QUASI-NORMAL RINGS

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The aim of the present paper is to exploit ideal-theoretic properties of a class of rings, called quasi-normal for lack of a proper designation, which includes integrally closed domains (i.e. normal domains) and Gorenstein rings. Broadly speaking, they are commutative noetherian rings in which principal ideals have a unique representation as an intersection of irreducible ideals. One could also say that quasi-normality occurs when one substitutes "discrete valuation ring" by "one-dimensional Gorenstein ring" in the usual characterization of normal domains. Such rings were first introduced in [4], where a non-intrinsic definition was used and solely for the purpose of studying reflexive modules.

Here we propose, by analogy with the normal case, to describe two classes of ideals which play, in general, interesting roles: the class of reflexive ideals and that of closed ideals. The main tools are Rees' theory of the grade of an ideal [3] and portions of Bass' survey of the basic properties of Gorenstein rings.

1. Quasi-normal rings

Throughout, all rings are commutative and noetherian. Also, unspecified modules are assumed finitely generated. Before we begin our journey we recall some basic definitions. For an irreducible ideal I in a ring it is understood that I is not an intersection of properly larger ideals. In the noetherian case any ideal I can be written as an intersection $J_1 \cap \cdots \cap J_n$ of irreducible ideals without superfluous ones; there might be several such representations but the integer n is invariant. After [3], we say that the ideal I has grade n if it contains an R-sequence of length n but no R-sequence of length $n + 1$. Finally, for basic facts and terminology on commutative noetherian rings, we refer, without mention, to [2]. Leading to our main object are the following

LEMMA 1.1. *Let I and J be primary ideals, P and Q their corresponding primes. If $P \subset Q$ but are distinct, there exists a primary ideal J', properly contained in J, with $I \cap J = I \cap J'$.*

Proof. We can assume $I \cap J = (0)$. Let n be an integer such that $Q^n \subset J$; then $Q^{(n)}$, the n-th symbolic power of Q, is contained in J. Now $Q^{(n+1)} \neq Q^{(n)}$ for otherwise, localizing at Q, $Q_Q^{n+1} = Q_Q^n$ and $Q_Q^n = (0)$ by the Nakayama's lemma. Since height $Q \geq 1$ this is impossible. Now take $J' = Q^{(n+1)}$.

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PROPOSITION 1.2. *Let R be a ring and I an ideal which admits a unique representation as an intersection of irreducible ideals. Then I has no embedded primes and all of its primary components are irreducible.*

Proof. We can assume that $I = (0)$. By (1.1) there are no embedded primes. Now if P is any minimal prime all we have to do is to show that the null ideal in R_P is irreducible. If not, there would exist more than one representation $(0) = Q_1 \cap \cdots \cap Q_n$, Q_i irreducible, in R_P, which could easily be lifted to a primary component of (0) in R.

Altogether we have

THEOREM 1.3. *For a noetherian ring the following are equivalent:*

1a. *For every prime ideal P of height ≤ 1, R_P is a Gorenstein ring.*

b. *For every prime ideal P of height ≥ 2,* grade $PR_P \geq 2$.

2. *The null ideal or any ideal generated by a nonzero divisor has a unique representation as an intersection of irreducible ideals.*

Proof. (1) $\Rightarrow$ (2). Immediately we have that the null ideal has no embedded components. Thus for every associated prime P of the null ideal, R_P is a zero-dimensional Gorenstein ring and by [1] (0) is irreducible in R_P and hence all primary components of (0) in R are irreducible and the uniqueness follows. If a is a nonzero divisor, by passing to $R/(a)$ we get a similar result for (a).

(2) $\Rightarrow$ (1). Follows from same reasoning plus (1.2) and the fact [1] that in dimension one a local Gorenstein ring is characterized in the following way: the maximal ideal does not consist entirely of zero divisors and these elements generate irreducible principal ideals.

DEFINITION 1.4. A quasi-normal ring is a commutative noetherian ring satisfying the equivalent conditions of (1.3).

Remarks. In case R is a domain the definition of quasi-normality is that of the introduction but allowing zero divisors, at a small scale, extends appreciably the class of such rings and the various ways of obtaining other quasi-normal rings from given ones. When R is a normal domain, any primary component of a nonzero principal ideals (a) is a symbolic power of some prime P with $R_P =$ discrete valuation ring; such component is then irreducible. Hence, normal $\Rightarrow$ quasi-normal. Also, if R is a normal domain one can see that for every finitely generated abelian group G, the group algebra $R[G]$ is quasi-normal. Another class of examples is given by Gorenstein rings of arbitrary dimension.

2. Reflexive ideals

Let R be a commutative ring and K its total ring of quotients. If I is any R-submodule of K containing a nonzero divisor, $\mathrm{Hom}_R(I, R)$ can be identified with

$$I^{-1} = \{x \in K : xI \subset R\}.$$

Such a I is said to be reflexive if $I = (I^{-1})^{-1}$ and this amounts to saying that the second dual map

$$I \to \text{Hom}_R\ (\text{Hom}_R\ (I, R), R)$$

is an isomorphism. From this it follows that if I is reflexive it remains so under localizations, thus underplaying the role of K. In this section we describe those ideals of a quasi-normal ring which are reflexive; in the case of a normal domain, as it is well known, an ideal is reflexive if and only if all of its associated primes are of height 1. We get a similar description here.

