ON THE ZEROS OF A CLASS OF DIRICHLET SERIES I

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1. Introduction

The purpose of this paper is to show that many theorems concerning the distribution of zeros for the Riemann zeta-function $\zeta(s)$ can be generalized to a large class of Dirichlet series [1]. For the most part, our results are concerned with the distribution of zeros in a certain vertical strip. The proofs are similar to those that have been given for $\zeta(s)$. Most of the corresponding theorems for $\zeta(s)$ can be found in [10].

DEFINITION 1. Let $\{\lambda_n\}$ and $\{\mu_n\}$ be two sequences of positive numbers tending to ∞, and $\{a(n)\}$ and $\{b(n)\}$ two sequences of complex numbers not identically zero. Let

$$\Delta(s) = \prod_{k=1}^{N} \Gamma(\alpha_k s + \beta_k),$$

where N is a positive integer, $\alpha_k > 0$, and β_k is an arbitrary complex number. Consider the functions φ and ψ representable as Dirichlet series

$$\varphi(s) = \sum_{n=1}^{\infty} a(n)\lambda_n^{-s}, \qquad \psi(s) = \sum_{n=1}^{\infty} b(n)\mu_n^{-s}, \qquad s = \sigma + it,$$

with finite abscissae of absolute convergence σ_a and σ_a^*, respectively. If r is real, we say that φ and ψ satisfy the functional equation

$$\Delta(s)\varphi(s) = \Delta(r-s)\psi(r-s) \tag{1.1}$$

if there exists in the s-plane a domain D which is the exterior of a compact set S, such that in D,

(i) φ is holomorphic,
(ii) $\varphi(s) = \Delta(r-s)\psi(r-s)/\Delta(s)$, $\sigma < r - \sigma_a^*$,
(iii) there exists a constant $K > 0$ such that

$$\varphi(s) = O(\exp|s|^K),$$

as $|s|$ tends to ∞.

Throughout the sequel we set $A = \sum_{k=1}^{N} \alpha_k$. If C denotes a simple closed curve, let $I(C)$ denote the interior of C and let $I'(C) = I(C) \cup C$. Finally, B always designates an unspecified positive constant, not necessarily the same with each occurrence.

2. Summary of results

THEOREM 1. *There exists a positive integer m such that*

$$-(m + j + \beta_k)/\alpha_k, \qquad k = 1, \cdots, N, j = 0, 1, 2, \cdots,$$

Received March 21, 1968.

are simple zeros of φ. *Moreover, the remaining zeros of* φ *belong to a vertical strip*, $\sigma_1 < \sigma < \sigma_2$.

This is, of course, a classical result for several Dirichlet series whose coefficients are of number theoretical interest. Lekkerkerker [7] has proven the result for $\Delta(s) = \Gamma(s)$. In the sequel the zeros of φ outside the strip, $\sigma_1 < \sigma < \sigma_2$, will be called the trivial zeros.

THEOREM 2. *The number of zeros of* φ *in the vertical strip*, $\sigma_1 < \sigma < \sigma_2$, *is infinite, and the distance between ordinates of successive zeros is bounded.*

THEOREM 3. *Let* $N(T)$ *denote the number of zeros of* φ *in* $D \cap I(R)$, *where* R *denotes the rectangle with vertices* σ_1, σ_2, $\sigma_1 + iT$ *and* $\sigma_2 + iT$. *If* h *is any positive number, no matter how large,*

$$N(T + h) - N(T) = O(\log T),$$

where $O = O(h)$.

COROLLARY 4. *The multiplicity of a zero of* φ *does not exceed* $O(\log T)$.

THEOREM 5. *Let* $\rho = \beta + i\gamma$ *run through the zeros of* φ. *Then,*

$$\varphi'(s)/\varphi(s) = \sum_{|t-\gamma|\leq 1} 1/(s - \rho) + O(\log t), \tag{2.1}$$

uniformly for $\sigma_1 - 1 \leq \alpha \leq \sigma_2 + 1$.

THEOREM 6. *We have*

$$\log \varphi(s) = \sum_{|t-\gamma|\leq 1} \log\,(s - \rho) + O(\log t),$$

uniformly for $\sigma_1 - 1 \leq \sigma \leq \sigma_2 + 1$, *where* $-\pi < \arg\,(s - \rho) \leq \pi$.

THEOREM 7. *There exists a positive constant* K *such that each interval* $(T, T + 1)$ *contains a value of* t *for which*

$$|\varphi(s)| > t^{-K},$$

where $\sigma_1 - 1 \leq \sigma \leq \sigma_2 + 1$. *Furthermore, if* $H > 1$ *is arbitrary, then*

$$|\varphi(s)| > T^{-KH},$$

where $\sigma_1 - 1 \leq \sigma \leq \sigma_2 + 1$, $T \leq t \leq T + 1$, *except possibly on a set of* t *values of measure* $1/H$.

The proofs of Theorems 6 and 7 will be omitted since they resemble the corresponding proofs for $\zeta(s)$ [10, pp. 185–186] with only obvious changes being necessary.

THEOREM 8. *For* $T > 0$ *sufficiently large,* φ *has a zero* $\beta + i\gamma$ *such that*

$$|\gamma - T| < B/(\log\log\log T).$$

THEOREM 9. *For any fixed* $h > 0$, *no matter how small,*

$$N(T + h) - N(T) > B \log T,$$

where $B = B(h)$.

There is no difficulty in constructing a proof along the same lines as that given for $\zeta(s)$ in [10, pp. 194–196], and so the proof of Theorem 9 will be omitted.

