CARTER SUBGROUPS AND FITTING HEIGHTS OF FINITE SOLVABLE GROUPS

BY

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Let G be a finite solvable group having Fitting height h (as defined in [7] or in §1 below). Let H be a Carter subgroup of G and l be the length of a composition series of H. We shall establish the correctness of a conjecture of John Thompson (at the end of [7]) by proving that

$$h \leq 10(2^l - 1) - 4l. \tag{0.1}$$

This is the result of Theorem 8.5 below, and the rest of this paper is a proof of that theorem.

The upper bound for h given by (0.1) is almost certainly too large. The work of Shamash and Shult [6] leads one to conjecture that there is some constant K such that

$$h \leq Kl, \tag{0.2}$$

for all finite solvable groups G. The methods of this paper unfortunately cannot give an upper bound whose order of magnitude is less than 2^l. This is caused by our very naive approach. Essentially we choose a normal subgroup P of prime order in H and a suitable chain $A_1, \cdots, A_h$ of H-invariant sections of G. Obviously either P centralizes $A_1, \cdots, A_{[h/2]}$ or there exists a subchain $A_k, A_{k+1}, \cdots, A_{k+[h/2]}$ such that P does not centralize A_k. In the latter case we construct (and this is the hard part of the proof) an H-invariant chain $D_{k+j}, D_{k+j+1}, \cdots, D_{k+[h/2]}$ of sections of $A_{k+j}, A_{k+j+1}, \cdots, A_{k+[h/2]}$ (respectively) such that j is bounded and P centralizes each D_i. In either case we obtain a chain of length "almost" $h/2$ of sections of G on which H/P acts, and which satisfies suitable axioms so that the process can be repeated (using a normal subgroup of prime order in H/P, etc.) Obviously no method based on this process can give an upper bound smaller than 2^l.

There are many technical complications in the proof due to the difficulty of handling the case $|P| = 3$ (among other things). But basically it is a straightforward application of the methods of Hall and Higman [3]. The few new concepts which are used are grouped together in Sections 1, 2 and 3. They are the notions of *Fitting chains* (which are the "correct" chains of sections $A_1, \cdots, A_n$ of G), of *weak equivalence* (which is used in place of equivalence in Fitting chains because it is impossible to verify the latter after

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a complicated construction), of *ample representations* (which are just the ones which are "good" in the Hall-Higman theory) and of the class $\mathfrak{A}$ of groups (which contains all the useful special groups and is closed under formation of non-trivial sections). There is also something called an *augmented Fitting chain* which was just introduced to handle the case $|P| = 3$.

The titles of the sections indicate pretty well the outline of the argument. §1 is the obligatory list of notations. In §2 we introduce Fitting chains and state the basic theorems we shall prove about them. In §3 we prove some elementary facts about ample representations. In §4 we study closely a certain situation of two steps in a chain in which a non-ample representation appears. From this we conclude (in Theorem 4.20) that ample representations always appear in our chains after a bounded number of steps. Then we show in §5 that, knowing we have ample representations in one step of our chain, we can find "enough" ample representations at the next step. The arguments here break down if $|P| = 3$. But in that case we have an augmented Fitting chain. In §6 we use the additional structure to find "enough" ample representations when $|P| = 3$. In §7, we put the results of the preceding sections together, add a few new ones, and prove the basic theorems of §2. Finally in §8 we prove (0.1) from the established results of §2.

1. Notation

Let G be any finite group. We denote by

$Z(G)$ *the center of* G,

G' *the derived group of* G,

$\Phi(G)$ *the Frattini subgroup of* G (*i.e., the intersection of all maximal subgroups of* G),

$F(G)$ *the Fitting subgroup of* G (*i.e., the largest normal nilpotent subgroup of* G),

$\mathrm{Aut}(G)$ *the automorphism group of* G.

The *Fitting series* $F_n(G)$, $n = 0, 1, 2, \cdots$, is defined inductively by

$$F_0(G) = \{1\}$$

$$F_n(G) \textit{ is the inverse image in } G \textit{ of } F(G/F_{n-1}(G)), \qquad \textit{for} \qquad n \geq 1.$$

Evidently each $F_n(G)$ is a characteristic subgroup of G. If G is solvable, then there is some integer $h \geq 0$ such that $F_h(G) = G$. We call the least such integer h the *Fitting height* of G and denote it by $h(G)$.

If each S_i, $i = 1, \cdots, k$, is an element or a subset of G, then $\langle S_1, \cdots, S_k\rangle$ will denote the subgroup of G generated by $S_1, \cdots, S_k$.

If $\sigma, \tau \in G$, then we define

$$\sigma^\tau = \tau^{-1}\sigma\tau$$

$$[\sigma, \tau] = \sigma^{-1}\tau^{-1}\sigma\tau = \sigma^{-1}\sigma^\tau.$$

For any $\tau_1, \cdots, \tau_n \in G$ and any integers $a_1, \cdots, a_n$, we define

$$\sigma^{a_1\tau_1+\cdots+a_n\tau_n} = (\sigma^{a_1})^{\tau_1}(\sigma^{a_2})^{\tau_2} \cdots (\sigma^{a_n})^{\tau_n}, \quad \textit{for all} \quad \sigma \in G.$$

Thus $[\sigma, \tau] = \sigma^{-1+\tau}$, for all $\sigma, \tau \in G$. If $\rho_1, \cdots, \rho_m$ are also elements of G and $b_1, \cdots, b_m$ are integers, then we define

$$\sigma^{(a_1\tau_1+\cdots+a_n\tau_n)(b_1\rho_1+\cdots+b_m\rho_m)} = [\sigma^{a_1\tau_1+\cdots+a_n\tau_n}]^{b_1\rho_1+\cdots+b_m\rho_m} \quad \textit{for all } \sigma \in G.$$

Thus

$$[\sigma, \tau, \rho] = [[\sigma, \tau], \rho] = \sigma^{(-1+\tau)(-1+\rho)}, \quad \textit{for all } \sigma, \tau, \rho \in G.$$

Obviously this definition can be repeated to define $\sigma^{f_1\cdots f_t}$, where each f_i has the form $a_1\tau_1 + \cdots + a_n\tau_n$, for some integers $a_1, \cdots, a_n$ and some elements $\tau_1, \cdots, \tau_n$ of G.

If A, B are two subgroups of G, then $[A, B]$ will denote the subgroup generated by all $[\sigma, \tau]$, where $\sigma \in A$, $\tau \in B$. We define $[A, B]^n$, for all integers $n \geq 0$, by

$$[A, B]^0 = A, \qquad [A, B]^n = [[A, B]^{n-1}, B], \quad \textit{for} \quad n > 0.$$

Thus $[A, B]^2 = [A, B, B] = [[A, B], B]$.