LEMMA 2.1. *Let R be a noetherian ring and I an ideal containing a nonzero divisor. Then* grade $I \geq 2$ *if and only if* $I^{-1} = R$.

Proof. From $0 \to I \to R \to R/I \to 0$ we get the exact sequence

$$0 \to \text{Hom}_R\ (R/I, R) \to R^{-1} \to I^{-1} \to \text{Ext}_R^1\ (R/I, R) \to 0$$

or

$$0 \to R \xrightarrow{\textit{inclusion}} I^{-1} \to \text{Ext}_R^1\ (R/I, R) \to 0$$

since I contains some nonzero divisor. Thus $I^{-1} = R$ iff $\text{Ext}_R^1\ (R/I, R) = (0)$, i.e. iff grade $I \geq 2$ by [3].

We next look at the primary decomposition of reflexive ideals.

PROPOSITION 2.2. *Let I be a reflexive ideal. Then any associated prime has grade* 1.

Proof. Let P be such a prime. It is enough to show that locally P has grade 1 and thus assume R local. We can describe the relationship between I and P by saying that there exists $a \notin I$ so that $P = I : a = \{x \in R : xa \in I\}$. One can even pick a to be a nonzero divisor: If $a' = a + r$, $r \in I$, then also $P = I : a'$. Now if r is a nonzero divisor, the distinct elements $a + r^i$, $i \geq 1$, cannot all be zero divisors for otherwise two of them, say $a + r^i$ and $a + r^j$, $j > i$, would be contained in a same prime associated to (0). But then $a + r^j - a - r^i = r^i(1 - r^{j-i})$ would be a zero divisor which is impossible since r^i is a nonzero divisor and $1 - r^{j-i}$ is a unit. We thus have $Pa \subset I$, with a a nonzero divisor not contained in I. Inverting this relation twice we get: $P^{-1}a^{-1} \supset I^{-1}$ and $(P^{-1})^{-1}a \subset I$. Hence, by (2.1), $P^{-1} = R$ if grade $P \geq 2$ and $a \in I$, a contradiction.

As a consequence we have

COROLLARY 2.3. *Let P be a prime ideal containing a nonzero divisor. Then P is reflexive if and only if* grade $P_P = 1$.

Proof. If $P \neq (P^{-1})^{-1}$ by localizing at P we get $(P_P)^{-1} = R_P$ and so grade $P_P \geq 2$. The converse follows from (2.2).

We can now give the promised description of reflexive ideals in a quasi-normal ring. Possibly it is characteristic of quasi-normality.

THEOREM 2.4. *Let R be a quasi-normal ring. Then the ideal I is reflexive if and only if all of its associated primes are of height* 1.

Proof. Recall that grade and height one denote the same thing here. Let I be unmixed of grade 1 and consider the inclusion

$$0 \to I \to J \to C \to 0$$

with $J = (I^{-1})^{-1}$. Let P be a prime ideal of height ≤ 1. Localizing at P it results that $C_P = (0)$ since over a zero or one-dimensional Gorenstein ring all ideals are reflexive [1]. Let P be an associated prime of C; the preceding shows that grade $P \geq 2$. Applying $\operatorname{Hom}_R(R/P, \ )$ to the above sequence we get

$$0 \to \operatorname{Hom}_R(R/P, I) \to \operatorname{Hom}_R(R/P, J) \to \operatorname{Hom}_R(R/P, C) \to \operatorname{Ext}^1_R(R/P, I).$$

Since P contains nozero divisors, $\operatorname{Hom}_R(R/P, I) = \operatorname{Hom}_R(R/P, J) = (0)$.

On the other hand

$$0 \to I \to R \to R/I \to 0$$

gives rise to

$$0 \to \operatorname{Hom}_R(R/P, R/I) \to \operatorname{Ext}^1_R(R/P, I) \to \operatorname{Ext}^1_R(R/P, R).$$

Now, since grade $P \geq 2$, $\operatorname{Ext}^1_R(R/P, R) = (0)$ and $\operatorname{Hom}_R(R/P, R/I) \neq (0)$ would mean that there was $a \notin I$ so that $Pa \subset I$ and P would be contained in an associated prime of I. This is impossible since they are all of height 1. To resume we proved that $\operatorname{Hom}_R(R/P, C) = (0)$, which is clearly impossible by the choice of P and thus $C = (0)$. The converse follows from (2.2).

3. Closed ideals

We begin with

DEFINITION 3.1. An ideal I of a ring R is said to be closed if $\operatorname{Hom}_R(I, I) = R$.

