HOMOTOPICALLY DISTINCT ACTIONS OF FINITE GROUPS

BY

C. B. THOMAS[1]

In this paper we study the following problem: given a finite group G of order r and period $(m+1)$ in cohomology, together with a generator $\mathbf{l}$ of $H^{m+1}(G, Z)$, does there exist a finite CW-complex Y with fundamental group isomorphic to G, universal cover homotopically equivalent to S^n and Eilenberg-MacLane invariant $\mathbf{l}$? (At least up to sign this invariant is equivalent to the first k-invariant of a Postnikov system, and is defined in [2].) The problem is motivated by the following result of R. G. Swan; see [7].

THEOREM. *Let G be a finite group of order r and let $d = (r, \varphi(r))$, φ being the Euler function. Suppose G has cohomological period $m+1$. Then there is a finite simplicial complex Σ^n of dimension $n = d(m+1) - 1$ having the homotopy type of S^n, and on which G acts simplicially without fixed points.*

With the orbit complex $Y = \Sigma^n/G$ we can associate an Eilenberg-MacLane invariant $\mathbf{l}$, which together with G defines its homotopy type. Our aim is to vary the construction of Y so as to obtain different generators of $H^{n+1}(G, \mathbf{Z})$ as l-invariants, and hence homotopically distinct G-actions. In particular we shall show that when G has cyclic Sylow subgroups, every generator of $H^{n+1}(G, \mathbf{Z})$ can be geometrically realised as an l-invariant. We emphasise that our results are only of interest in the category of finite CW-complexes, without this restriction Swan himself notes that d can be replaced by 1 and all abstract homotopy types easily realised.

The author wishes to express his gratitude to Professor P. J. Hilton for his help and encouragement at all stages of the preparation of this paper.

1. Simplifying Swan's construction

Throughout we work with complexes of finitely generated, projective left $\mathbf{Z}(G)$-modules. If the modules are free, the complex can be geometrically realised as a CW-complex—indeed as a polyhedron—provided that its truncation in dimension two defines the chains of a 1-connected complex admitting G as a group of free operators. Under this assumption purely algebraic arguments have an immediate geometric interpretation for the polyhedral realisations.

Following [7] we define a periodic projective (free) resolution of length v for G to be an exact sequence of projective (free) G-modules,

$$0 \to \mathbf{Z} \to W_{v-1} \to \cdots \to W_0 \to \mathbf{Z} \to 0.$$

Received October 13, 1967.

[1] Work on this paper was supported by a National Science Foundation grant.

Construct a projective resolution for $\mathbf{Z}$ over $\mathbf{Z}(G)$ by starting with the chain complex of the universal cover of a 2-complex with G as fundamental group, and extending to higher dimensions by takng kernels. Truncate this resolution in dimension m

$$0 \to A \to C_m \to C_{m-1} \to \cdots \to C_0 \to \mathbf{Z} \to 0. \qquad (\alpha)$$

The crucial step in Swan's argument is the proof that there exists a projective module P such that $A + \mathbf{Z}(G) \cong \mathbf{Z} + P$. Using this isomorphism we replace (α) by

$$0 \to \mathbf{Z} \to C_m \xrightarrow{(\partial,g)} C_{m-1} + P \xrightarrow{(\partial,0)} \cdots \to C_0 \to \mathbf{Z} \to 0 \qquad (\beta)$$

Here $\mathbf{Z}(G)$ has been absorbed into C_m, and g is a G-retraction of (new) C_m onto P.

Let $\tilde{K}^\circ(G)$ be the projective class group of the group ring $\mathbf{Z}(G)$. Since G is finite, $\tilde{K}^\circ(G)$ is finite, see [6, Proposition 9.1]; denote the exponent of $\tilde{K}^\circ(G)$ by c.

LEMMA 1.1 *There exists a periodic free resolution γ_f for G of period $c(m+1)$*.

