REMARKS ON ALMGREN'S INTERIOR REGULARITY THEOREMS

BY

HARLEY FLANDERS[1]

1. Introduction

In [1] Almgren gives some interesting and deep results on minimal hypersurfaces in $\mathbf{E}^4$ and $\mathbf{E}^5$. His proofs are based on two lemmas which in turn are proved by extensive calculations. It is the purpose of this note to give alternate versions of the calculations based on differential forms and moving frames. These seem to us to be simple and more natural than the local coordinate method which Almgren uses. We also include some material on the n-dimensional situation and a result of J. Simons.

Let Σ be an oriented smooth surface smoothly immersed in $\mathbf{S}^3$, the standard unit sphere $|\mathbf{x}| = 1$ in $\mathbf{E}^4$. We denote by $\mathbf{x}$ the moving point of Σ so that $\mathbf{x}$ is a function on Σ to $\mathbf{S}^3$ which may also be considered as the outward unit normal to $\mathbf{S}^3$ at $\mathbf{x}$. Locally we let $\mathbf{e}_1$, $\mathbf{e}_2$ be a right-handed orthonormal moving frame for the tangent space to the image of Σ. Finally we select a unit vector $\mathbf{e}$ so that $\mathbf{x}$, $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}$ is a right-handed orthonormal frame in $\mathbf{E}^4$. Thus $\mathbf{e}$ is the normal to Σ in $\mathbf{S}^3$. (The vectors $\mathbf{x}$, $\mathbf{e}$ are Almgren's f, g respectively.) Because of orthonormality we have

$$d\begin{bmatrix} \mathbf{x} \\ \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e} \end{bmatrix} = \Omega \begin{bmatrix} \mathbf{x} \\ \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e} \end{bmatrix} \tag{1}$$

where

$$\Omega = \begin{bmatrix} 0 & \sigma_1 & \sigma_2 & 0 \\ -\sigma_1 & 0 & \varpi & -\omega_1 \\ -\sigma_2 & -\varpi & 0 & -\omega_2 \\ 0 & \omega_1 & \omega_2 & 0 \end{bmatrix} \tag{2}$$

is a skew-matrix of one-forms on Σ (locally). The only special entry is the zero at the end of the first row due to the fact that $d\mathbf{x}$ is in the tangent plane to Σ which is spanned by $\mathbf{e}_1$ and $\mathbf{e}_2$. Exterior differentiation of (1) leads to the integrability conditions

$$d\Omega = \Omega^2$$

which we spell out as

$$d\sigma_1 = \varpi\sigma_2, \qquad d\sigma_2 = -\varpi\sigma_1, \tag{3}$$

$$d\omega_1 = \varpi\omega_2, \qquad d\omega_2 = -\varpi\omega_1, \tag{4}$$

Received December 8, 1967.

[1] This research was supported by a National Science Foundation grant.

$$\sigma_1 \omega_1 + \sigma_2 \omega_2 = 0, \tag{5}$$

$$d\varpi + \omega_1 \omega_2 + \sigma_1 \sigma_2 = 0. \tag{6}$$

(We omit the usual $\wedge$ for exterior multiplication.) The relation (5) asserts the symmetry of the matrix expressing ω_1, ω_2 as linear combinations of σ_1 and σ_2. (Clearly σ_1, σ_2 are independent so any differential on Σ can be expressed in terms of them.) Thus

$$\omega_1 = r\sigma_1 + s\sigma_2, \qquad \omega_2 = s\sigma_1 + t\sigma_2. \tag{7}$$

The total (Gaussian) curvature of Σ is the scalar K given by

$$d\varpi + K\sigma_1 \sigma_2 = 0. \tag{8}$$

The curvatures of Σ relative to the immersion in S^3 are the coefficients of the characteristic polynomial of

$$\begin{pmatrix} r & s \\ s & t \end{pmatrix}.$$

The first or mean curvature is

$$K_1 = \tfrac{1}{2}(r + t) \tag{9}$$

and the second curvature is

$$K_2 = rt - s^2. \tag{9'}$$

Clearly $\omega_1 \omega_2 = K_2 \sigma_1 \sigma_2$ so that (6) is the same as

$$K = 1 + K_2. \tag{10}$$

Henceforth we assume that Σ is immersed as a relative minimal surface, i.e., that $K_1 = 0$. Thus equations (7), (9′) become