By $A \leq G$ we mean "A is a subgroup of G" as opposed to $A \subseteq G$, which means "A is a subset of G". By $A \trianglelefteq G$ we mean "A is a normal subgroup of G".

A *section* of G is a factor group A/B where $B \trianglelefteq A \leq G$. The section A/B *equals* another section C/D if and only if $A = C$ and $B = D$. A subgroup E of G *covers* the section A/B if $(E \cap A)B = A$ and *avoids* A/B if $E \cap A = E \cap B$.

If G is solvable, then $l(G)$ is defined to be the length of a composition series of G. If we write the order $|G|$ as a product of (not necessarily distinct) primes: $|G| = p_1 \cdots p_l$, then $l(G) = l$.

If G is a non-trivial p-group, for some prime p, we write $p = p(G)$.

Let F be any field. We denote by $F[G]$ the group algebra of G over F. By an "$F[G]$-*module*", we understand a right $F[G]$-module on which the identity of $F[G]$ acts as the identity transformation and which is finite-dimensional as a vector space over F.

If V is an $F[G]$-module and H is any subgroup of G, then V_H will denote the restriction of V to an $F[H]$-module. If U is any $F[H]$-module, then U^G will denote the $F[G]$-module induced from U.

An $F[G]$-module V is *trivial* if G centralizes it. It is *completely reducible* if it is a direct sum of irreducible $F[G]$-submodules. If $|G|$ is relatively prime to the characteristic of F, then every $F[G]$-module is completely reducible.

For any $F[G]$-module V, there exists some $F[G]$-composition series $\{0\} = V_0 < V_1 < \cdots < V_n = V$. We call the composition factors V_i/V_{i-1}, $i = 1, \cdots, n$, the *irreducible* $F[G]$-*components* of V. Of course, these irreducible components are unique up to order and $F[G]$-isomorphism.

An $F[G]$-module V is called *primary* if all of its irreducible $F[G]$-components are isomorphic to each other.

If V is an $F[G]$-module and H is a subgroup of G, then

$[V, H]$ *is the F-subspace spanned by all* $v(\sigma - 1)$, $v \in V$, $\sigma \in H$.

$[V, H]^0 = V$.

$[V, H]^n = [[V, H]^{n-1}, H]$, *for all* $n > 0$.

$C_V(H)$ *is the F-subspace of all* $v \in V$ *such that* $v\sigma = v$ *for all* $\sigma \in H$.

Evidently $[V, H]$, $[V, H]^n$ and $C_V(H)$ are all $F[H]$-submodules of V_H.

For any integer $n \geq 1$ and any $F[G]$-module V, we define the $F[G]$-module $n \times V$ by:

$$n \times V = \overbrace{V \oplus \cdots \oplus V}^{n}.$$

If V is any $F[G]$-module, then the dual F-vector space $\mathrm{Hom}_F(V, F)$ is made into the *dual $F[G]$-module* by

$$(f\sigma)(v) = f(v\sigma^{-1}), \quad \text{for all } f \in \mathrm{Hom}_F(V, F), \quad \sigma \in G, \quad v \in V.$$

We say that two $F[G]$-modules V, U are *weakly $F[G]$-equivalent* if each non-trivial irreducible $F[G]$-component of V is $F[G]$-isomorphic to an irreducible $F[G]$-component of U and vice versa, i.e., V, U have the same non-trivial irreducible components with possibly different multiplicities. Obviously weak $F[G]$-equivalence is an equivalence relation among $F[G]$-modules. Furthermore, it satisfies:

(1.1) *If U, V are weakly $F[G]$-equivalent $F[G]$-modules and $H \leq G$, then U_H, V_H are weakly $F[H]$-equivalent.*

Indeed, any non-trivial irreducible $F[H]$-component of U_H must be $F[H]$-isomorphic to an $F[H]$-component of some non-trivial $F[G]$-component of U, and hence to an $F[H]$-component of V. Statement (1.1) follows immediately from this.

Another remark about weak equivalence has to do with field extensions:

(1.2) *Let E be a finite algebraic extension field of F and U, V be weakly $E[G]$-equivalent $E[G]$-modules. Then U, V, considered as $F[G]$-modules, are weakly $F[G]$-equivalent.*

Indeed, any non-trivial $F[G]$-component of U must be $F[G]$-isomorphic to an $F[G]$-component of some non-trivial $E[G]$-component of U, and hence to an $F[G]$-component of some $E[G]$-component of V. The statement follows directly from this.

An *action* of a group K on a group G will be a homomorphism of K into Aut (G). Since we seldom need consider two different actions of K on G, we usually write "$(K$ on $G)$" to denote that action of K on G which is being considered at a given point in the argument. If $\sigma \in K$, then we write τ^σ for the image of $\tau \in G$ under the automorphism of G which is the image of σ in

Aut (G). We may always form the semidirect product KG in which τ^{σ} becomes $\sigma^{-1}\tau\sigma$, for all $\sigma \in K$, $\tau \in G$. This enables us to define $[G, K]$ and $[G, K]^n$ as usual. We may also define the centralizers $C_G(K)$ of K in G and $C_K(G)$ of G in K. For the latter we usually use the alternative notation Ker (K on G) $= C_K(G)$, since it is the kernel of the representation of K on G given by (K on G).

We denote the image of K in Aut (G) by K_G. Often we also consider K_G to be the section $K/$Ker (K on G) of K. This identification seldom causes confusion.

If G is an abelian group, we denote by G^+ the group G written additively. When G is an elementary abelian p-group (i.e., when G is abelian with prime exponent p), we make G^+ into a vector space over the field Z_p of p elements in the natural way. If another group K acts on G, then G^+ becomes a $Z_p[K]$-module.

Suppose a group K acts on a finite solvable group G. Then each K-composition factor A/B of G is an elementary abelian p-group, for some prime p. So $[A/B]^+$ is an irreducible $Z_p[K]$-module, which we call *an irreducible component of* (K on G). If K also acts on another finite solvable group H, then (K on G) and (K on H) are *weakly equivalent* if each nontrivial irreducible component of (K on G) is K-isomorphic to an irreducible component of (K on H) and vice versa. Obviously this is an equivalence relation among K-groups. As in (1.1) we have

(1.3) *If* (K on G) *is weakly equivalent to* (K on H) *and* $L \leq K$, *then* (L on G) *is weakly equivalent to* (L on H),

where, of course, the actions of L are restricted from those of K.

Suppose that a group K acts on a group G. A section A/B of G is *K-invariant* if both A and B are K-invariant subgroups of G. We also say that "K normalizes A/B". In this case K acts naturally on the factor group A/B. To say that a section C/D of K *normalizes* A/B means that C normalizes A/B and $D \leq$ Ker (C on A/B). Then C/D acts naturally on A/B.