Proof. Let Q be a projective inverse to P. Splice together c copies of β and form an exact sequence γ of length $c(m+1)$. Define families of exact sequence $\{\delta_{i,t}\}$ $\{\gamma_i\}$ as follows: $\delta_{i,t}$ is the trivial exact sequence of length $c(m+1)$ having all terms except the $(i+1, i)$ pair trivial. For i odd this pair is id : $tP \to tP$, and for i even id : $tQ \to tQ$.

$$\gamma_{m-2} = \gamma, \quad \gamma_i = \text{direct sum of } \gamma_{i-1} \text{ and } \delta_{i,t},$$
$$\text{where} \quad t(m+1) - 2 \le i \le (t+1)(m+1) - 3^{\cdot}$$

All we are doing is alternately adding copies of P and Q to γ in order to make the terms free. Eventually we reach

$$\gamma_{c(m+1)-3} : 0 \to \mathbf{Z} \to C_m \to C_{m-1} + cP \to \text{lower dimensional terms.}$$

But $[cP] = c[P] = 0$ in $\tilde{K}^\circ(G)$, because c is the exponent. ([ ] denotes the stable equivalence class of P.) Therefore there are finitely generated free modules F, F' such that $cP + F' \cong F$. Define a last direct sum of complexes:

$$\gamma_f : 0 \to \mathbf{Z} \to C_m + F' \to C_{m-1} + F \to \cdots \to C_0 \to \mathbf{Z} \to 0$$

γ_f will be our basic periodic free resolution for the group G.

The argument of Lemma 1.1 shortens the construction of a periodic free resolution for G of finite type in [7] considerably, but at the price of replacing r or $d = (r, \varphi(r))$ by the usually larger integer c. However this makes no difference, if our primary concern is to construct a compact polyhedron with l-invariant defined by a given generator of $H^{m+1}(G, \mathbf{Z})$.

2. Resolutions over subgroups of G

LEMMA 2.1. *Given a subgroup $H \subset G$, and a free resolution of $\mathbf{Z}$ over $\mathbf{Z}(H)$ admitting geometric realisation, the resolution can be embedded in a G-resolution also admitting geometric realisation.*

Proof. Take the orbit space of the 1-connected 2-dimensional complex defined by the given H-resolution truncated in dimension two, and assume without loss of generality that there is only one 0-cell. This complex gives a presentation of H, the 1-cells defining the generators and the 2-cells the relations. Furthermore the truncated chain complex in the universal cover coincides with that defined using the free differential calculus on this presentation of H, see Reidemeister [5].

There is a presentation of G which includes the presentation of H we have just defined. A subset of the relations added describes how H embeds in G. Use this presentation to construct a free $\mathbf{Z}(G)$-resolution of $\mathbf{Z}$. The boundary homomorphisms can be considered finite matrices with entries from $\mathbf{Z}(G)$, write the original boundary matrices as ${}^k\bar{M}^{k-1}$. [The entries in ${}^k\bar{M}^{k-1}$ are now considered to lie in $\mathbf{Z}(G)$ rather than in $\mathbf{Z}(H)$.] Since we are working with modules over a non-commutative ring, a little care is necessary in writing the operations. Both the scalar multiplication and the abstract boundary map $\partial_k : C_k \rightarrow C_{k-1}$ are written on the *left*, but the matrix ${}^kM^{k-1}$ operates on the *right*. This corresponds to making $\mathbf{Z}(G)$ a left module, when the map "right multiplication by ν" is a module homomorphism.

Formally the G-resolution is started using the free differential calculus, and continued by taking kernels repeatedly. The first two boundary matrices are

$$ {}^1M^0 = \begin{pmatrix} {}^1N^0 \\ {}^1\bar{M}^0 \end{pmatrix} \quad \text{and} \quad {}^2M^1 = \begin{pmatrix} {}^2N^1 & {}^2P^1 \\ 0 & {}^2\bar{M}^1 \end{pmatrix}. $$