THEOREM 10. *Let c and d be the least positive integers such that* $a(c) \neq 0$ *and* $b(d) \neq 0$, *respectively. Let* $N_i(T)$, $i = 1, 2$, *denote the number of zeros of* φ *outside S which lie in the strips* $\sigma_1 < \sigma < \sigma_2$, $0 < t < T$ and $\sigma_1 < \sigma < \sigma_2$, $-T < t < 0$, *respectively. Then,*

$$N_i(T)$$

$$= (A/\pi)T \log T - (T/2\pi)(\log \lambda_c \mu_d - 2 \textstyle\sum_{k=1}^{N} \alpha_k \log \alpha_k + 2A) + O(\log T). \tag{2.2}$$

Von Mangoldt first gave the proof of the above formula for $\zeta(s)$. However, Backlund later gave another proof, and it is essentially his method which we employ in our proof. Landau [5, p. 534] has proven Theorem 10 for Dirichlet L-functions. Potter and Titchmarsh [8] have proven the theorem for a class of Epstein zeta-functions. Lekkerkerker [7] has proven the result when $\Delta(s) = \Gamma(\mu s)$, where $\mu > 0$.

THEOREM 11. *Let* $\varphi = \psi$, $a(n)$ *be real and* β_k *be real*, $k = 1, \cdots, N$. *Suppose also that* $(\sigma_a - \frac{1}{2}r)A < \frac{1}{2}$. *Then, the number of zeros of* φ *on the critical line* $\sigma = \frac{1}{2}r$ *is infinite.*

The corresponding theorem for $\zeta(s)$ was first proven by Hardy. The method we use for Theorem 11 is that used by Landau in his proof of the theorem for $\zeta(s)$ [6, p. 83]. The conclusion of Theorem 11 is valid, of course, for other subclasses of Dirichlet series in Definition 1. Potter and Titchmarsh [8] have proven the theorem for a class of Epstein zeta-functions and Kober [4] for a somewhat larger class of the same. Hecke [3] and Lekkerkerker [7] have proven the result for large classes of Dirichlet series when $\Delta(s) = \Gamma(s)$. Hecke [3, p. 95] and Lekkerkerker [7, p. 59] have pointed out that the theorem can only hold for a restricted subset of the series given in Definition 1 and have given examples of Dirichlet series with no zeros on $\sigma = \frac{1}{2}r$. It is interesting to note that entirely different methods must be used to prove the theorem for different classes of Dirichlet series. The conditions of Theorem 11 are satisfied by $\zeta(s)$, but not, in general, by the other classes mentioned above.

THEOREM 12. *Suppose that* β_k *is real*, $k = 1, \cdots, N$. *Let*

$$\chi(s) = \Delta(r - s)/\Delta(s).$$

Then, for $|t|$ *large enough and* $\sigma > \frac{1}{2}r$,

$$|1/\chi(s)| > 1. \tag{2.3}$$

This theorem was first proven by Spira [9] and then by Dixon and Schoenfeld [2] for $\zeta(s)$.

COROLLARY 13. *For* $|t|$ *large enough and* $\sigma > \frac{1}{2}r$,

$$|\psi(r-s)| > |\varphi(s)|,$$

except at the zeros of $\varphi(s)$.

COROLLARY 14. *Let* $f(s)$ *be a Dirichlet series of signature* $(1, r, \gamma)$ (cf. [3] *or* [7]). *If* $|t| \geq 6.8$ *and* $\sigma > \frac{1}{2}r$, *then*

$$|f(r-s)| > |f(s)|,$$

except at the zeros of $f(s)$.

3. Preliminary results

We first give three forms of Stirling's formula.

For Re $s > 0$ [12, p. 251],

$$\log \Gamma(s) = (s - \tfrac{1}{2}) \log s - s + O(1), \tag{3.1}$$

as $|s|$ tends to ∞. For the proof of Theorem 12 we shall need the more precise result [2],

$$\log \Gamma(s) = \left(s - \frac{1}{2}\right) \log s - s + \frac{1}{2} \log 2\pi + \frac{1}{12s} - 2\int_0^\infty \frac{P_3(x)\, dx}{(s+x)^3}, \tag{3.2}$$

where $P_3(x)$ is a function with period 1 which is equal to

$$x(2x^2 - 3x + 1)/12$$

on [0, 1]. On this interval

$$6|P_3(x)| \leq \tfrac{1}{8}. \tag{3.3}$$

By periodicity (3.3) is valid for all $x \geq 0$. (3.2) is valid in the s-plane cut along the negative real axis.

A direct consequence of Stirling's formula is [10, p. 68]

$$\Gamma(\sigma + it) = t^{\sigma+it-\frac{1}{2}} e^{-\frac{1}{2}\pi t - it + \frac{1}{2}i\pi(\sigma - \frac{1}{2})} (2\pi)^{\frac{1}{2}} (1 + O(t^{-1})), \tag{3.4}$$

as t tends to ∞. A similar formula may be given for $t < 0$ and t tending to $-\infty$ by using the fact that $\Gamma(\sigma - it) = \overline{\Gamma(\sigma + it)}$.

LEMMA 3.1. φ *is of finite order in any half-plane* $\sigma \geq \eta$.

Proof. Let σ be fixed. For $\sigma > \sigma_a^*$, $\psi(\sigma + it) = O(1)$ as $|t|$ tends to ∞. Thus, from the functional equation for $\sigma < r - \sigma_a^*$,

$$\varphi(s) = O\left(\frac{\Delta(r-s)}{\Delta(s)} \psi(r-s)\right) = O\left(\frac{\Delta(r-s)}{\Delta(s)}\right) = O(|t|^{(r-2\sigma)A}), \tag{3.5}$$

by (3.4), as $|t|$ tends to ∞. As $\varphi(s) = O(1)$ for $\sigma > \sigma_a$, it follows from property (iii) and a Phragmen-Lindelöf theorem [11, p. 180] that φ is of finite order in any half-plane $\sigma \geq \eta$.

