_j\}_{j=1}^k$ of B will be said to be a full Υ-orthogonal subset if every element of Υ has value one at some b_j, $j = 1, 2, \cdots, k$.

Suppose now that V is a linear Hausdorff space and $B = \{b_i\}_{i \epsilon I}$ is an I-basis for V with continuous co-ordinate functionals. Denote the collection of co-ordinate functionals for B, $\{l(-, x)\}_{x \epsilon X}$, by Υ. Since $l(b_i, x)$ is $\chi_{\mu^{-1}(b_i)}(x)$ the pair (B, Υ) satisfies condition I of Definition 4. It follows from the definition of Υ that if $\{b_j\}_{j=1}^k$ is Υ-orthogonal then $\{\mu^{-1}(b_j)\}_{j=1}^k$ is a disjoint collection of non-void sets of $\mathcal{P}$. Conversely if $\{E_l\}_{l=1}^q$ is a disjoint collection of non-void sets of $\mathcal{P}$ then $\{\mu(E_l)\}_{l=1}^q$ is a Υ-orthogonal subset of B. Furthermore if $\{D_j\}_{j=1}^k$ is a $\mathcal{P}$-subdivision of $\bigcup_{l=1}^q E_i$ which refines $\{E_l\}_{l=1}^q$ then $\{\mu(E_l)\}_{l=1}^q$ is contained in sp $\{\mu(D_j)\}_{j=1}^k$. It therefore follows that (B, Υ) satisfies condition II of Definition 4 and so is bows. Also if $\{D_j\}_{j=1}^k$ is a $\mathcal{P}$-subdivision of X then $\{\mu(D_j)\}_{j=1}^k$ is a full Υ-orthogonal subset of B. There is a partial converse to this result.

THEOREM 4. *Suppose V is a linear Hausdorff space which is either barrelled or has the t-property, $B = \{b_i\}_{i \epsilon I}$ is a fundamental collection of distinct elements of V, $\Upsilon = \{f_x\}_{x \epsilon X}$ is a subset of V^*, and (B, Υ) is bows. If B contains a full Υ-orthogonal subset and there exists a function F on B onto X such that*

(a) *for each* $x \in X$ *and each full* Υ*-orthogonal subset* B_1 *of* B *there exists a full* Υ*-orthogonal subset* B_2 *of* B *with* $B_1 < B_2$ *and* $F(b_i) = x$ *for some* $b_i \in B_2$;

(b) $\left\{ \sum_{j=1}^{k} f_{F(b_j)}(y) b_j : \{b_j\}_{j=1}^{k} \text{ a full } \Upsilon\text{-orthogonal subset of } B \right\}$ *is bounded for each* y *in* V;

then B *is an* I*-basis with continuous coordinate functionals for* V.

Proof. For each b_i let E_{b_i} be $\{x \in X : f_x(b_i) = 1\}$ and denote $\{E_{b_i} : b_i \in B\}$ by $\mathcal{P}'$. Since singleton subsets of B are Υ-orthogonal it follows from Proposition 3 [6] that $E_{b_i} = E_{b_j}$ if and only if $b_i = b_j$. The definition of Υ-orthogonality implies that $\{b_j\}_{j=1}^{k}$ is Υ-orthogonal if and only if $\{E_{b_j}\}_{j=1}^{k}$ is a disjoint collection. Therefore, since B contains a full Υ-orthogonal subset there exists a $\mathcal{P}'$-subdivision of X. It is an immediate consequence of proposition 3 [6] that if $\{b_j\}_{j=1}^{k}$ is a full Υ-orthogonal set, $\{b_l\}_{l=1}^{p}$ is Υ-orthogonal and $\{b_j\}_{j=1}^{k} < \{b_l\}_{l=1}^{p}$ then $\{b_l\}_{l=1}^{p}$ is a full Υ-orthogonal set. Moreover, if $\{b_l\}_{l=1}^{p}$ is a Υ-orthogonal set then $\{b_j\}_{j=1}^{k} < \{b_l\}_{l=1}^{p}$ if and only if $\{E_{b_l}\}_{l=1}^{p}$ refines $\{E_{b_j}\}_{j=1}^{k}$. Therefore condition II of Definition 4 implies that the $\mathcal{P}'$-subdivisions of X are directed by refinement. A similar argument shows that if E is in $\mathcal{P}'$ then there is a $\mathcal{P}'$-subdivision of X to which E belongs. Therefore, by lemma 1, $\mathcal{P} = \mathcal{P}' \cup \{\emptyset\}$ is a proto-algebra. It is clear that the function μ defined by

$$\mu(E_{b_i}) = b_i, \qquad E_{b_i} \in \mathcal{P}', \qquad \mu(\emptyset) = N$$

is a p-volume on $\mathcal{P}$, and the function φ on $\mathcal{P}'$ defined by $\varphi = F \circ \mu$ is a choice function on $\mathcal{P}'$. If a function l on $V \times X$ is defined by

$$l(y, x) = f_x(y), \qquad \forall y \in V,\ \forall x \in X$$

then condition (b) and the Banach-Steinhaus theorem imply that

$$y = \varphi \int_X l(y, -)\, d\mu, \qquad \forall y \in V,$$

since this relation holds for all y in sp B. The uniqueness of this representation is an immediate consequence of condition (a) and Definition 2. This completes the proof.

It should be noted that if the pointwise boundedness hypothesis of condition (b) in theorem 4 is replaced by an equi-continuity condition then it need not be assumed that V is barrelled or has the t-property.

Finally, if (B, Υ) is a biorthogonal collection then B contains a full Υ-orthogonal set only if it is finite. Thus a necessary condition that an I-basis B be biorthogonal to the associated co-ordinate functionals is that B be finite. It is easy to show that this condition is also sufficient.

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