${}^1N^0$ describes the generators of G outside H. ${}^2N^1$ refers to the differentiation on the new relations with respect to the new generators, and ${}^2P^1$ to differentiation with respect to the old. The 0 block matrix in the lower left corner of ${}^2M^1$ comes from the fact that relations inside H are functions only of the generators of H. ${}^3M^2$, ${}^4M^3$, $\cdots$ are defined by taking C_3, C_4, $\cdots$ to be the free $\mathbf{Z}(G)$-modules generated by the kernels at steps C_2, C_3, $\cdots$. At each stage the boundary map can be written in the form

$$ {}^kM^{k-1} = \begin{pmatrix} {}^kN^{k-1} & {}^kP^{k-1} \\ 0 & {}^k\bar{M}^{k-1} \end{pmatrix}. $$

The block submatrix ${}^kP^{k-1}$ is needed to cover that part of ker $({}^{k-1}\bar{M}^{k-2})$defined by elements of $\mathbf{Z}(G)$ outside $\mathbf{Z}(H)$, and to kill cross terms originally introduced at stage two by the embedding relations for H. Since the right hand end of this algebraic complex is defined using the free differential calculus, there is a geometric realisation.

Remark. The existence of the embedding given by this lemma is stated without proof in [4, page 658]. The same paper contains a brief treatment of the free differential calculus.

If we apply the first step of the Swan argument to the resolution just constructed, we obtain a periodic projective resolution of the form below.

$$\begin{tikzcd}
 & & \bar{C}_m \arrow[r] \arrow[ddddr, dashed] & \bar{C}_{m-1} \arrow[r] & \bar{C}_{m-2} & \\
 & & \oplus & \oplus & \oplus & \\
0 \arrow[r] & \mathbf{Z} \arrow[r] & C_m \arrow[r] \arrow[uur] \arrow[ddr, dashed, "g"] & C_{m-1} \arrow[r] \arrow[uur] & C_{m-2} & \cdots \\
 & & \oplus & \oplus & & \\
 & & F \arrow[r, dashed, shift left] & P \arrow[l, dashed, shift left, "h"] \arrow[uur, dashed, "0"'] & &
\end{tikzcd} \qquad (*)$$

Let h embed P as a direct summand of the "m-chains". Suppose that the long spliced sequence of length $c(m+1)$ has been made free as in Lemma 1.1; a fortiori the algebraic complex is a periodic free H-resolution, in which the direct summands $\bar{C}_k$, $0 \leq k \leq m$ are replaced by direct sums of copies of themselves indexed by the cosets G/H.

Recall that the Eilenberg-MacLane invariant 1^{n+1} is defined to be the primary obstruction to extending an equivariant chain map from the n-skeleton of a reference acyclic complex W to a given chain complex C_*Y equivariantly over the $(n+1)$-chains of W. $\tilde{Y}^{c(m+1)-1}$, as constructed above, has an Eilenberg MacLane invariant with respect to both G and H. Denote these by

$$\mathbf{b} = 1_G^{n+1} \in H^{n+1}(G, H_n \tilde{Y}) \quad \text{and}$$

$$\mathbf{a} = 1_H^{n+1} \in H^{n+1}(H, H_n \tilde{Y}), \quad n = c(m+1) - 1.$$

If $\mu : H \to G$ denotes inclusion, $\mu_* \mathbf{b} = \mathbf{a}$.

The embedding construction of the lemma can be performed simultaneously for any family of *pairwise disjoint* subgroups $H_1, H_2, \cdots, H_h$ of G. For clarity consider two subgroups H_1, H_2 such that $H_1 \cap H_2 = \{1\}$. The matrices of the G-boundary homomorphisms will have the form

$${}^kM^{k-1} = \begin{pmatrix} {}^k\bar{M}^{k-1} & 0 \\ {}^kP^{k-1} & {}^kQ^{k-1} \\ 0 & {}^k\bar{\bar{M}}^{k-1} \end{pmatrix},$$

where the upper left and lower right blocks are defined by H_1- and H_2-resolutions. After we have completed the Swan construction, there are three l-invariants related by

$$\mu_{i\,*}\,\mathbf{b} = \mathbf{a}_i, \qquad i = 1, 2.$$

3. Resolutions for cyclic groups

In this short paragraph we collect together various facts and notations for cyclic groups, which we need for the main theorem. Different periodic free resolutions of $\mathbf{Z}$ over $\mathbf{Z}_p(\xi)$ are given by the equivariant CW decompositions of S^m (m odd) which describe the Lens Spaces. Thus $L^m(p;\, q,\, 1, \cdots, 1)$ gives

$$0 \to \mathbf{Z} \to C_m \xrightarrow{\xi^q - 1} C_{m-1} \xrightarrow{N_\xi} C_{m-2} \cdots \xrightarrow{\xi - 1} C_0 \to \mathbf{Z} \to 0.$$