LEMMA 3.2 [10, p. 49]. *Let f be holomorphic and*

$$|f(s)/f(s_0)| < e^M, \quad M > 1,$$

on $I'(C)$, *where* $C = \{s : |s - s_0| = r\}$. *Then,*

$$|f'(s)/f(s) - \textstyle\sum_\rho 1/(s - \rho)| < BM/r, \quad |s - s_0| \leq r/4,$$

where ρ *runs through the zeros of* $f(s)$ *such that* $|\rho - s_0| \leq \frac{1}{2}r$.

LEMMA 3.3 [10, p. 62]. *Let* $F(x)$ *and* $G(x)$ *be real functions on* $[a, b]$ *such that*

(i) $G(x)/F'(x)$ *is monotonic,*
(ii) $F''(x) \geq r > 0$ *or* $F''(x) \leq -r < 0$,
(iii) $|G(x)| \leq M, M > 0$.

Then,

$$\left| \int_a^b G(x) e^{iF(x)}\, dx \right| \leq 8M/\sqrt{r}.$$

4. Proofs of the theorems

Proof of Theorem 1. Let c and d be the least positive integers such that $a(c) \neq 0$, $b(d) \neq 0$, respectively. Since φ and ψ converge in some half-plane, we can choose $\alpha > \max(0, \sigma_a, \sigma_a^*)$ so that

$$\textstyle\sum_{n=c+1}^{\infty} |a(n)| \lambda_n^{-\alpha} \leq \frac{1}{2}|a(c)| \lambda_c^{-\alpha}, \tag{4.1}$$

$$\textstyle\sum_{n=d+1}^{\infty} |b(n)| \mu_n^{-\alpha} \leq \frac{1}{2}|b(d)| \mu_d^{-\alpha}.$$

Thus, for $\sigma \geq \alpha$,

$$|\varphi(s)| \geq |a(c)| \lambda_c^{-\sigma} - \textstyle\sum_{n=c+1}^{\infty} |a(n)| \lambda_n^{-\sigma} \geq \frac{1}{2}|a(c)| \lambda_c^{-\sigma}.$$

Similarly, for $\sigma \geq \alpha$,

$$|\psi(s)| \geq \tfrac{1}{2}|b(d)| \mu_d^{-\sigma}. \tag{4.2}$$

Thus φ and ψ are free of zeros and holomorphic in the half-plane $\sigma \geq \alpha$. Also, since $\sigma_a > \frac{1}{2}r + 1/4A$ [1, p. 111], $r - \alpha < \alpha$. Now, $\Delta(s)$ has simple poles at $s = -(n + \beta_k)/\alpha_k$, $k = 1, \cdots, N$, $n = 0, 1, 2, \cdots$. It follows that if we let m be the least positive integer such that

$$-(m + Re\beta_k)/\alpha_k < r - \alpha, \qquad k = 1, \cdots, N,$$

$\varphi(s)$ has simple zeros at $s = -(m + j + \beta_k)/\alpha_k$, $k = 1, \cdots, N, j = 0, 1, 2, \cdots$. The remainder of the zeros must lie in the strip $r - \alpha < \sigma < \alpha$.

Proof of Theorem 2. Let c and α be as given in the proof of Theorem 1. Without loss of generality we assume $\lambda_c = 1$, for the zeros of $\varphi(s)$ are the same as those for $\lambda_c^{-s}\varphi(s)$.

Now, let $M = \max\{|\text{Re}\, a(c)|, |\text{Im}\, a(c)|\} > 0$. Suppose $M = \text{Re}\, a(c)$. Then choose $\alpha_0 \geq \alpha$ large enough so that

$$\begin{aligned}\operatorname{Re}\varphi(s) &= \operatorname{Re} a(c) + \{\operatorname{Re} a(c+1)\cos(t\log\lambda_{c+1}) \\ &\quad + \operatorname{Im} a(c+1)\sin(t\log\lambda_{c+1})\}\lambda_{c+1}^{-\sigma} + \cdots \\ &> \operatorname{Re} a(c) - |\operatorname{Re} a(c+1)\cos(t\log\lambda_{c+1}) \\ &\quad + \operatorname{Im} a(c+1)\sin(t\log\lambda_{c+1})|\lambda_{c+1}^{-\sigma} - \cdots \\ &> 0,\end{aligned}$$

for $\sigma \geq \alpha_0$. Similarly, if $M = \operatorname{Im} a(c)$, $\alpha_0 \geq \alpha$ can be chosen large enough so that $\operatorname{Im}\varphi(s) > 0$ for $\sigma \geq \alpha_0$. If $M = -\operatorname{Re} a(c)$ or $-\operatorname{Im} a(c)$, $\alpha_0 \geq \alpha$ can be chosen large enough so that $\operatorname{Re}\varphi(s) < 0$ or $\operatorname{Im}\varphi(s) < 0$, accordingly, for $\sigma \geq \alpha_0$. Thus, for all cases we may define a branch of $\log\varphi$ for $\sigma \geq \alpha_0$,

$$(4.3) \qquad \log\varphi(s) = \log|\varphi(s)| + i\arg\varphi(s),$$

where $\arg\varphi(s)$ ranges over an interval of length no greater than π. Hence, for $\sigma \geq \alpha_0$,

$$(4.4) \qquad |\log\varphi(s)| < B.$$

For $\sigma < \alpha_0$ we define $\log\varphi(s)$ as the analytic continuation of (4.3) along the line segment $(\sigma + it, \alpha_0 + it)$, provided that φ is holomorphic and $\varphi(s) \neq 0$ on this segment.