Let a group K act on a group G and another group L act on both K and G. We say that (K on G) is *L-invariant* if $(\sigma^{\tau}) = (\sigma^{\rho})^{\tau^{\rho}}$, for all $\sigma \in G$, $\tau \in K$, $\rho \in L$. In that case we may form the "triple semi-direct product" LKG.

If K acts on G and L acts on K, then (K on G) is *weakly L-invariant* if the actions (K on G) and $(K \text{ on } G)^{\sigma}$, the latter given by

$$\tau \rightarrow (K \text{ on } G)(\tau^{\sigma^{-1}}) \quad \textit{for} \quad \tau \in K,$$

are weakly equivalent for all $\sigma \in L$. We define weak L-invariance similarly for $F[K]$-modules V over any field F, using weak $F[K]$-equivalence.

We define $\mathfrak{A}$ to be the family of all finite groups A satisfying:

(1.4a) *A is a non-trivial p-group, for some prime p.*
(1.4b) $\Phi(A) \leq Z(A)$.

(1.4c) $\Phi(\Phi(A)) = \{1\}$.
(1.4d) *If p is odd, then A has exponent p.*

Evidently all special groups A (in the sense of [3]) lie in $\mathfrak{A}$ provided they are non-trivial and satisfy (1.4d). However $\mathfrak{A}$ obviously has the following important property which special groups lack:

(1.5) *Any non-trivial section B/C of a group $A \in \mathfrak{A}$ also lies in $\mathfrak{A}$.*

If $A \in \mathfrak{A}$, we define $\bar{A}$ to be the $Z_{p(A)}$-vector space $[A/\Phi(A)]^+$. It follows easily from (1.4b) that the map f_A defined by

$$f_A(\sigma\Phi(A), \tau\Phi(A)) = [\sigma, \tau], \quad \textit{for all } \sigma, \tau \in A, \tag{1.6}$$

is an alternating, bilinear map of $\bar{A} \times \bar{A}$ into $\Phi(A)^+$ (note that $\Phi(A)^+$ is also a $Z_{p(A)}$-vector space by (1.4c)). It is clear from (1.6) that the radical of f_A (i.e., the set of all $\bar{\sigma} \in \bar{A}$ such that $f_A(\bar{\sigma}, \bar{A}) = \{0\}$) is precisely $[Z(A)/\Phi(A)]^+$.

2. Fitting chains

The simplest way of thinking about the Fitting height of a finite solvable group G is to consider chains $A_1, \cdots, A_t$ of sections of G satisfying the following conditions:

(2.1a) *Each A_i, $i = 1, \cdots, t$, is a non-trivial p_i-group, for some prime p_i.*
(2.1b) *A_i normalizes A_{i+1}, for $i = 1, \cdots, t-1$.*
(2.1c) Ker $(A_i \text{ on } A_{i+1}) = \{1\}$, *for $i = 1, \cdots, t-1$.*
(2.1d) *$p_i \neq p_{i+1}$ for $i = 1, \cdots, t-1$.*

It is easy to verify that the Fitting height $h(G)$ is merely the maximum of the lengths t of all such chains of sections of G (see Lemma 8.2 below for part of the argument).

The basic idea behind our proof of Thompson's conjecture is that one should forget about the group G and consider only chains $A_1, \cdots, A_t$ of groups, each acting on the next, which satisfy axioms similar to (2.1). From this point of view the Carter subgroup H of G becomes a group outside the chain acting on each A_i and leaving invariant each action (A_i on A_{i+1}). Under certain conditions, which Carter subgroups and appropriate chains of sections of G can be shown to satisfy, we prove that the length t of such a chain must be bounded as a function of $l(H)$.

To make this program more explicit, we first consider the axioms which our chains $A_1, \cdots, A_t$ must satisfy. Obviously we want the groups A_i to have as uncomplicated a structure as possible. The Hall-Higman theory suggests that we take them to be special. However, the class of special groups is not closed under subgroups and epimorphic images, which makes it awkward to use in complicated constructions. So we choose the A_i instead from the class $\mathfrak{A}$, which does have the desired closure properties by (1.5) and contains enough special groups for our purposes.

A little experimentation soon demonstrates that we cannot allow the actions (A_{i-1} on A_i) and (A_i on A_{i+1}) to be completely independent of each other. It is tempting to make the representation (A_i on $\bar{A}_{i+1}$) invariant under A_{i-1}. However, in practice this condition is much too difficult to verify after a construction. So we only insist that (A_i on $\bar{A}_{i+1}$) be weakly A_{i-1}-invariant, which turns out to be sufficient in general to establish what we need.

Another axiom suggested by the Hall-Higman theory is that A_i centralize $\Phi(A_{i+1})$. This condition turns out to be vital in many of our proofs.

Combining the above ideas, we define a *Fitting chain* to consist of groups $A_1, \cdots, A_t$ and actions (A_i on A_{i+1}), for $i = 1, \cdots, t - 1$, satisfying:

(2.2a) *$A_i \in \mathfrak{A}$, for $i = 1, \cdots, t$.*
(2.2b) *$p(A_i) \neq p(A_{i+1})$, for $i = 1, \cdots, t - 1$.*
(2.2c) *$[\Phi(A_{i+1}), A_i] = \{1\}$, for $i = 1, \cdots, t - 1$.*
(2.2d) *Ker (A_i on A_{i+1}) $= \{1\}$, for $i = 1, \cdots, t - 1$.*
(2.2e) *(A_{i+1} on $\bar{A}_{i+2}$) is weakly A_i-invariant, for $i = 1, \cdots, t - 2$.*

Usually we speak of "the Fitting chain $A_1, \cdots, A_t$" leaving the actions (A_i on A_{i+1}) to be understood.

Suppose that $A_1, \cdots, A_t$ is a Fitting chain and that D_i is a section of A_i, for $i = 1, \cdots, t$. If the action of A_i on A_{i+1} induces an action of D_i on D_{i+1}, for $i = 1, \cdots, t - 1$, and if these actions make $D_1, \cdots, D_t$ a Fitting chain, we say that $D_1, \cdots, D_t$ is a *Fitting subchain* of $A_1, \cdots, A_t$. Notice that some of the axioms (2.2) for $D_1, \cdots, D_t$ are free by

Proposition 2.3. *Let $A_1, \cdots, A_t$ be a Fitting chain. Then sections D_i of A_i, for $i = 1, \cdots, t$, will form a Fitting subchain if and only if they satisfy*:
(2.4a) *$D_1 \neq \{1\}$.*
(2.4b) *D_i normalizes D_{i+1}, for $i = 1, \cdots, t - 1$.*
(2.4c) *Ker (D_i on D_{i+1}) $= \{1\}$ for $i = 1, \cdots, t - 1$.*
(2.4d) *(D_{i+1} on $\bar{D}_{i+2}$) is weakly D_i-invariant, for $i = 1, \cdots, t - 2$.*

Proof. If $D_1, \cdots, D_t$ is a Fitting subchain it certainly satisfies (2.4) by (2.2) and (1.4a).