Here $(p, q) = 1$, $N_\xi = 1 + \xi + \cdots + \xi^{p-1}$ is the norm element in $\mathbf{Z}(\mathbf{Z}_p(\xi))$. Suppose $qq' \equiv 1 \bmod p$. If we use $L^\infty(p;\, 1,\, 1, \cdots \cdot, \cdots)$ to define the reference complex W, direct computation shows that the cocycle

$$l^{m+1}(L(p;\, q,\, 1, \cdots, 1))$$

equals q'.

In $\mathbf{Z}(\mathbf{Z}_p(\xi))$ write

$$\omega = 1 + \xi^q + \cdots + \xi^{q(q'-1)} \quad \text{and} \quad \omega' = 1 + \xi + \cdots + \xi^{q-1}.$$

One easily verifies that

(i) $\omega(\xi^q - 1) = \xi - 1$, $\omega'(\xi - 1) = \xi^q - 1$, and
(ii) $\omega . \omega' = \omega' \cdot \omega = 1 + N_\xi$.

4. Embedding resolutions for cyclic subgroups

By means of an intuitively simple but computationally messy argument we prove:

THEOREM 4.1. *Let $\mathbf{Z}_p(\xi)$ be a cyclic subgroup of G, a finite group with cohomological period $m + 1$. Then there exists a periodic free resolution of $\mathbf{Z}$ over $\mathbf{Z}(G)$ of dimension $c(m + 1) - 1 = n$ with Eilenberg-MacLane invariant* $\mathbf{b}$, *such that* $\mathbf{b}$ *restricts to any chosen generator in* $H^{n+1}(\mathbf{Z}_p(\xi), \mathbf{Z})$.

Observe first that if $\mathbf{Z}$ is equivariantly embedded in a free $\mathbf{Z}(G)$-module on k generators, then the image of 1 is a multiple of the norm N_G. We distinguish carefully between the norm N_G of the whole group G, and the norm N_ξ of some cyclic subgroup. With $\mathbf{Z}_p(\xi) = H$ apply the embedding construction of §2, using first the periodic free resolution for $\mathbf{Z}_p(\xi)$ defined by $L^m(p;\, 1,\, 1, \cdots 1)$ and second that defined by $L(p;\, q,\, 1, \cdots, 1)$. Truncation in dimension m gives

$$\begin{tikzcd}
& & \bar{e}_m \arrow[rr, "\partial^1_m = \xi - 1"] & & \bar{e}_{m-1} \arrow[rr, "N_\xi"] & & {} \\
0 \arrow[r] & A_1 \arrow[r] & \oplus & & \oplus & & \cdots \\
& & C_m \arrow[rr, "\theta_m"'] \arrow[uurr, "\nu"] & & C_{m-1} \arrow[rr, "\theta_{m-1}"'] \arrow[uurr] & & {} \\
& & & \Delta^1_m & & \Delta_{m-1} &
\end{tikzcd} \qquad (1)$$

and

$$\begin{tikzcd}
& & \bar{e}_m \arrow[rr, "\partial^q_m = \xi^q - 1"] & & \bar{e}_{m-1} \arrow[rr, "N_\xi"] & & {} \\
0 \arrow[r] & A_q \arrow[r] & \oplus & & \oplus & & \cdots \\
& & C_m \arrow[rr, "\theta_m"'] \arrow[uurr, "\nu"] & & C_{m-1} \arrow[rr, "\theta_{m-1}"'] \arrow[uurr] & & {} \\
& & & \Delta^q_m & & \Delta_{m-1} &
\end{tikzcd} \qquad (\mathrm{q})$$

$\Delta^*_m = ((\partial^*_m, 0) + (\nu, \theta_m))$, etc. and $A_1 =$ the kernel at stage m in (1) is generated by pairs (x, y) such that, either

(i) $x = 0$ and both $\nu y = 0$, $\theta_m y = 0$, or
(ii) $\partial^1_m x = 0$ and $y = 0$, or
(iii) $\partial^1_m x + \nu y = 0$ and $\theta_m y = 0$, but $\nu y \neq 0$.