$$\omega_1 = r\sigma_1 + s\sigma_2, \qquad \omega_2 = s\sigma_1 - r\sigma_2, \tag{11}$$

$$K_2 = -r^2 - s^2. \tag{12}$$

2. The first lemma

The invariant adjoint operator $*$, essentially rotation 90° in the tangent plane to Σ is given by $*\sigma_1 = \sigma_2$, $*\sigma_2 = -\sigma_1$ and linearity. By (11),

$$*\omega_1 = -\omega_2, \qquad *\omega_2 = \omega_1. \tag{13}$$

Next we set

$$\varpi = a\sigma_1 + b\sigma_2 \tag{14}$$

so that

$$d\sigma_1 = a\sigma_1 \sigma_2, \qquad d\sigma_2 = b\sigma_1 \sigma_2 \tag{15}$$

by (3). By (4) and (11) we have

$$d\omega_1 = -(ar + bs)\sigma_1\sigma_2, \quad d\omega_2 = (-as + br)\sigma_1\sigma_2. \tag{16}$$

We now set

$$dr = r_1\sigma_1 + r_2\sigma_2, \quad ds = s_1\sigma_1 + s_2\sigma_2. \tag{17}$$

We differentiate (11):

$$\begin{aligned} d\omega_1 &= dr\,\sigma_1 + ds\,\sigma_2 + r\,d\sigma_1 + s\,d\sigma_2 \\ &= -r_2\sigma_1\sigma_2 + s_1\sigma_1\sigma_2 + ra\sigma_1\sigma_2 + sb\sigma_1\sigma_2 \\ &= (-r_2 + s_1 + ar + bs)\sigma_1\sigma_2 \end{aligned}$$

and similarly

$$d\omega_2 = (-s_2 - r_1 + as - br)\sigma_1\sigma_2.$$

Comparing to (16) we have

$$r_1 + s_2 = 2(as - br), \quad r_2 - s_1 = 2(ar + bs). \tag{18}$$

Remark. We note two formulas which will not be used subsequently. These are both the result of direct computation. First we have

$$d * d\mathbf{x} = -2\sigma_1\sigma_2\mathbf{x} - 2K_1\sigma_1\sigma_2\mathbf{e}$$

so that under our hypothesis $K_1 = 0$ we have for the invariant Laplacian on Δ on $\mathbf{\Sigma}$, $\Delta\mathbf{x} + 2\mathbf{x} = 0$. Similarly, $\Delta\mathbf{e} - 2K_2\mathbf{e} = 0$.

We now introduce an isothermal coordinate system on $\mathbf{\Sigma}$ locally. This is a coordinate system u, v such that $\mathbf{x}_u$, $\mathbf{x}_v$ are orthogonal and of the same length $k > 0$. Thus

$$\mathbf{x}_u = k\mathbf{e}_1, \qquad \mathbf{x}_v = k\mathbf{e}_2 \tag{19}$$

defines a moving tangent frame $\mathbf{e}_1$, $\mathbf{e}_2$ and we work relative to this frame. We have

$$\sigma_1 = k\,du, \qquad \sigma_2 = k\,dv,$$

$$\begin{aligned} d\sigma_1 &= -k_v\,du\,dv = -(k_v/k^2)\sigma_1\sigma_2 \\ d\sigma_2 &= k_u\,du\,dv = (k_u/k^2)\sigma_1\sigma_2, \end{aligned} \tag{20}$$

so by (15),

$$a = -k_v/k^2, \qquad b = k_u/k^2,$$

$$\begin{aligned} \varpi &= (-k_v\sigma_1 + k_u\sigma_2)/k^2, \\ *\varpi &= -(k_u\sigma_1 + k_v\sigma_2)/k^2, \\ &= -(k_u\,du + k_v\,dv)/k, \\ *\varpi &= -dk/k = -d(\ln k). \end{aligned} \tag{21}$$