Conversely, suppose that $D_1, \cdots, D_t$ satisfies (2.4). Then (2.2a) and (2.4a) imply $D_1 \in \mathfrak{A}$, by (1.5). Suppose we know that $D_i \in \mathfrak{A}$, for some $i = 1, \cdots, t - 1$. Then $D_i \neq \{1\}$, by (1.4a). So (2.4b, c) imply that $D_{i+1} \neq \{1\}$. Since D_{i+1} is a section of A_{i+1}, this, (1.5), and (2.2a) give $D_{i+1} \in \mathfrak{A}$. By induction, (2.2a) holds for $D_1, \cdots, D_t$.

Clearly $p(D_i) = p(A_i)$ and $p(D_{i+1}) = p(A_{i+1})$. Therefore (2.2b) for $A_1, \cdots, A_t$ implies (2.2b) for $D_1, \cdots, D_t$.

Because A_i centralizes $\Phi(A_{i+1})$ (by (2.2c)), so does D_i. If $D_{i+1} = E/F$, then it follows that D_i centralizes $\Phi(E) \leq E \cap \Phi(A_{i+1})$. Since $\Phi(D_{i+1})$ is the image in D_{i+1} of $\Phi(E)$, it is centralized by D_i. So (2.2c) holds for $D_1, \cdots, D_t$.

Finally (2.2d, e) for $D_1, \cdots, D_t$ are just (2.4c, d). Therefore (2.2) holds for $D_1, \cdots, D_t$, which proves the proposition.

We say that a group H *acts* on a Fitting chain $A_1, \cdots, A_t$ if H acts on each group A_i, $i = 1, \cdots, t$, leaving invariant each action (A_i on A_{i+1}), for $i = 1, \cdots, t-1$. A Fitting subchain $D_1, \cdots, D_t$ is then *H-invariant* if each D_i, $i = 1, \cdots, t$, is an H-invariant section of A_i. In that case H clearly acts on the Fitting chain $D_1, \cdots, D_t$ in the natural manner.

The first two of our three basic theorems concern the situation in which

(2.5a) *A group H acts on a Fitting chain $A_1, \cdots, A_t$,*
(2.5b) *H has a normal subgroup P of prime order p,*
(2.5c) $[A_1, P] \neq \{1\}$.

The theorems, whose proofs will follow later (see §7), are:

THEOREM 2.6. *If $t \geq 3$ and p does not divide $\prod_{i=1}^{t} |A_i|$, then there exists an H-invariant Fitting subchain $D_3, D_4, \cdots, D_t$ of $A_3, \cdots, A_t$ such that P centralizes each D_i, $i = 3, \cdots, t$.*

THEOREM 2.7. *If $t \geq 4$ and $p \geq 5$, then there exists an H-invariant Fitting subchain $D_4, D_5, \cdots, D_t$ of $A_4, \cdots, A_t$ such that P centralizes each D_i, $i = 4, \cdots, t$.*

Assuming these two theorems, we now prove

THEOREM 2.8. *Let H be a finite group acting on a Fitting chain $A_1, \cdots, A_t$ such that no non-trivial section of any A_i, $i = 1, \cdots, t$, is centralized by H. Assume further that H is a supersolvable group whose order is not divisible by 6.*

Proof. We use induction on $l(H)$. If $l(H) = 0$, then $H = \{1\}$. Since no non-trivial section of any A_i, $i = 1, \cdots, t$, is centralized by H, each A_i must be $\{1\}$. By (2.2a) and (1.4a), this implies that $t = 0 = 3(2^0 - 1)$. So the theorem is true in this case.

Now suppose that $l = l(H) > 0$ and that the theorem is true for all smaller values of $l(H)$. Since H is supersolvable it has a normal subgroup P whose order is the largest prime p dividing $|H|$ (see Theorem VI, 9.1 of [4]).

Suppose that P centralizes $A_1, \cdots, A_s$, for some integer $s = 1, \cdots, t$. Then H/P acts on the Fitting chain $A_1, \cdots, A_s$. Obviously H/P, and $A_1, \cdots, A_s$, satisfy all the hypotheses of the theorem with $l(H/P) = l - 1$. So induction tells us that $s \leq 3(2^{l-1} - 1)$.

If $t \leq 3(2^{l-1} - 1) + 3$, then

$$t \leq 3(2^{l-1} - 1) + 3 + 3(2^{l-1} - 1) = 3(2^l - 1)$$

and the theorem is true. So assume that $t > 3(2^{l-1} - 1) + 3$. The argument of the preceding paragraph gives us an integer

$$s = 1, 2, \cdots, 3(2^{l-1} - 1) + 1$$

such that P does not centralize A_s. Furthermore, the length $t - s + 1$ of the Fitting chain $A_s, A_{s+1}, \cdots, A_t$ is at least

$$[3(2^{l-1} - 1) + 4] - [3(2^{l-1} - 1) + 1] + 1 = 4.$$

If $p \geq 5$, then Theorem 2.7 applied to H, P and $A_s, \cdots, A_t$ gives us an H-invariant Fitting subchain $D_{s+3}, D_{s+4}, \cdots, D_t$ of $A_{s+3}, \cdots, A_t$ which is centralized by P. Evidently H/P, and $D_{s+3}, \cdots, D_t$, satisfy the hypotheses of the theorem with $l(H/P) = l - 1$. So induction tells us that

$$t - (s + 3) + 1 \leq 3(2^{l-1} - 1).$$

Hence

$$\begin{aligned} t &\leq s + 3(2^{l-1} - 1) + 2 \\ &\leq [3(2^{l-1} - 1) + 1] + 3(2^{l-1} - 1) + 2 = 3(2^l - 1), \end{aligned}$$

and the theorem is true in this case.

If $p = 2$ or 3, then H is a p-group. Because H centralizes no non-trivial section of the $p(A_i)$-group A_i, for $i = 1, \cdots, t$, the primes p and $p(A_i)$ must be distinct. Hence Theorem 2.6 applies to H, P, and $A_s, \cdots, A_t$, giving us an H-invariant Fitting subchain $D_{s+2}, D_{s+3}, \cdots, D_t$ of $A_{s+2}, \cdots, A_t$ which is centralized by P. By induction

$$t - (s + 2) + 1 \leq 3(2^{l-1} - 1).$$

So $t \leq s + 3(2^{l-1} - 1) + 1 < 3(2^l - 1)$, which finishes the proof of the theorem.

The second sentence of Theorem 2.8 looks very suspicious. It seems reasonable to make the

CONJECTURE 2.9. *There is a function g from the non-negative integers into themselves such that $t \leq g(l(H))$ whenever a finite group H acts on a Fitting chain $A_1, \cdots, A_t$ and centralizes no non-trivial section of any A_i, $i = 1, \cdots, t$.* One might even hope that g can be chosen so that $g(l) = O(l)$ as $l \to \infty$.