Next, let β be a positive real number chosen so that $\alpha_0 - \beta < r - \alpha_0$. Consider a system of four concentric circles C_1, C_2, C_3 and C_4 with center $\alpha_0 + 1 + iT$ and radii $1, \beta + 1, \beta + 2$, and $\beta + 3$, respectively. Here $|T|$ is chosen large enough so that $I'(C_4) \subset D$ and none of the trivial zeros lies in $I'(C_4)$.

Suppose that $\varphi(s) \neq 0$ on $I'(C_4)$ so that $\log\varphi(s)$ is holomorphic on $I'(C_4)$. Let M_2 and M_3 denote the maximum moduli of $\log\varphi(s)$ on C_2 and C_3, respectively. By Lemma 3.1 $\operatorname{Re}\varphi(s) = O(\log T)$ for s on $I'(C_4)$. Hence, by (4.4) and the Borel-Carathéodory theorem [11, p. 175],

$$M_3 = O(\log T).$$

Next, we apply Hadamard's 3 circles theorem [11, p. 172] to C_1, C_2 and C_3 to obtain

$$M_2 \leq B(\log T)^{\rho},$$

where $\rho = \log(\beta + 1)/\log(\beta + 2) < 1$. In particular,

$$(4.5) \qquad \varphi(\alpha_0 - \beta + iT) = 0(\exp\{\log^{\rho} T\}) = O(T^{\epsilon}),$$

where $\epsilon > 0$, since $\rho < 1$.

On the other hand, by our choice of β and (4.2),

$$|\psi(r - \alpha_0 + \beta - iT)| \geq \tfrac{1}{2}|b(d)|\mu_d^{-\alpha_0} = K,$$

say. Hence, by (1.1) and (3.4),

$$(4.6)\quad \begin{aligned} |\varphi(\alpha_0 - \beta + iT)| &\geq K\,|\Delta(r - \alpha_0 + \beta - iT)/\Delta(\alpha_0 - \beta + iT)| \\ &\geq B\,|T|^{(r+2\beta-2\alpha_0)A}. \end{aligned}$$

As $r + \beta - 2\alpha_0 > 0$ and $\beta > 0$, $r + 2\beta - 2\alpha_0 > 0$. Thus, (4.6) is a contradiction to (4.5), and $\varphi(s)$ must have at least one zero on $I'(C_4)$. The last statement of the theorem easily follows from the proof.

Proof of Theorem 3. Let $r_h = \{(\sigma_2 - \sigma_1 + 1)^2 + h^2\}^{1/2}$ and define r_k similarly for $k > h$. Consider a circle C of radius r_k and center $\sigma_2 + 1 + iT$, where T is chosen large enough so that $I'(C) \subset D$. Then, clearly,

$$(4.7)\quad N(T + h) - N(T) \leq n(r_h),$$

where $n(x)$ denotes the number of zeros of φ in the circle of radius x and center $\sigma_2 + 1 + iT$. By Jensen's theorem [11, p. 126] and Lemma 3.1,

$$(4.8)\quad \begin{aligned} \int_0^{r_k} \frac{n(x)}{x}\,dx &= \frac{1}{2\pi}\int_0^{2\pi} \log|\varphi(\sigma_2 + 1 + iT + r_k e^{i\theta})|\,d\theta \\ &\quad - \log|\varphi(\sigma_2 + 1 + iT)| \\ &< B \log T. \end{aligned}$$

On the other hand,

$$(4.9)\quad \int_0^{r_k} \frac{n(x)}{x}\,dx \geq \int_{r_h}^{r_k} \frac{n(x)}{x}\,dx \geq n(r_h)\int_{r_h}^{r_k}\frac{dx}{x} = Bn(r_h).$$

Combining (4.7), (4.8) and (4.9), we obtain the conclusion of the theorem.

Proof of Theorem 5. In Lemma 3.2 put

$$f = \varphi, \qquad s_0 = \sigma_2 + 1 + iT \quad \text{and} \quad r = 4(\sigma_2 - \sigma_1 + 2).$$

Here T is chosen large enough so that $I'(C) \subset D$. By Lemma 3.1 we may take $M = B \log T$. Thus,

$$(4.10)\quad \frac{\varphi'(s)}{\varphi(s)} = \sum_{|\rho - s_0| \leq \frac{1}{2}r} \frac{1}{s - \rho} + O(\log T),$$

where $|s - s_0| \leq \sigma_2 - \sigma_1 + 2$. In particular, (4.10) is valid for

$$\sigma_1 - 1 \leq \sigma \leq \sigma_2 + 1.$$

For these values of σ, clearly, we may replace T by t in (4.10). Also, any term that appears in (4.10), but not (2.1), is bounded, and by Theorem 3 the number of such terms is no greater than

$$N(T + \tfrac{1}{2}r) - N(T - \tfrac{1}{2}r) = O(\log t).$$

Proof of Theorem 8. We give only the beginning of the proof, for after a certain point the details are precisely the same as the corresponding theorem for $\zeta(s)$ [10, p. 191–193].

We choose T large enough so that $I'(C_{k\nu})$, where $C_{k\nu}$ is defined below, contains none of the trivial zeros and $I'(C_{k\nu}) \subset D$. Also choose α_0 as in the proof of Theorem 2.

Suppose $\varphi(s)$ has no zeros in $T - \delta \leq t \leq T + \delta$, where $\delta < \frac{1}{2}$. Then $f(s) = \log \varphi(s)$ is holomorphic for $T - \delta \leq t \leq T + \delta$, where $f(s)$ is given its principal value for $\sigma \geq \alpha_0$. Let $C_{1\nu}$, $C_{2\nu}$, $C_{3\nu}$ and $C_{4\nu}$ be four concentric circles with center $\alpha_0 + 1 - \nu\delta/4 + iT$ and radii $\delta/4$, $\delta/2$, $3\delta/4$ and δ, respectively. Here $\nu = 0, 1, 2, \cdots, n$, where $n = [4(\alpha_0 - \sigma_1 + 2)/\delta] + 1$. Thus, the centers of the circles with center $\alpha_0 + 1 - n\delta/4$ lie on or to the left of $\sigma = \sigma_1 - 1$. Proceed now exactly as in [10].