We set

$$U = k^2 r, \qquad V = -k^2 s. \tag{22}$$

Then by (21), (14) and (17),

$$\begin{aligned} dU &= 2k\,dk\,r + k^2\,dr \\ &= -2k^2 r(*\varpi) + k^2\,dr \\ &= -2k^2 r(-b\sigma_1 + a\sigma_2) + k^2(r_1\,\sigma_1 + r_2\,\sigma_2) \\ &= k^2[(r_1 + 2br)\sigma_1 + (r_2 - 2ar)\sigma_2], \\ dV &= -2k\,dk\,s - k^2\,ds \\ &= 2k^2 s(*\varpi) - k^2\,ds \\ &= 2k^2 s(-b\sigma_1 + a\sigma_2) - k^2(s_1\,\sigma_1 + s_2\,\sigma_2) \\ &= -k^2[(s_1 + 2bs)\sigma_1 + (s_2 - 2as)\sigma_2]. \end{aligned}$$

By (18) we have

$$*dU = dV. \tag{23}$$

We set

$$w = u + iv, \qquad W = U + iV. \tag{24}$$

The equation (23) is simply the Cauchy-Riemann equations asserting that W is an analytic function of w. To see how this transforms when we pass to an overlapping isothermal coordinate system we note that

$$W = 2\frac{\partial \mathbf{x}}{\partial w}\cdot\frac{\partial \mathbf{e}}{\partial w}. \tag{25}$$

For

$$\partial\mathbf{x}/\partial w = \tfrac{1}{2}(\mathbf{x}_u - i\mathbf{x}_v) = \tfrac{1}{2}k(\mathbf{e}_1 - i\mathbf{e}_2)$$

by (19). On the other hand, by (1), (2), (11), and (20),

$$\begin{aligned} d\mathbf{e} &= \omega_1\,\mathbf{e}_1 + \omega_2\,\mathbf{e}_2 \\ &= k[(r\,du + s\,dv)\mathbf{e}_1 + (s\,du - r\,dv)\mathbf{e}_2] \\ &= k[(r\mathbf{e}_1 + s\mathbf{e}_2)\,du + (s\mathbf{e}_1 - r\mathbf{e}_2)\,dv], \end{aligned}$$

so that

$$\mathbf{e}_u = k(r\mathbf{e}_1 + s\mathbf{e}_2), \qquad \mathbf{e}_v = k(s\mathbf{e}_1 - r\mathbf{e}_2),$$

$$\frac{\partial \mathbf{e}}{\partial w} = \frac{k}{2}[(r\mathbf{e}_1 + s\mathbf{e}_2) - i(s\mathbf{e}_1 - r\mathbf{e}_2)],$$

$$2\frac{\partial \mathbf{x}}{\partial w}\cdot\frac{\partial \mathbf{e}}{\partial w} = \frac{k^2}{2}[(r - is) + (-is + r)]$$

and (25) follows.

If $x, y, z = x + iy$, $X, Y, Z = X + iY$ are the corresponding quantities in another isothermal coordinate system, then

$$Z = 2\frac{\partial \mathbf{x}}{\partial z}\cdot\frac{\partial \mathbf{e}}{\partial z} = 2\left(\frac{\partial \mathbf{x}}{\partial w}\frac{dw}{dz}\right)\cdot\left(\frac{\partial \mathbf{e}}{\partial w}\frac{dw}{dz}\right)$$

and so

$$Z = W(dw/dz)^2. \tag{26}$$

We now give Almgren's Lemma 1 and its proof.

If $\mathbf{\Sigma} = \mathbf{S}^2$ *is smoothly immersed as a minimal surface* $\mathbf{S}^3$, *then the immersion is an imbedding onto an equator of* $\mathbf{S}^3$.

For (26) shows that there is a quadratic differential $W[dw]^2 = Z[dz]^2$ on the Riemann sphere $\mathbf{\Sigma}$. But the only quadratic differential on a closed Riemann surface of genus zero is 0, hence

$$W = 0, \quad U = 0, \quad V = 0, \quad r = 0, \quad s = 0,$$

$$\omega_1 = \omega_2 = 0, \quad d\mathbf{e} = 0, \quad \mathbf{e} = \text{constant}.$$

Thus the immersion maps $\mathbf{S}^2$ into (hence onto) the equator $\mathbf{S}^2$ of $\mathbf{S}^3$ orthogonal to the constant unit vector $\mathbf{e}$. This covering map must be one-one.

3. The second lemma

We pass to Almgren's Lemma 2. The proof will be based on the formulas of Section 1.