If X_1 is the submodule of $\mathbf{Z}(G).\bar{e}_m$ spanned by the x satisfying (iii), X_1 does not meet the submodule generated by N_ξ. (Otherwise $\partial^1_m x = 0$ and we do not have a cross term.) Let A_q and X_q be the corresponding modules in (q), again X_q does not meet N_ξ. Because of the two void intersections, X_1 and X_q are isomorphic to $\bar{X}_1$ and $\bar{X}_q$ respectively in the quotient module $\mathbf{Z}(G)/\mathbf{Z}(G).N_\xi$. Relation (ii) at the end of §3 implies that postmultiplication by ω and ω' in $\mathbf{Z}(G)$ defines inverse automorphisms in $\mathbf{Z}(G)/\mathbf{Z}(G).N_\xi$ and relation (i) that $\bar{X}_q\,\omega' = \bar{X}_1$. Since $A_1 \cap C_m = A_q \cap C_m$ and kernel $(\xi - 1) =$ kernel $(\xi^q - 1)$ in $\mathbf{Z}(G).\bar{e}_m$, it follows that $A_1 \cong A_q$. Hence the *same* projective module P with its retraction g can be used for the modification of both (1) and (q). Both periodic projective resolutions are described by the diagram $(*)$ from §2. The embedding of $\mathbf{Z}$ is the same in both cases, since the only way in which they might differ is through mixed terms $\eta_1 \oplus \eta_2$ in $\bar{e}_m \oplus C_m$. But there can be none such, for $\mathbf{Z}$ is equivariant, and $(\xi^q - 1)$ and $(\xi - 1)$ both kill all normed elements.

We next compute a chain obstruction which is effectively the same as the

Eilenberg-MacLane invariant. By an abuse of notation we shall refer to it as such, even though as yet because of the module P, there is no geometric realisation. The proof makes clear that the eventual replacement of P by a free module using the method of §1 will not change either the chain map or to obstruction to extending it.

LEMMA 4.2. *Using the resolution for* $\mathbf{Z}$ *over* $\mathbf{Z}_p(\xi)$ *defined by* $L^{\infty}(p; 1, \cdots, 1, \cdots)$ *as a reference complex, let the projective complex* (1) *have* $\mathrm{l}^{m+1} = \mathbf{a}_p$ *over* $\mathbf{Z}_p$. *Then the complex* (q) *has* $\mathrm{l}^{m+1} = q'.\mathbf{a}_p$ *over* $\mathbf{Z}_p$.

Proof. Carry out the construction of a chain map from $\tilde{L}^{\infty}(p)$ to (1). Up to dimension $(m - 1)$ the identity embedding will do, but in dimension m we must take care of the retraction g. If Δ_m and $'\Delta_m$ are the boundary homomorphisms in the modified and unmodified complex, see $(*)$,

$$'\Delta_m \, \bar{e}_m = \bar{\partial}_m \, \bar{e}_m + g\bar{e}_m .$$

If h embeds P in the "m-chains", h followed by Δ_m maps P', the part of P over $\bar{C}_m \oplus C_m$ to zero, because $P' \subset A_1 = \operatorname{Ker} \Delta_m$. So in dimension m define the chain map by $\bar{e}_m \to \bar{e}_m - hg\bar{e}_m$. The primary obstruction to extending this map is $\mathbf{a}_p$.