Proof of Theorem 10. Let α be given as in the proof of Theorem 1. Choose T_0 and $T > T_0$ so that the lines $t = T_0$ and $t = T$ contain no zeros of φ and so that S lies within the rectangle with vertices $r - \alpha \pm iT_0$ and $\alpha \pm iT_0$. Let R denote the rectangle with vertices $r - \alpha + iT_0$, $\alpha + iT_0$, $\alpha + iT$ and $r - \alpha + iT$. R is free of zeros of φ. Lastly, let N_0 denote the number of zeros of φ outside S but within the rectangle given by $0 < t < T_0$, $\sigma_1 < \sigma < \sigma_2$. Thus,

$$\begin{aligned} N_1(T) - N_0 &= \frac{1}{2\pi i}\int_R \frac{d}{ds}\log\varphi(s)\,ds \\ &= \frac{1}{2\pi i}\left\{\int_{r-\alpha+iT_0}^{\alpha+iT_0} + \int_{\alpha+iT_0}^{\alpha+iT} + \int_{\alpha+iT}^{r-\alpha+iT} + \int_{r-\alpha+iT}^{r-\alpha+iT_0}\right\}\frac{d}{ds}\log\varphi(s)\,ds \\ &= \frac{1}{2\pi i}\,\mathrm{Im}\,\{I_1 + I_2 + I_3 + I_4\}. \end{aligned}$$

We examine each integral in turn. As I_1 is independent of T, $I_1 = O(1)$. Next,

$$\begin{aligned} (4.11) \qquad I_2 &= \log\varphi(s)\,\Big|_{\alpha+iT_0}^{\alpha+iT} \\ &= \log a(c)\lambda_c^{-s}\,\Big|_{\alpha+iT_0}^{\alpha+iT} \\ &\quad + \log\{1 + \textstyle\sum_{n=c+1}^{\infty} a^{-1}(c)a(n)(\lambda_n/\lambda_c)^{-s}\}\,\Big|_{\alpha+iT_0}^{\alpha+iT}, \end{aligned}$$

where we take the variation in any branch of the logarithm along the straight line segment $(\alpha + iT_0, \alpha + iT)$. Let

$$f(s) = \textstyle\sum_{n=c+1}^{\infty} a^{-1}(c)a(n)(\lambda_n/\lambda_c)^{-s}.$$

By (4.1), it follows that for $\sigma \geq \alpha$, $|f(s)| \leq \frac{1}{2}$. Hence, the argument of $1 + f(s)$ ranges over an interval of length less than π, and so the imaginary part of the second term of (4.11) is at most π. An easy calculation shows that the first term in (4.11) is $i(T_0 - T)\log\lambda_c$. Hence,

$$\mathrm{Im}\,I_2 = -T\log\lambda_c + O(1).$$

By a similar argument,

$$(4.12) \qquad \mathrm{Im}\int_{\alpha-iT_0}^{\alpha-iT}\frac{d}{ds}\log\psi(s)\,ds = T\log\mu_d + O(1).$$

For the estimation of I_3 define

$$\varphi_1(s) = e^{i(\gamma+T\log\lambda_c)}\varphi(s),$$

where γ is chosen so that $a(c)e^{i\gamma} > 0$. Let q be the number of zeros of $\operatorname{Re}\{\varphi_1(s)\}$ on $(r-\alpha+iT,\, \alpha+iT)$. These zeros subdivide this line segment into at most $q+1$ subintervals, in each of which $\operatorname{Re}\{\varphi_1(s)\}$ is of constant sign. On each subinterval the variation of $\operatorname{Im}\{\log\varphi_1(s)\}$ is at most π. Since $\arg\varphi(s)$ and $\arg\varphi_1(s)$ differ only by a constant,

$$|\operatorname{Im} I_3| = |\operatorname{Im}\log\varphi(s)\}|\,\big|_{\alpha+iT}^{r-\alpha+iT} \leq (q+1)\pi.$$

To estimate q we define

$$f(z) = \tfrac{1}{2}\{\varphi_1(z+iT) + \overline{\varphi_1(\bar{z}+iT)}\},$$

and note that if $z = \sigma$ is real,

$$f(\sigma) = \tfrac{1}{2}\{\varphi_1(\sigma+iT) + \overline{\varphi_1(\sigma+iT)}\} = \operatorname{Re}\{\varphi_1(\sigma+iT)\}. \tag{4.13}$$

Without loss of generality assume that

$$\rho = T - T_0 > 4(\alpha - \tfrac{1}{2}r).$$

If z is such that $|z-\alpha| < \rho$, then $\operatorname{Im}(z+iT) > T - \rho = T_0$. Since $\varphi(s)$ is holomorphic for $t > T_0$, $\varphi(z+iT)$ is holomorphic within $|z-\alpha| < \rho$. It follows that $\overline{\varphi(\bar{z}+iT)}$, and hence $f(z)$, is holomorphic within $|z-\alpha| < \rho$ as well. By (4.13), the definition of γ, and (4.1)

$$f(\alpha) > \tfrac{1}{2}\lambda_c^{-\alpha}|a(c)|.$$

We are thus in a position to apply Jensen's theorem. Let

$$r_0 = 4(\alpha - \tfrac{1}{2}r), \qquad r_1 = \tfrac{1}{2}r_0,$$

and $n(x)$ the number of zeros of f within $|z-\alpha| \leq x$. Then,

$$\begin{aligned} n(r_1)\int_{r_1}^{r_0}\frac{dx}{x} &\leq \int_0^{r_0} n(x)\,\frac{dx}{x} \\ &= \frac{1}{2\pi}\int_0^{2\pi}\log|f(r_0e^{i\theta}+\alpha)|\,d\theta - \log|f(\alpha)|. \end{aligned} \tag{4.14}$$