We now assume $\mathbf{\Sigma}$ is a compact oriented surface of positive genus. Thus the Euler characteristic satisfies $\chi \leq 0$. By (10) and the Gauss-Bonnet formula

$$\iint_{\Sigma} K_2\,\sigma_1\,\sigma_2 = \iint_{\Sigma} (K - 1)\sigma_1\,\sigma_2 \leq -\iint_{\Sigma} \sigma_1\,\sigma_2 = -A \tag{27}$$

where A is the total area of $\mathbf{\Sigma}$.

We assume $\mathbf{\Sigma}$ is imbedded in $\mathbf{S}^3$ as a minimal surface. The cone based on $\mathbf{\Sigma}$ with apex 0 is the set of points $u\mathbf{x}$ where $\mathbf{x} \in \mathbf{\Sigma}$, $0 \leq u \leq 1$. As we shall see shortly, except for its singularity at 0, this is a minimal hypersurface in $\mathbf{\Sigma}^4$. The assertion of Almgren's Lemma 2 is the following.

There is a radial variation of the cone leaving 0 *and its boundary* $\mathbf{\Sigma}$ *fixed, but decreasing volume.*

Such a variation is given by $\mathbf{X} = u\mathbf{x} + hF(u)\mathbf{e}$ where $F(u)$ is a smooth function on [0, 1] with supp $\phi \subset (0, 1)$. To find the volume V of our three dimensional variation of the cone we use the formula

$$6\, dV\, \mathbf{n} = [d\mathbf{X}, d\mathbf{X}, d\mathbf{X}]. \tag{28}$$

Here $\mathbf{n}$ is the unit normal to the variation at $\mathbf{X}$, and the right hand side is the vector product in $\mathbf{E}^4$ extended to vectors with one-form coefficients in the obvious way (cf. Chern [2].) By (1) and (2).

$$\begin{aligned} d\mathbf{X} &= \mathbf{x}\, du + u(\sigma_1 \mathbf{e}_1 + \sigma_2 \mathbf{e}_2) + hF'\, du\, \mathbf{e} + hF(\omega_1 \mathbf{e}_1 + \omega_2 \mathbf{e}_2) \\ &= du\, \mathbf{x} + (u\sigma_1 + hF\omega_1)\mathbf{e}_1 + (u\sigma_2 + hF\omega_2)\mathbf{e}_2 + hF'\, du\, \mathbf{e}. \end{aligned}$$

Noting that $du \wedge du = 0$, (28) yields

$$\begin{aligned} dV\, \mathbf{n} = du(u\sigma_1 + hF\omega_1)(u\sigma_2 + hF\omega_2)\mathbf{e} \\ - (u\sigma_1 + hF\omega_1)(u\sigma_2 + hF\omega_2)hF'\, du\, \mathbf{x}. \end{aligned}$$

By (11) and (12), $\sigma_1 \omega_2 - \sigma_2 \omega_1 = 0$, $\omega_1 \omega_2 = K_2 \sigma_1 \sigma_2$, hence

$$\begin{aligned} (u\sigma_1 + hF\omega_1)(u\sigma_2 + hF\omega_2) &= u^2\sigma_1 \sigma_2 + hF(\sigma_1 \omega_2 - \sigma_2 \omega_1) + h^2F^2\omega_1 \omega_2 \\ &= (u^2 + h^2F^2K_2)\sigma_1 \sigma_2 . \end{aligned}$$

We have

$$dV\, \mathbf{n} = (u^2 + h^2F^2K_2)(du\, \sigma_1 \sigma_2)(\mathbf{e} - hF'\, \mathbf{x}). \tag{29}$$

Thus since $\mathbf{x}$ and $\mathbf{e}$ are orthogonal unit vectors,

$$\begin{aligned} dV &= (u^2 + h^2F^2K_2)(1 + h^2F'^2)^{1/2}\, du\, \sigma_1 \sigma_2 \\ &= (u^2 + h^2F^2K_2)[1 + \tfrac{1}{2}h^2F'^2 + O(h^4)]\, du\, \sigma_1 \sigma_2 \\ &= [u^2 + h^2(F^2K_2 + \tfrac{1}{2}u^2F'^2) + O(h^4)]\, du\, \sigma_1 \sigma_2 . \end{aligned}$$