By an example which is too complicated to give here I can show that Theorem 2.7 does not hold for $p = 3$. So we are forced to consider more complicated chains of groups in order to prove Thompson's conjecture by this method when $|H|$ is divisible by 6. The idea is to make the connection between (A_i on A_{i+1}) and (A_{i+1} on A_{i+2}) stronger when $p(A_{i+1}) = 3$ and to leave everything else alone.

We define an *augmented Fitting chain* to be a Fitting chain $A_1, \cdots, A_t$ together with certain additional groups, actions, and epimorphisms. We say that an index $i = 1, \cdots, t$ is *relevant* if $1 \leq i \leq t - 2$ and $p(A_{i+1}) = 3$. For each relevant index i, we have an additional group B_i, an action of B_i on A_{i+2}, and an epimorphism η_i of B_i onto A_i (which defines an action of B_i on A_{i+1} via (A_i on A_{i+1})) satisfying:

(2.10a) *B_i is a $p(A_i)$-group.*

(2.10b) *(A_{i+1} on A_{i+2}) is B_i-invariant.*

(2.10c) *If* $i \leq t - 3$ *then* $(A_{i+2}$ on $\bar{A}_{i+3})$ *is weakly* B_i*-invariant.*

We usually write "the augmented Fitting chain $A_1, \cdots, A_t, \{B_i\}$" leaving the actions and the epimorphisms η_i to be understood.

Suppose that $A_1, \cdots, A_t, \{B_i\}$ is an augmented Fitting chain, that $D_1, \cdots, D_t$ is a Fitting subchain of $A_1, \cdots, A_t$, and that C_i is a section of B_i, for each relevant i. If η_i induces an epimorphism of C_i onto D_i and C_i normalizes D_{i+2} for each relevant i, and if $D_1, \cdots, D_t, \{C_i\}$ with these epimorphisms and actions form an augmented Fitting chain, then we call $D_1, \cdots, D_t, \{C_i\}$ an *augmented Fitting subchain* of $A_1, \cdots, A_t, \{B_i\}$. As in Proposition 2.3, we need not verify all the properties of $D_1, \cdots, D_t, \{C_i\}$.

PROPOSITION 2.11. *Let* $A_1, \cdots, A_t, \{B_i\}$ *be an augmented Fitting chain,* D_j *be a section of* A_j, *for* $j = 1, \cdots, t$, *and* C_i *be a section of* B_i, *for all relevant* i. *Then* $D_1, \cdots, D_t, \{C_i\}$ *form an augmented Fitting subchain if and only if they satisfy:*

(2.12a) $D_1 \neq \{1\}$.

(2.12b) D_j *normalizes* D_{j+1}, *for* $j = 1, \cdots, t - 1$.

(2.12c) η_i *induces an epimorphism of* C_i *onto* D_i *for all relevant* i.

(2.12d) C_i *normalizes* D_{i+2}, *for all relevant* i.

(2.12e) Ker $(D_j$ on $D_{j+1}) = \{1\}$, *for* $j = 1, \cdots, t - 1$.

(2.12f) $(D_{i+1}$ on $\bar{D}_{i+2})$ *is weakly* D_i*-invariant, if* $i = 1, \cdots, t - 2$ *and* $p(A_{i+1}) \neq 3$.

(2.12g) $(D_{i+2}$ on $\bar{D}_{i+3})$ *is weakly* C_i*-invariant for all relevant* $i \leq t - 3$.

Proof. It is clear that (2.12) holds whenever $D_1, \cdots, D_t, \{C_i\}$ is an augmented Fitting subchain.

Suppose that (2.12) holds. We have enough groups, epimorphisms, and actions to form an augmented Fitting subchain $D_1, \cdots, D_t, \{C_i\}$. So we need only check the various axioms.

Let $i = 1, \cdots, t - 2$ with $p(A_{i+1}) = 3$. The B_i-invariance of $(A_{i+1}$ on $A_{i+2})$, together with (2.12c, d), implies that $(D_{i+1}$ on $D_{i+2})$ is C_i-invariant. So (2.10b) holds for our subchain. Since D_{i+1} centralizes $\Phi(D_{i+2})$ by (2.2c), this clearly implies that $(D_{i+1}$ on $\bar{D}_{i+2})$ is weakly D_i-invariant. So (2.12f) is satisfied for all $i = 1, \cdots, t - 2$. This and (2.12a, b, e) are conditions (2.4). Therefore $D_1, \cdots, D_t$ is a Fitting subchain by Proposition 2.3. Obviously $p(C_i) = p(B_i) = p(A_i) = p(D_i)$, for all relevant i. And (2.12g) is (2.10c) for the subchain. Hence $D_1, \cdots, D_t, \{C_i\}$ satisfies (2.10) and the proposition is true.

A group H *acts* on an augmented Fitting chain $A_1, \cdots, A_t, \{B_i\}$ if it acts on each group A_j, $j = 1, \cdots, t$, and on B_i, for each relevant i, so that all the actions and epimorphisms of the chain are H-invariant. An augmented Fitting subchain $D_1, \cdots, D_t, \{C_i\}$ is then H*-invariant* if each D_j, $j = 1, \cdots, t$, and C_i, for each relevant i, is an H-invariant section. In that case H acts on the augmented Fitting chain $D_1, \cdots, D_t, \{C_i\}$ in the natural manner.

The third basic theorem, whose proof will follow later (see §7) is

THEOREM 2.13. *Let H be a group acting on an augmented Fitting chain $A_1, \cdots, A_t, \{B_i\}$. Suppose that P is a normal subgroup of order 3 in H such that $[A_1, P] \neq \{1\}$. If $t \geq 6$, then there is an H-invariant augmented Fitting subchain $D_6, \cdots, D_t, \{C_i\}$ of $A_6, \cdots, A_t, \{B_i\}$ such that P centralizes each D_j and C_i.*

Assuming the three Theorems 2.6, 2.7 and 2.13, we now prove the following result from which we shall later derive a proof of Thomson's conjecture (see §8).

THEOREM 2.14. *Let a finite group H act on an augmented Fitting chain $A_1, \cdots, A_t, \{B_i\}$ so that H centralizes no non-trivial section of any A_j, $j = 1, \cdots, t$. Assume further that H is a supersolvable group with a normal 3-Sylow subgroup M. Then $t \leq 5(2^{l(H)} - 1)$.*

Proof. We use induction on $|M|$. If $|M| = 1$, then H and $A_1, \cdots, A_t$ satisfy the hypotheses of Theorem 2.8. That theorem tells us that $t \leq 3(2^{l(H)} - 1) \leq 5(2^{l(H)} - 1)$. So this theorem is true if $|M| = 1$.