By Lemma 3.1,

$$\varphi(s) = O(t^B), \qquad \sigma \geq \alpha - r_0, \qquad t \geq T_0.$$

Hence,

$$f(r_0e^{i\theta}+\alpha) = O(T^B).$$

Thus, by (4.14),

$$n(r_1) = O(\log T).$$

Now, the zeros of $\operatorname{Re}\{\varphi_1(s)\}$ on $(r-\alpha+iT,\, \alpha+iT)$ are those of $f(z)$ on $(r-\alpha,\, \alpha)$. Since $(r-\alpha,\, \alpha)$ is contained with the circle $|z-\alpha| = r_1$,

$$q \leq n(r_1) \quad\text{and}\quad \operatorname{Im} I_3 = O(\log T).$$

Lastly, by the functional equation (1.1),

$$I_4 = \{\log \Delta(s) - \log \Delta(r - s) - \log \psi(r - s)\}\Big|_{r-\alpha+iT_0}^{r-\alpha+iT}.$$

By (3.1),

$$\log \Delta(s)\Big|_{r-\alpha+iT_0}^{r-\alpha+iT}$$
$$= \sum_{k=1}^N \{\log \Gamma(\alpha_k r - \alpha_k \alpha + i\alpha_k T + \beta_k) - \log \Gamma(\alpha_k r - \alpha_k \alpha + i\alpha_k T_0 + \beta_k)\}$$
$$= \sum_{k=1}^N (\alpha_k r - \alpha_k \alpha + i\alpha_k T + \beta_k - \tfrac{1}{2}) \log (\alpha_k r - \alpha_k \alpha + i\alpha_k T + \beta_k)$$
$$- \sum_{k=1}^N (\alpha_k r - \alpha_k \alpha + i\alpha_k T + \beta_k) + O(1).$$

Similarly,

$$\log \Delta(r - s)\Big|_{r-\alpha+iT_0}^{r-\alpha+iT} = \sum_{k=1}^N (\alpha_k \alpha - i\alpha_k T + \beta_k - \tfrac{1}{2}) \log (\alpha_k \alpha - i\alpha_k T + \beta_k)$$
$$- \sum_{k=1}^N (\alpha_k \alpha - i\alpha_k T + \beta_k) + O(1).$$

Using (4.12), we have

$$I_4 = \sum_{k=1}^N (\alpha_k r - \alpha_k \alpha + i\alpha_k T + \beta_k - \tfrac{1}{2}) \log (\alpha_k r - \alpha_k \alpha + i\alpha_k T + \beta_k)$$
$$- \sum_{k=1}^N (\alpha_k r - i\alpha_k T + \beta_k - \tfrac{1}{2}) \log (\alpha_k \alpha - i\alpha_k T + \beta_k)$$
$$- 2iTA - iT \log \mu_d .$$

Now,

$$\log (\alpha_k r - \alpha_k \alpha + i\alpha_k T + \beta_k) = \log (i\alpha_k T) + O(T^{-1})$$
$$= \log \alpha_k + \log T + \tfrac{1}{2}\pi i + O(T^{-1}),$$

since $\alpha_k > 0$. A similar result holds for $\log (\alpha_k \alpha - i\alpha_k T + \beta_k)$, and so,

$$I_4 = 2iTA \log T + 2iT \sum_{k=1}^N \alpha_k \log \alpha_k - 2iTA - iT \log \mu_d + O(\log T).$$

Combining the values for the four integrals, we have (2.2), $i = 1$. As the right-hand side of (2.2) is continuous in T and as any line $t = T$ containing zeros of φ can be approximated arbitrarily closely by a line $t = T'$ containing no zeros of φ, the aforementioned restriction on T is unnecessary.

If $\beta + i\gamma$, $\gamma < 0$, is not a zero of $\Delta^{-1}(s)$, then $\beta + i\gamma$ is a zero of $\varphi(s)$ if and only if $r - \beta - i\gamma$ is a zero of $\psi(s)$. Since (2.2), $i = 1$, holds for ψ as well and is symmetric in c and d, (2.1) is valid for $i = 2$ also.

Proof of Theorem 11. Define $\chi(s)$ as in the statement of Theorem 12. Clearly,

$$|\chi(\tfrac{1}{2}r + it)| = 1 \tag{4.15}$$

Also, define

$$R(s) = \Delta(s)\varphi(s).$$

From the functional equation it follows that $R(\tfrac{1}{2}r + it) = R(\tfrac{1}{2}r - it)$. Since

$a(n)$ and β_k, $k = 1, \cdots, N$, are real, $R(\frac{1}{2}r + it)$ is a real-valued function of t. Next, let

$$\theta = -\tfrac{1}{2} \arg \chi(\tfrac{1}{2}r + it),$$

so that

$$\chi(\tfrac{1}{2}r + it) = e^{-2i\theta}.$$

Lastly, let

$$\begin{aligned} Z(t) &= e^{i\theta}\varphi(\tfrac{1}{2}r + it) \\ &= \{\chi(\tfrac{1}{2}r + it)\}^{-1/2}\varphi(\tfrac{1}{2}r + it) \\ &= \{\Delta(\tfrac{1}{2}r + it)/\Delta(\tfrac{1}{2}r - it)\}^{1/2}\varphi(\tfrac{1}{2}r + it) \\ &= R(\tfrac{1}{2}r + it)/|\Delta(\tfrac{1}{2}r + it)|. \end{aligned}$$

Hence, $Z(t)$ is a real function of t, and

$$|Z(t)| = |\varphi(\tfrac{1}{2}r + it)|. \tag{4.16}$$

As in Landau's proof, we shall compare the behaviors of the two integrals

$$\int_T^{2T} |Z(t)|\, dt, \qquad \int_T^{2T} Z(t)\, dt,$$

where T is chosen large enough so that $\sup_{s \in S} \{t\} < T$.