Integrating leads to the desired variation formula

$$\begin{aligned} V = \left(\int_0^1 u^2\, du\right)\left(\iint_\Sigma \sigma_1 \sigma_2\right) + h^2\Bigg[\left(\int_0^1 F^2\, du\right)\left(\iint_\Sigma K_2\, \sigma_1 \sigma_2\right) \\ + \tfrac{1}{2}\left(\int_0^1 u^2F'^2\, du\right)\left(\iint_\Sigma \sigma_1 \sigma_2\right)\Bigg] + O(h^4). \end{aligned} \tag{30}$$

Thus the first variation of V vanishes, verifying that our cone is a minimal hypersurface, and the second variation is

$$\tfrac{1}{2}\frac{d^2V}{dh^2}\bigg|_0 = \left(\iint_\Sigma K_2\, \sigma_1 \sigma_2\right)\left(\int_0^1 F(u)^2\, du\right) + \frac{A}{2}\left(\int_0^1 u^2\, F'(u)^2\, du\right). \tag{31}$$

In view of (27) we have

$$\tfrac{1}{2}\frac{d^2V}{dh^2}\bigg|_0 \leq \frac{A}{2}\left[\int_0^1 u^2F'(u)^2\, du - 2\int_0^1 F(u)^2\, du\right]. \tag{32}$$

We choose F to be the piecewise linear function whose polygonal graph has the vertices $(0, 0)$, $(\varepsilon, 1)$, $(1, 0)$, where ε is a small fixed number. One easily finds

$$\int_0^1 u^2 F'(u)^2\, du = \tfrac{1}{3} + O(\varepsilon),$$

$$\int_0^1 F(u)^2\, du = \tfrac{1}{3} + O(\varepsilon),$$

hence

$$\left.\frac{d^2V}{dh^2}\right|_0 \leq A(-\tfrac{1}{3} + O(\varepsilon)) < -\frac{A}{4} < 0$$

for ε sufficiently small. Finally, one rounds off the corners on the graph of F to create a smooth variation with smaller volume.

4. The second variation in $\mathbf{E}^{n+2}$

In this section we sketch the situation in higher dimensions. We assume $\mathbf{\Sigma}$ is an n-dimensional oriented manifold imbedded in $\mathbf{S}^{n+1} \subset \mathbf{E}^{n+2}$ as a minimal hypersurface in $\mathbf{S}^{n+1}$. We select a right-handed moving orthonormal frame $\mathbf{x}, \mathbf{e}_1, \cdots, \mathbf{e}_n, \mathbf{e}$ where $\mathbf{x}$ is the moving point on $\mathbf{\Sigma}$ and $\mathbf{e}_1, \cdots, \mathbf{e}_n$ is a right-handed orthonormal frame in the tangent plane to $\mathbf{\Sigma}$ at $\mathbf{x}$. The equations of structure are

$$(33) \qquad d\begin{bmatrix} \mathbf{x} \\ \mathbf{e}_1 \\ \vdots \\ \mathbf{e}_n \\ \mathbf{e} \end{bmatrix} = \begin{bmatrix} 0 & \sigma & 0 \\ -{}^t\sigma & \tilde{\omega} & {}^t\omega \\ 0 & -\omega & 0 \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{e}_1 \\ \vdots \\ \mathbf{e}_n \\ \mathbf{e} \end{bmatrix}$$

where

$$\sigma = (\sigma_1, \cdots, \sigma_n), \qquad \omega = (\omega_1, \cdots, \omega_n),$$

$$\varpi = \| \varpi_{ij} \|, \quad n \times n \text{ skew},$$

are matrices of one-forms. The integrability conditions $d\Omega = \Omega^2$, where Ω is the 3×3 matrix of blocks above, are

$$(34) \qquad \begin{aligned} d\sigma &= \sigma\varpi, & \sigma\, {}^t\omega &= 0, \\ d\omega &= \omega\varpi, & (d\varpi - \varpi^2) + {}^t\sigma\sigma + {}^t\omega\omega &= 0. \end{aligned}$$

We write $\omega = \sigma A$, $A = {}^tA$, with the matrix A of functions symmetric by (34) The relative curvatures $K_1, K_2, \cdots$ are given by

$$(35) \qquad | tI + A | = t^n + \binom{n}{1} K_1 t^{n-1} + \binom{n}{2} K_2 t^{n-2} + \cdots + K_n .$$