Now we assume that $|M| > 1$ and that this theorem is true for all smaller values of $|M|$. Since H is supersolvable it has a normal subgroup P of prime order p. We may even choose P to be contained in the normal 3-Sylow subgroup M of H. So $p = 3$.

Suppose that P centralizes $A_1, \cdots, A_s$, for some integer $s = 1, \cdots, t$. If $p(A_{i+1}) = 3$, for some $i = 1, \cdots, s-2$, then the P-invariance of (B_i on A_{i+2}), together with the fact that P centralizes A_{i+2}, implies that $[B_i, P]$ centralizes A_{i+2}. Furthermore, the facts that P centralizes $\eta_i(B_i) = A_i$ and leaves η_i invariant imply that $\eta_i([B_i, P]) = \{1\}$. So there is a natural action of $B_i/[B_i, P]$ on A_{i+2} and a natural epimorphism of $B_i/[B_i, P]$ onto A_i. Since P is normal in H, the subgroup $[B_i, P]$ is H-invariant. Hence so are the action ($B_i/[B_i, P]$ on A_{i+2}) and the epimorphism of $B_i/[B_i, P]$ onto A_i. It follows that $A_1, \cdots, A_s, \{B_i/[B_i, P]\}$ is an augmented Fitting chain on which H/P acts. Since H/P, M/P and $A_1, \cdots, A_s, \{B_i/[B_i, P]\}$ satisfy our hypothesis with $|M/P| < |M|$, we know by induction that $s \leq 5(2^{l-1} - 1)$, where $l = l(H) = l(H/P) + 1$.

If $t \leq 5(2^{l-1} - 1) + 5$, the theorem is certainly true. So assume that $t > 5(2^{l-1} - 1) + 5$. The preceding paragraph proves that there exists an $s = 1, \cdots, 5(2^{l-1} - 1) + 1$ such that P does not centralize A_s. The length $t - s + 1$ of the augmented Fitting chain $A_s, A_{s+1}, \cdots, A_t, \{B_i\}$ is at least

$$[5(2^{l-1} - 1) + 6] - [5(2^{l-1} - 1) + 1] + 1 = 6.$$

So Theorem 2.13 gives us an augmented Fitting subchain $D_{s+5}, D_{s+6}, \cdots, D_t$, $\{C_i\}$ of $A_{s+5}, \cdots, A_t, \{B_i\}$ such that P centralizes each D_j and C_i. Applying the present theorem by induction to H/P, M/P and $D_{s+5}, \cdots, D_t$, $\{C_i\}$, we see that

$$t - (s + 5) + 1 \leq 5(2^{l-1} - 1).$$

Hence

$$t \leq (s+5) - 1 + 5(2^{l-1} - 1) \leq 5(2^{l-1} - 1) + 6 - 1 + 5(2^{l-1} - 1)$$
$$= 5(2^l - 1).$$

This completes the proof of the theorem.

3. Ample representations

We begin with some elementary observations about the situation in which

(3.1a) *PA is the semi-direct product of a group P of prime order p acting on a group $A \in \mathfrak{A}$,*

(3.1b) *F is a field of prime characteristic $q \neq p(A)$ which is a splitting field for all subgroups of PA,*

(3.1c) *V is an irreducible $F[PA]$-module.*

The first observation is

PROPOSITION 3.2. *If* (3.1) *holds and $[Z(A_V), P] \neq \{1\}$, then V is induced from an irreducible $F[A]$-module U.*

Proof. Apply Clifford's theorem (see Theorem V, 17.3 of [4]) to V and the inverse image B of $Z(A_V)$ in A. Since F is a splitting field for B and $B_V = Z(A_V)$ is abelian, there is a linear F-character λ of B and an irreducible $F[C_{PA}(\lambda)]$-module U such that V is induced from U and any $\sigma \in B$ acts on U as scalar multiplication by $\lambda(\sigma) \in F$. Because A centralizes B_V, it fixes λ. By (3.1a), A is maximal in PA. Therefore $C_{PA}(\lambda)$ is either A or PA. The latter possibility implies that $U = V$ and that P centralizes $B/\text{Ker}\,\lambda = B_V = Z(A_V)$, contradicting our hypotheses. Hence $C_{PA}(\lambda) = A$ and the proposition is true.

COROLLARY 3.3. *Let $C = C_A(P)$. Then both $C_V(P)$ and $[V, P]^{p-1}$ are non-zero $F[C]$-submodules of V. Furthermore, both of them are weakly $F[C]$-equivalent to V.*

Proof. From $V = U^{PA}$ we conclude that $V_{P\times C}$ is $F[P \times C]$-isomorphic to the outer Kronecker product $F[P] \otimes U_C$ of the regular $F[P]$-module with U_C. We always have $C_{F[P]}(P) \neq \{0\}$ and $[F[P], P]^{p-1} \neq \{0\}$. It follows that both

$$C_V(P) \simeq C_{F[P]}(P) \otimes U_C \quad \text{and} \quad [V, P]^{p-1} \simeq [F[P], P]^{p-1} \otimes U_C$$

are non-zero, and that all of V_C, $C_V(P)$, $[V, P]^{p-1}$ are multiples of U_C as $F[C]$-modules. So they are weakly $F[C]$-equivalent to each other, which proves the corollary.

Another observation which we shall use repeatedly is

PROPOSITION 3.4. *Suppose that* (3.1) *holds, that $p \neq p(A)$, and that P centralizes $Z(A_V)$ but not A_V. Then $[A_V, P]$ is an extra-special group and $\Phi([A_V, P]) = \Phi(A_V)$ is centralized by P. The group PA_V is the central product* *of its subgroups $P[A_V, P]$ and $C_{A_V}(P)$, which intersect in $\Phi(A_V)$. Regarding PA_V as the natural image of $P[A_V, P] \times C_{A_V}(P)$, the module V is the outer Kronecker product $W \otimes_F U$ of an irreducible $F[P[A_V, P]]$-module W and an irreducible $F[C_{A_V}(P)]$-module U. Furthermore ($[A_V, P]$ on W) is faithful.*

Proof. Our hypotheses insure that A_V is not abelian. So (1.5) tells us that $A_V \in \mathfrak{A}$. Since P centralizes $Z(A_V)$, it centralizes $\Phi(A_V)$ (by 1.4b)), which is non-trivial since it contains A_V'. Therefore $\Phi(A_V) \leq Z(PA_V)$. Because V is an irreducible $F[PA_V]$-module on which $\Phi(A_V)$ is faithfully represented, this implies that $\Phi(A_V)$ is cyclic. Hence $\Phi(A_V)$ has order $p(A) = p(A_V)$ by (1.4c).