Let c be given as in the proof of Theorem 1. Define

$$\varphi_c(s) = \lambda_c^s \varphi(s).$$

Thus, by (4.16),

$$\begin{aligned} \int_T^{2T} |Z(t)|\, dt &= \int_T^{2T} |\lambda_c^{-r/2-it}\varphi_c(\tfrac{1}{2}r + it)|\, dt \\ &\geq \lambda_c^{-r/2} \left| \int_T^{2T} \varphi_c(\tfrac{1}{2}r + it)\, dt \right|. \end{aligned} \tag{4.17}$$

Also,

$$\begin{aligned} i\int_T^{2T} \varphi_c(\tfrac{1}{2}r + it)\, dt &= \int_{r/2+iT}^{r/2+2iT} \varphi_c(s)\, ds \\ &= \left(\int_{r/2+iT}^{\sigma_a+1+iT} + \int_{\sigma_a+1+iT}^{\sigma_a+1+2iT} + \int_{\sigma_a+1+2iT}^{r/2+2iT} \right) \varphi_c(s)\, ds \end{aligned}$$

by Cauchy's theorem.

As usual, define

$$\mu(\sigma) = \inf\{\xi : \varphi(s) = O(|t|^\xi)\}.$$

From (3.5) and the general theory of $\mu(\sigma)$ [11, p. 299], we find that for $\frac{1}{2}r \leq \sigma \leq \sigma_a$,

$$\mu(\sigma) \leq (\sigma_a - \sigma)A. \tag{4.18}$$

Thus,

$$i \int_T^{2T} \varphi_c(\tfrac{1}{2}r + it)\, dt = \left[s - \sum_{n=c+1}^{\infty} \frac{a(n)(\lambda_n/\lambda_c)^{-s}}{\log(\lambda_n/\lambda_c)} \right]_{s=\sigma_a+1+iT}^{s=\sigma_a+1+2iT} + O\left(\int_{r/2}^{\sigma_a+1} T^{(\sigma_a - r/2)A+\epsilon} \right), \quad \epsilon > 0$$

$$= iT + O(T^{(\sigma_a - r/2)A+\epsilon}).$$

Since $(\sigma_a - \frac{1}{2}r)A < \frac{1}{2}$, we have shown by (4.17) that

$$\int_T^{2T} |Z(t)|\, dt > BT. \tag{4.19}$$

Now, let C denote the rectangle with sides $\sigma = \frac{1}{2}r$, $\sigma = \sigma_a + \delta$, $t = T$ and $t = 2T$, where $\delta > 0$ is chosen so small that

$$(\sigma_a + \delta - \tfrac{1}{2}r)A < \tfrac{1}{2}.$$

By Cauchy's theorem,

$$\int_C \{\chi(s)\}^{-1/2} \varphi(s)\, ds = 0. \tag{4.20}$$

We proceed to estimate the integrals along the two horizontal sides and the right side. By (3.4),

$$\Gamma(\alpha_k s + \beta_k)/\Gamma(\alpha_k\{r - s\} + \beta_k) = C_k(\alpha_k t)^{\alpha_k(2\sigma - r + 2it)} e^{-2i\alpha_k t}(1 + O(t^{-1})),$$

where C_k is a constant. Hence,

$$\{\chi(s)\}^{-1/2} = \textstyle\prod_{k=1}^{N} C_k^{1/2} (\alpha_k t)^{(\alpha_k/2)(2\sigma - r + 2it)} e^{-i\alpha_k t}(1 + O(t^{-1})). \tag{4.21}$$

From (4.21) and (4.18) we have

$$\{\chi(s)\}^{-1/2} \varphi(s) = O(t^{(\alpha_a - r/2)A+\varepsilon})$$

for $\frac{1}{2}r \le \sigma \le \sigma_a$, and

$$\{\chi(s)\}^{-1/2} \varphi(s) = O(t^{(\sigma_a + \delta - r/2)A+\varepsilon})$$

for $\sigma_a \le \sigma \le \sigma_a + \delta$. The integrals along the sides $t = T$ and $t = 2T$ are therefore

$$O(T^{(\sigma_a + \delta - r/2)A+\varepsilon}).$$

The integral along the right-hand side is

$$i \int_T^{2T} \prod_{k=1}^{N} C_k^{1/2} (\alpha_k t)^{\alpha_k(\sigma_a + \delta - r/2 + it)} e^{-i\alpha_k t} \varphi(\sigma_a + \delta + it)(1 + O(t^{-1}))\, dt.$$

The contribution of the O-term is

$$O(t^{(\sigma_a + \delta - r/2)A}).$$

The other part of the integral is a constant multiple of

$$\sum_{n=1}^{\infty} a(n)\lambda_n^{-\sigma_a-\delta} \int_T^{2T} t^{(\sigma_a+\delta-r/2)A} \exp\left\{it\left(\sum_{k=1}^{N} \alpha_k \log \alpha_k t - A - \log \lambda_n\right)\right\} dt.$$

We now employ Lemma 3.3 with

$$F(t) = t(\textstyle\sum_{k=1}^{N} \alpha_k \log \alpha_k t - A - \log \lambda_n) \quad \text{and} \quad G(t) = t^{(\sigma_a+\delta-r/2)A}.$$