If we replace (1) by (q) as image complex, the new chain map in dimension m is $\bar{e}_m \to (\bar{e}_m - hg\bar{e}_m).\omega$, where ω is the same element from $\mathbf{Z}(\mathbf{Z}_p(\xi))$ as before. Since $N_{\xi}.\omega$ is multiplication by q' $(qq' \equiv 1 \bmod p)$, the new obstruction is $q'.\mathbf{a}_p$.

Form a free periodic resolution of length $c(m + 1) = n + 1$ by splicing $(c - 1)$ copies of (1) onto the left hand end of (q). Since the critical projective module in dimension $(m - 1)$ is P in both cases, the argument of §1 applies. The long resolution defines a CW-complex $\tilde{Y}^n$, which has true Eilenberg-MacLane invariants over G and $\mathbf{Z}_p(\xi)$. Compute the $\mathbf{Z}_p(\xi)$-invariants using Lemma 4.2; clearly we obtain $(\mathbf{a}_p)^c$ and $q'.(\mathbf{a}_p)^c$. This concludes the proof of the theorem.

5. Groups with cyclic Sylow subgroups

LEMMA 5.1. *Let* $H \subset G$ *be a subgroup of a group with cohomological period* $m + 1$. *A fortiori* H *has period* $m + 1$, *and the restriction map*

$$\mu^* : H^{m+1}(G, \mathbf{Z}) \to H^{m+1}(H, \mathbf{Z})$$

is epic.

Proof. Use the periodicity isomorphism to lift the epimorphism

$$\hat{H}^{\circ}(G, \mathbf{Z}) \to \hat{H}^{\circ}(H, \mathbf{Z})$$

to dimension $(m + 1)$. Here the symbol $(\wedge)$ refers to Tate cohomology, see [1, Chapter XII].

THEOREM 5.2. *Let G be a finite group having all its Sylow subgroups cyclic. A periodic free resolution of $\mathbf{Z}$ over G can be constructed with dimension $c(m+1)-1=n$ and* l*-invariant defined by any generator of $H^{m+1}(G, \mathbf{Z})$.*

Proof. It is a standard result in finite group theory, [3, page 146], that G has a presentation:

$$\{\xi, \eta : \xi^p = \eta^r = 1, \eta^{-1}\xi\eta = \xi^t, (p, r) = 1, t^r \equiv 1 \pmod p), (t-1, p) = 1\}.$$

Apply Theorem 4.1 to the disjoint subgroups generated by ξ and η, and construct $\tilde{Y}$ with l-invariant $\mathbf{b}$ in $H^{n+1}(G, \mathbf{Z})$. Identify this group with $\mathbf{Z}_{pr}$, when $\mathbf{b}$ is such that

$$b = q + up, \ (p, q) = 1 \quad \text{and} \quad b = s + vr, \ (r, s) = 1.$$

Here q is the l-invariant of $L^n(p; q', \cdot, \cdots, 1)$ and s the l-invvriant of $L^n(r; s', 1, \cdots, 1)$, whose chain complexes have been used to define that of $\tilde{Y}$. The pair of equations defines b uniquely in $\mathbf{Z}_{pr}$, and any generator arises from a pair (q, s).

REFERENCES

1. H. CARTAN AND S. EILENBERG, *Homological algebra,* Princeton, 1956.
2. S. EILENBERG AND S. MACLANE, *Homology of spaces with operators II,* Trans. Amer. Math. Soc., vol. 65 (1949), pp. 49–99.
3. M. HALL, JR., *The theory of groups,* Macmillan, New York, 1959.
4. R. C. LYNDON, *Cohomology of groups with a single defining relation,* Ann. of Math., vol. 52 (1950), pp. 650–665.
5. K. REIDEMEISTER, *Complexes and homotopy chains,* Bull. Amer. Math. Soc., vol. 56 (1950), pp. 297–307.
6. R. G. SWAN, *Induced representations and projective modules,* Ann. of Math., vol. 71 (1960), pp. 552–578.
7. R. G. SWAN, *Periodic resolutions for finite groups,* Ann. of Math., vol. 72 (1960), pp. 267–291.

CORNELL UNIVERSITY
ITHACA, NEW YORK
UNIVERSITY OF HULL
HULL, ENGLAND