From $p \neq p(A_V)$ and $[A_V, P] \neq \{1\}$ we conclude that

$$\bar{A}_V = [\bar{A}_V, P] \oplus C_{\bar{A}_V}(P) \quad \text{and} \quad [\bar{A}_V, P] \neq \{0\}.$$

Since P centralizes $\Phi(A_V)$ and leaves the form f_{A_V} of (1.6) invariant, the two subspaces $[\bar{A}_V, P]$ and $C_{\bar{A}_V}(P)$ of $\bar{A}_V$ must be f_{A_V}-perpendicular. Therefore A_V is the central product of the inverse images L of $[\bar{A}_V, P]$ and K of $C_{\bar{A}_V}(P)$ with $L \cap K = \Phi(A_V)$.

The radical $[Z(A_V)/\Phi(A_V)]^+$ of f_{A_V} is contained in $C_{\bar{A}_V}(P)$ by hypothesis. So the restriction of f_{A_V} to $[\bar{A}_V, P] \times [\bar{A}_V, P]$ is non-singular. Since $\Phi(A_V)$ is cyclic of order $p(A)$ and $[\bar{A}_V, P] \neq \{0\}$, we conclude that L is extra-special with $\Phi(L) = \Phi(A_V)$. Obviously L contains $[A_V, P]$. But $[A_V, P]$ covers $\bar{L} \simeq [\bar{A}_V, P]$. Therefore $L = [A_V, P]$ and the first statement of the proposition is true.

Since $p \neq p(A)$ and P centralizes both $\Phi(A_V)$ and $K/\Phi(A_V) = C_{\bar{A}_V}(P)$, the group P centralizes K. It follows that $K = C_{A_V}(P)$. So A_V is the central product of $[A_V, P]$ and $C_{A_V}(P)$. Because P normalizes $[A_V, P]$ and centralizes $C_{A_V}(P)$, the second statement of the proposition follows directly from this.

The third statement of the proposition comes immediately from the second, since F is a splitting field for all the groups involved. Finally, any $\sigma \in [A_V, P] - \{1\}$ acts non-trivially on W since $\sigma \times 1 \in P[A_U, P] \times C_{A_V}(P)$ has the image $\sigma \in PA_V$ which acts non-trivially on V. So the entire proposition is true.

The following fact is well known (see [3] or Theorem (IV.9) of [2]):

(3.5) *Under the hypotheses of Proposition* 3.4, *there is a regular $F[P]$-submodule of W_P unless p is a Fermat prime, $p(A) = 2$, and $[\bar{A}_V, P] \simeq \overline{[A_V, P]}$ is an irreducible $Z_2[P]$-module.*

We use this to prove the following corollary to Proposition 3.4.

Corollary 3.6. *Let $C = C_A(P)$. Unless p is a Fermat prime, $p(A) = 2$, and $[\bar{A}_V, P]$ is an irreducible $Z_2[P]$-module, the subspaces $C_V(P)$ and $[V, P]^{p-1}$ are both non-zero $F[C]$-submodules of V and are both weakly $F[C]$-equivalent to V.*

Proof. Assume we are not in the exceptional case. Then (3.5) gives us an $F[P]$-submodule of W_P isomorphic to $F[P]$. Clearly $C_V \leq C_{A_V}(P)$. So the proposition tells us that $V_{P\times C}$ is $F[P \times C]$-isomorphic to the outer Kronecker product $W_P \otimes U_C$, which contains a submodule isomorphic to $F[P] \otimes U_C$. Since neither $C_{F[P]}(P)$ nor $[F[P], P]^{p-1}$ is $\{0\}$, we conclude that $C_V(P) \simeq C_W(P) \otimes U_C$ and $[V, P]^{p-1} \simeq [W, P]^{p-1} \otimes U_C$ are both non-zero. Furthermore all the modules V_C, $C_V(P)_C$, and $([V, P]^{p-1})_C$ are isomorphic to positive multiples of U_C, and hence are weakly $F[C]$-equivalent to each other. So the corollary is true.

When $p = p(A)$, we use a different approach to get a result similar to Corollary 3.6.

PROPOSITION 3.7. *Suppose that* (3.1) *holds with* $p = p(A) \geq 3$ *and* $[A_V, P]^{p-1} \neq \{1\}$. *Let* $C = C_{[A,P]^{p-1}}(P)$. *Then* $C_V(P)$ *is a non-zero* $F[C]$-*submodule of* V *and is weakly* $F[C]$-*equivalent to* V.

Proof. If $[Z(A_V), P] \neq \{1\}$, the result follows immediately from (1.1) and Corollary 3.3, since $C \leq C_A(P)$. So we may assume that $[Z(A_V), P] = \{1\}$.

Under this assumption we first prove that

$$[A_V, P]^{p-1} \leq Z([A_V, P]^{p-2}). \tag{3.8}$$

For this it suffices by (1.6) to show that

$$f_{A_V}([\bar{A}_V, P]^{p-1}, [\bar{A}_V, P]^{p-2}) = \{0\}.$$

But $[\bar{A}_V, P]^n = \bar{A}_V(\pi - 1)^n$, for all $n \geq 0$, where π is any generator of P. If $\alpha, \beta \in \bar{A}_V$, we use $p \geq 3$ and the fact that P centralizes $f_{A_V}(\bar{A}_V, \bar{A}_V) \leq Z(A_V)^+$ to compute

$$\begin{aligned} f_{A_V}(\alpha(\pi - 1)^{p-1}, \beta(\pi - 1)^{p-2}) &= f_{A_V}(\alpha(\pi - 1)^{p-1}(\pi^{-1} - 1), \beta(\pi - 1)^{p-3}) \\ &= f_{A_V}(-\alpha(\pi - 1)^p\pi^{-1}, \beta(\pi - 1)^{p-3}). \end{aligned}$$

But $\bar{A}_V$ is a vector space over a field Z_p of characteristic $p = p(A)$. So $\alpha(\pi - 1)^p = \alpha(\pi^p - 1) = 0$. Therefore (3.8) holds.

Let U be any non-trivial irreducible $F[C]$-submodule of V. Since $C \leq [A, P]^{p-1}$, there exists some irreducible $F[[A, P]^{p-1}]$-submodule W of V containing an $F[C]$-submodule isomorphic to U. So we may assume that $U \leq W$. Since C is non-trivial on U, it is non-trivial on W. Therefore

$$\mathrm{Ker}\,([A, P]^{p-1} \text{ on } W) < [A, P]^{p-1}.$$

So there must exist some element $\sigma \in [A_V, P]^{p-2}$ and some $\pi \in P$ such that

$$[\sigma, \pi] \in [A_V, P]^{p-1} - \mathrm{Ker}\,([A_V, P]^{p-1} \text{ on } W).$$

It follows from (3.8) that $\langle \sigma, [A_V, P]^{p-1} \rangle$ is a P-invariant abelian subgroup of A_V. Let B be its inverse image in A. Then $[A, P]^{p-1} \leq B$. So there is some irreducible $F[PB]$-submodule Y of V containing an $F[[A, P]^{p-1}]$-submodule isomorphic to W. As before, we may assume that $W \leq Y$. Clearly B_Y is a homomorphic image of $B_V = \langle \sigma, [A_V, P]^{p-1} \rangle$ and hence is abelian. If P centralized B_Y, then $[\sigma, \pi]$ would lie in Ker (B_V on Y), contradicting the fact that $[\sigma, \pi]$ acts nontrivially on $W \leq Y$. Hence $[B_Y, P] \neq \{1\}$. So Corollary 3.3 applies to PB and Y. It tells us that Y and $C_Y(P)$ are weakly $F[C_B(P)]$-equivalent. Since $C \leq C_B(P)$, the modules Y and $C_Y(P)$ are weakly $F[C]$-equivalent (by (1.1)). Hence there is an irreducible $F[C]$-submodule of $C_Y(P) \leq C_V(P)$ which is isomorphic to the non-trivial $F[C]$-submodule U of Y.