If R is a normal domain then all nonzero ideals are closed since for any such ideal I, $\operatorname{Hom}_R(I, I)$ can be identified to the set of elements x in the field of quotients of R with $xI \subset I$. But $\operatorname{Hom}_R(I, I)$ is finitely generated and thus integral over R. Also, if the ideal I in the ring R is large enough, say, grade $I \geq 2$, then an argument similar to (2.1) would show I closed. Another example is given by an ideal I of a ring R, containing a nonzero divisor and having finite projective dimension, but this is less immediate. Here we determine the class of closed ideals for a quasi-normal ring.

We examine the lowest dimension first.

PROPOSITION 3.2. *Let I be an ideal of a one-dimensional Gorenstein ring. Then I is closed if and only if it is invertible.*

Proof. We can assume R to be local. Let I be a closed ideal. We must show that $I \cdot I^{-1} = R$. This is the same as showing that the mapping

$$\varphi : I \otimes \mathrm{Hom}_R(I, R) \to R$$

given by $\varphi(r \otimes f) = f(r)$ is an epimorphism. The image of φ is the so called trace ideal of the R-module I. Now let a be a nonunit, nonzero divisor in R. If we show that modulo (a), φ is an epimorphism, the claim will be sustained. Consider the exact sequence induced by multiplication by a

$$0 \to I \xrightarrow{a} I \to I/aI \to 0.$$

It leads to

$$0 \to \mathrm{Hom}_R(I, I) \xrightarrow{a} \mathrm{Hom}_R(I, I) \to \mathrm{Hom}_R(I, I/aI)$$

or $$0 \to R \xrightarrow{a} R \to \mathrm{Hom}_{R/(a)}(I/aI, I/aI).$$

Thus $\mathrm{Hom}_{R/(a)}(I/aI, I/aI)$ contains a submodule isomorphic to $R/(a)$. In particular, I/aI, as an $R/(a)$-module, has trivial annihilator. Since $R/(a)$ is self-injective [1] it follows that I/aI contains a direct summand isomorphic to $R/(a)$ and the trace ideal of the $R/(a)$-module I/aI is $R/(a)$.

Consider now the exact sequence

$$0 \to R \xrightarrow{a} R \to R/(a) \to 0.$$

Applying $\mathrm{Hom}_R(I, \quad)$ to it we get

$$0 \to \mathrm{Hom}_R(I, R) \xrightarrow{a} \mathrm{Hom}_R(I, R) \to \mathrm{Hom}_R(I, R/(a)) \to \mathrm{Ext}^1_R(I, R).$$

Since R has self-injective dimension 1, $\mathrm{Ext}^1_R(I, R) = (0)$ [1] and thus $\mathrm{Hom}_{R/(a)}(I/aI, R/(a)) = \mathrm{Hom}_R(I, R) \otimes R/(a)$ and from this it follows that the trace ideal of I maps onto that of the $R/(a)$-module I/aI. The desired conclusion follows then by the Nakayama's lemma. The converse is obvious.

The next technical fact needed is

PROPOSITION 3.3. *Let R be a ring and S an over-ring of R contained in the total ring of quotients of R. Then the conductor C of S with respect to R, i.e. the largest common ideal, is a reflexive ideal, as an R-module.*

Proof. Let $x \in (C^{-1})^{-1}$, $s \in S$ and let $y \in C^{-1}$. First, C^{-1} is an S-module: $syC = y(sC) \subset yC \subset R$. Next, $(sx)y = x(sy) \in (C^{-1})^{-1} \cdot C^{-1} \subset R$ and so $(C^{-1})^{-1}$ is also an S-module and $(C^{-1})^{-1} = C$.

We can now state and prove the main result in this section

THEOREM 3.4. *Let R be a quasi-normal ring. Then the ideal I is closed if and only if* grade $I \cdot I^{-1} \geq 2$.

Proof. *If* is closed, by (3.2), we have that I_P is principal for every prime ideal P of height 1. Thus $(I \cdot I^{-1})_P = I_P \cdot (I_P)^{-1} = R_P$ and $I \cdot I^{-1}$ lies outside of any such prime, i.e. grade $I \cdot I^{-1} \geq 2$.

Conversely, assume grade $I \cdot I^{-1} \geq 2$ and let $S = \mathrm{Hom}_R(I, I)$. Let C denote the conductor of S with respect to R. At every height 1 prime P, I_P is principal and thus for such primes $C_P = R_P$. This shows that grade $C \geq 2$. From (3.3) however C is a reflexive ideal and hence by (2.4) it is unmixed of grade 1. Thus $C = R$ and $S = R$ as wanted.

Remark. An example of a non-invertible closed ideal in a one-dimensional local domain can be obtained in the following way: Let S denote the power series ring in the variable t over the field K and R the subring of all power series without first or second degree terms. Let I be the ideal of R generated by t^3 and t^4; it is clear that I is not a principal ideal. $\mathrm{Hom}_R(I, I)$ is a subring of S containing R and since t^5 is not in I, by direct checking, one sees that a series $a + bt + ct^2 + \cdots$ is in $\mathrm{Hom}_R(I, I)$ only if $b = c = 0$ or, in other words, only if it already in R.

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