Since we are assuming $\mathbf{\Sigma}$ is minimal in $\mathbf{S}^{n+1}$, we have

$$(36) \qquad K_1 = 0.$$

The cone with apex 0 over $\mathbf{\Sigma}$ is a hypersurface in $\mathbf{E}^{n+2}$ given by $\mathbf{X} = u\mathbf{x}$,

$0 \leq u \leq 1$, $\mathbf{x} \in \Sigma$. We consider a variation of this cone of the form

$$\mathbf{X} = u\mathbf{x} + hF(u)\mathbf{e} \tag{37}$$

where h is a small parameter and supp $F(u) \subset (0, 1)$. The volume V of this variation is obtained from

$$(n+1)!\, dV\, \mathbf{n} = [d\mathbf{X}, \cdots, d\mathbf{X}], \tag{38}$$

where $\mathbf{n}$ is the unit normal to the new hypersurface. We have, using (33),

$$d\mathbf{X} = du\, \mathbf{x} + \sum (u\sigma_i + hF\omega_i)\mathbf{e}_i + hF'\, du\, \mathbf{e}.$$

By (34),

$$\begin{aligned} dV\,\mathbf{n} &= du(u\sigma_1 + hF\omega_1) \cdots (u\sigma_n + hF\omega_n)\mathbf{e} \\ &\qquad \pm hF'\, du(u\sigma_1 + hF\omega_1) \cdots (u\sigma_n + hF\omega_n)\mathbf{x} \\ &= du[u^n\sigma_1 \cdots \sigma_n + hFu^{n-1}\textstyle\sum \sigma_1 \cdots \sigma_{n-1}\,\omega_n \\ &\qquad + h^2F^2u^{n-2}\textstyle\sum \sigma_1 \cdots \sigma_{n-2}\,\omega_{n-1}\,\omega_n + O(h^3)](\mathbf{e} \pm hF'\mathbf{x}). \end{aligned}$$

Now we have

$$\begin{aligned} \textstyle\sum \sigma_1 \cdots \sigma_{n-1}\,\omega_n &= (\textstyle\sum a_{ii})\sigma_1 \cdots \sigma_n = nK_1\,\sigma_1 \cdots \sigma_n = 0, \\ \textstyle\sum \sigma_1 \cdots \sigma_{n-1}\,\omega_{n-1}\,\omega_n &= \textstyle\sum_{i<j} (a_{ii}\,a_{jj} - a_{ij}^2)\sigma_1 \cdots \sigma_n \\ &= \binom{n}{2} K_2\,\sigma_1 \cdots \sigma_n, \end{aligned}$$

hence

$$dV\,\mathbf{n} = \left[u^n + \binom{n}{2} h^2F^2u^{n-2}K_2 + O(h^3)\right](\mathbf{e} \pm hF'\mathbf{x})(du\,\sigma_1 \cdots \sigma_n). \tag{39}$$

Comparing lengths,

$$dV = \left[u^n + \binom{n}{2} h^2F^2u^{n-2}K_2 + O(h^3)\right][1 + h^2F'^2]^{1/2}(du\,\sigma_1 \cdots \sigma_n)$$

which gives us

$$\begin{aligned} dV = \Bigg\{u^n + h^2\left[\binom{n}{2} F^2u^{n-2}K_2 + \tfrac{1}{2}\,F'^2u^n\right] \\ + O(h^3)\Bigg\}(du\,\sigma_1 \cdots \sigma_n). \end{aligned} \tag{40}$$

This yields the desired formula for the second variation:

$$\begin{aligned} \left.\frac{d^2V}{dh^2}\right|_0 = 2\binom{n}{2}\left(\int_\Sigma K_2\,\sigma_1 \cdots \sigma_n\right)\left(\int_0^1 u^{n-2}F(u)^2\,du\right) \\ + \left(\int_\Sigma \sigma_1 \cdots \sigma_n\right)\left(\int_0^1 u^nF'(u)^2\,du\right) \end{aligned} \tag{41}$$

As an application we obtain a result of J. Simons [3]. This is the stability of the cone on $S^3 \times S^3 \subset S^7$.