Since

$$F'(t) = \textstyle\sum_{k=1}^{N} \alpha_k \log \alpha_k t - \log \lambda_n$$

and $F''(t) = A/t$, the hypotheses of Lemma 3.3 are clearly satisfied for T large enough. Hence, the above sum is

$$O(T^{(\sigma_a+\delta-r/2)A+1/2}).$$

Hence, by (4.20) we have shown

$$\int_{r/2+iT}^{r/2+2iT} \{\chi(s)\}^{-1/2}\varphi(s)\, ds = i\int_T^{2T} Z(t)\, dt$$
$$= O(T^{(\sigma_a+\delta-r/2)A+1/2}) = o(T),$$

since $(\sigma_a + \delta - \frac{1}{2}r)A < \frac{1}{2}$. Comparing this result with (4.19), we conclude that in every interval $(T, 2T)$ for T large enough, $Z(t)$ changes sign at least once. As the zeros of $Z(t)$ are those of $\varphi(\frac{1}{2}r + it)$, $\varphi(s)$ has an infinite number of zeros on $\sigma = \frac{1}{2}r$.

Proof of Theorem 12. For $t \neq 0$, $\chi(s)$ is holomorphic and $\chi(s) \neq 0$. Define for $t \neq 0$,

$$h(s) = -\log|\chi(s)|.$$

In order to prove (2.3) it is sufficient to show that $h(s) > 0$ for $\sigma > \frac{1}{2}r$.

Using the fact that $\Delta(s)$ is real on the real axis and thus takes conjugate values at conjugate points, we have by the mean value theorem,

$$h(s) = \log|\Delta(\sigma + it)| - \log|\Delta(r - \sigma + it)|$$

$$= 2\left(\sigma - \frac{1}{2}r\right)\left[\frac{\partial}{\partial\sigma}\log|\Delta(\sigma + it)|\right]_{\sigma=\sigma_1}, \tag{4.22}$$

where $r - \sigma < \sigma_1 < \sigma$. Now,

$$\frac{\partial}{\partial\sigma}\log|\Delta(\sigma + it)| = \operatorname{Re}\frac{d}{ds}\log\Delta(s)$$
$$= \operatorname{Re}\frac{d}{ds}\sum_{k=1}^{N}\Gamma(\alpha_k s + \beta_k).$$

Since β_k, $k = 1, \cdots, N$, is real and $t \neq 0$, we have from (3.2)

$$\log\Gamma(\alpha_k s + \beta_k) = (\alpha_k s + \beta_k - \tfrac{1}{2})\log(\alpha_k s + \beta_k) - (\alpha_k s + \beta_k) + \tfrac{1}{2}\log 2\pi$$
$$+ \frac{1}{12(\alpha_k s + \beta_k)} - 2\int_0^\infty \frac{P_3(x)\, dx}{(\alpha_k s + \beta_k + x)^3}.$$

Thus, by (3.3),

$$\frac{\partial}{\partial\sigma}\log|\Delta(\sigma+it)|$$

$$\begin{aligned}&= \operatorname{Re}\left[\sum_{k=1}^{N}\alpha_k\left\{\log(\alpha_k s+\beta_k)-\frac{1}{2(\alpha_k s+\beta_k)}\right.\right.\\ &\qquad\left.\left.-\frac{1}{12(\alpha_k s+\beta_k)^2}+6\int_0^\infty\frac{P_3(x)\,dx}{(\alpha_k s+\beta_k+x)^4}\right\}\right]\\ &\geq \sum_{k=1}^{N}\alpha_k\left\{\log|\alpha_k s+\beta_k|-\frac{1}{2|\alpha_k s+\beta_k|}\right.\\ &\qquad\left.-\frac{1}{12|\alpha_k s+\beta_k|^2}-\frac{I_k}{8}\right\},\end{aligned} \tag{4.23}$$

where

$$\begin{aligned}I_k &= \int_0^\infty\frac{dx}{\{(\alpha_k\sigma+\beta_k+x)^2+(\alpha_k t)^2\}^2}\\ &\leq \int_{-\infty}^{\infty}\frac{dy}{\{y^2+(\alpha_k t)^2\}^2}\\ &= 2\int_0^\infty\frac{dy}{\{y^2+(\alpha_k t)^2\}^2}=\frac{\pi}{2\alpha_k^3|t|^3}.\end{aligned} \tag{4.24}$$

Thus, by (4.22)–(4.24) we have shown that for $\sigma > \frac{1}{2}r$ and $s_1 = \sigma_1 + it$,

$$\frac{h(s)}{2(\sigma-\frac{1}{2}r)} > \sum_{k=1}^{N}\alpha_k\left\{\log|\alpha_k s_1+\beta_k|-\frac{1}{2|\alpha_k s_1+\beta_k|}-\frac{1}{12|\alpha_k s_1+\beta_k|^2}-\frac{\pi}{16\alpha_k^3|t|^3}\right\}.$$

It is easily seen that if $|t|$ is large enough, the right-hand side is positive, and this completes the proof.

Proofs of Corollaries 13 *and* 14. Corollary 13 is immediate from the functional equation (1.1).

If f has signature $(1, r, \gamma)$, then $\varphi(x) = (2\pi)^{-s}f(s)$. From the proof of Theorem 12, it is sufficient to choose $|t|$ large enough so that

$$\log|t|-\tfrac{1}{2}|t|-\tfrac{1}{12}|t|^2+\tfrac{1}{16}|t|^3-\log 2\pi > 0.$$

If $|t| \geq 6.8$, the above is greater than

$$1.918 - 0.074 - 0.002 - 0.001 - 1.838 = 0.003 > 0.$$

The author expresses his thanks to Professor Lowell Schoenfeld for suggesting a simplification on the author's initial proof of Theorem 12.

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