Let $\mathbf{y} \in S^3$ and let $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ be a moving frame on a neighborhood of $\mathbf{y}$. Similarly let $\mathbf{y}'$ be another point on S^3 and let $\mathbf{e}_1', \mathbf{e}_2', \mathbf{e}_3'$ be a moving frame for a neighborhood of $\mathbf{y}'$. Set

$$\mathbf{x} = (1/\sqrt{2})(\mathbf{y}, \mathbf{y}') \in S^7 \subset E^8. \tag{42}$$

On the two neighborhoods of S^3 we have respectively

$$d\mathbf{y} = \textstyle\sum_1^3 \sigma_i \mathbf{e}_i, \qquad d\mathbf{y}' = \sum_1^3 \sigma_j' \mathbf{e}_j'.$$

Hence

$$d\mathbf{x} = \textstyle\sum_1^3 (\sigma_i/\sqrt{2})(\mathbf{e}_i, 0) + \sum_1^3 (\sigma_j'/\sqrt{2})(0, \mathbf{e}_j'). \tag{43}$$

We deduce that an orthonormal basis of one-forms to $S^3 \times S^3$ is given by

$$\sigma = (1/\sqrt{2})(\sigma_1, \sigma_2, \sigma_3, \sigma_1', \sigma_2', \sigma_3'). \tag{44}$$

The vector field

$$\mathbf{e} = (1/\sqrt{2})(-\mathbf{y}, \mathbf{y}') \tag{45}$$

is normal to $\mathbf{x}$, $(\mathbf{e}_i, 0)$, and $(0, \mathbf{e}_j')$, hence

$$\mathbf{x}, (\mathbf{e}_1, 0), \cdots, (0, \mathbf{e}_3'), \mathbf{e}$$

is the moving frame we seek.

We have

$$d\mathbf{e} = (-1/\sqrt{2})\textstyle\sum_1^3 \sigma_i(\mathbf{e}_i, 0) + (1/\sqrt{2})\sum_1^3 \sigma_j'(0, \mathbf{e}_j') \tag{46}$$

and consequently

$$\omega = (1/\sqrt{2})(-\sigma_1, -\sigma_2, -\sigma_3, \sigma_1', \sigma_2', \sigma_3'). \tag{47}$$

This implies that the matrix A is

$$A = (1/\sqrt{2}) \operatorname{diag}\{-1, -1, -1, 1, 1, 1\}. \tag{48}$$

We readily compute $K_1 = 0$, $\binom{6}{2}K_2 = -3$. Thus we have $S^3 \times S^3$ imbedded in S^7 as a minimal hypersurface $\mathbf{\Sigma}$ and (41) specializes to

$$\left.\frac{d^2V}{dh^2}\right|_0 = |\mathbf{\Sigma}| \left(\int_0^1 u^6F'(u)^2\,du - 6\int_0^1 u^4F(u)^2\,du\right). \tag{49}$$

This is non-negative for any F, in fact we have the following result.

LEMMA. *If F is a C' function on* $[0, 1]$ *such that* supp $F \subset (0, 1)$, *then*

$$\int_0^1 u^6F'(u)^2\,du \geq (25/4)\int_0^1 u^4F(u)^2\,du.$$

Proof. Set $F(u) = u^{-5/2}G(u)$ so that supp $G \subset (0, 1)$. Then

$$\int_0^1 u^6 F'(u)^2 \, du = \int_0^1 \left[uG'^2 - 5GG' + \frac{25}{4}\frac{G^2}{u} \right] du.$$

Since $uG'^2 \geq 0$ and $\int_0^1 GG' \, du = \frac{1}{2}G^2 \mid_0^1 = 0$, we have

$$\int_0^1 u^6 F'(u)^2 \, du \geq (25/4) \int_0^1 G^2(du/u) = (25/4) \int_0^1 u^4 F(u)^2 \, du.$$

A variational approach to this inequality indicates that $\frac{25}{4}$ is the best constant.

REFERENCES

1. F. J. ALMGREN, JR., *Some interior regularity theorems for minimal surfaces and an extension of Bernstein's Theorem*. Ann. of Math., vol. 84 (1966), pp. 277–292.
2. S. S. CHERN, *Integral formulas for hypersurfaces in Euclidean space and their applications to uniqueness theorems*, J. Math. Mech., vol. 8 (1959), pp. 947–955, especially p. 949.
3. J. SIMONS, *Minimal cones, Plateau's problem, and the Bernstein conjecture*, Proc. Nat. Acad. Sci., vol. 58 (1967), pp. 410–411.

PURDUE UNIVERSITY
LAFAYETTE, INDIANA