We have shown that any non-trivial irreducible $F[C]$-submodule U of V is $F[C]$-isomorphic to a submodule of $C_V(P)$. The converse being obvious, this proves that $C_V(P)$ and V are weakly $F[C]$-equivalent.

By hypothesis there exists a non-trivial irreducible $F[[A, P]^{p-1}]$-submodule W of V. Constructing B and Y as above we see from Corollary 3.3 that $C_Y(P) \neq \{0\}$. So $C_V(P) \neq \{0\}$ and the proposition is proved.

In practice we must consider modules over fields which need not satisfy (3.1b). A simple ground field extension quickly reduces this more general case to the one we have been considering.

Suppose that (3.1a) holds, that E is any field of prime characteristic $q \neq p(A)$, and that V is any irreducible $E[PA]$-module. We call V *ample* if one of the following conditions holds:

(3.9a) *$p \neq p(A)$ and $[Z(A_V), P] \neq \{1\}$.*

(3.9b) *$p \neq p(A)$, $[Z(A_V), P] = \{1\}$, $[A_V, P] \neq \{1\}$ and we are not in the exceptional case in which $p(A) = 2$, p is a Fermat prime, and $[\bar{A}_V, P]$ is an irreducible $Z_2[P]$-module.*

(3.9c) *$p = p(A) \geq 3$ and $[A_V, P]^{p-1} \neq \{1\}$.*

These are, of course, just the hypotheses of Corollaries 3.3 and 3.6 and Proposition 3.7 made into axioms. From these results we easily prove

PROPOSITION 3.10. *Let V be an ample irreducible $E[PA]$-module. Let C be $C_A(P)$, if $p \neq p(A)$, or $C_{[A,P]^{p-1}}(P)$, if $p = p(A)$. Then $C_V(P)$ is a non-zero $E[C]$-submodule of V and is weakly $E[C]$-equivalent to V. If $p \neq p(A)$, then $[V, P]^{p-1}$ is also a non-zero $E[C]$-submodule of V weakly $E[C]$-equivalent to V.*

Proof. Since PA has only a finite number of subgroups, we may choose a finite algebraic extension field F of E so that it is a splitting field for all subgroups of PA. Let U be an irreducible $F[PA]$-submodule of the extension $F \otimes_E V$ of V to an $F[PA]$-module. Clearly $F \otimes_E V$, considered as a module over $E[PA]$, is isomorphic to $[F:E] \times V$. Since V is an irreducible $E[PA]$-module, we conclude that the restriction U_E of U to an $E[PA]$-module satisfies

$$U_E \simeq n \times V \quad (\textit{as } E[PA]\textit{-modules}), \tag{3.11}$$

for some positive integer n.

If $p \neq p(A)$, let $Z = C_V(P)$ or $[V, P]^{p-1}$. If $p = p(A)$, let $Z = C_V(P)$. We must prove that Z is a non-zero $E[C]$-submodule of V weakly $E[C]$-equivalent to V. Let $Y = C_U(P)$, if $Z = C_V(P)$, and $Y = [U, P]^{p-1}$, if $Z = [V, P]^{p-1}$. Evidently the isomorphism (3.11) always carries Y onto $n \times Z$. Therefore it suffices to prove that Y is a non-zero $F[C]$-submodule of U weakly $E[C]$-equivalent to $U_E \simeq n \times V$.

One conclusion from (3.11) is that Ker (PA on U) = Ker (PA on V). Hence $A_U = A_V$, as factor groups of A. Therefore PA, F, and U satisfy the hypotheses of Corollary 3.3 or Corollary 3.6 or Proposition 3.7, if (3.9a) or (3.9b) or (3.9c), respectively, hold. The fact that V is ample says that one of (3.9a, b, c) is satisfied. So the cited results tell us that Y is a non-zero $F[C]$-submodule of U which is weakly $F[C]$-equivalent to U. Since the weak $F[C]$-equivalence of Y and U implies their weak $E[C]$-equivalence by (1.2), the proposition is proved.

COROLLARY 3.12. *If V is an irreducible $E[PA]$-module, $[Z(A_V), P] = \{1\}$, and $[A_V, P] \neq \{1\}$, then $[A_V, P]$ is an extra-special group, $\Phi([A_V, P]) = \Phi(A_V)$ is centralized by P, and PA_V is the central product of $P[A_V, P]$ and $C_{A_V}(P)$ which intersect in $\Phi(A_V)$.*

Proof. Apply the first paragraph of the above argument to V. Then (3.11) implies that $A_V = A_U$. Since F, PA and U satisfy (3.1), Proposition 3.4 applies to them, giving this corollary.

The following proposition sometimes helps to prove a module ample. As before, (3.1a) holds, E is any field of prime characteristic $q \neq p(A)$, and V is any irreducible $E[PA]$-module. Let B be some non-trivial P-invariant subgroup of A. Then PB also satisfies (3.1a) (by (1.5)).

PROPOSITION 3.13. *Suppose there is an ample irreducible $E[PB]$-component W of V. Then V is an ample $E[PA]$-module.*

Proof. Since W is a component of V, the factor group B_W is naturally a section of A_V.

Suppose that $p = p(A)$. Because $p = p(B)$ and W is ample, (3.9c) must hold for P and B_W. Therefore $p \geq 3$ and $[B_W, P]^{p-1} \neq \{1\}$. Since B_W is a section of A_V, this implies that $[A_V, P]^{p-1} \neq \{1\}$. So (3.9c) holds for P and A_V. I.e., V is ample.

Suppose that $p \neq p(A)$ and that V is not ample. Since W is ample, $[B_W, P]$ is not $\{1\}$. Because B_W is a section of A_V, this implies that $[A_V, P] \neq \{1\}$. Neither (3.9a) nor (3.9b) can hold for PA_V. So $[A_V, P] \neq \{1\}$ forces PA_V to lie in the exceptional case of (3.9b).

From (3.9b) and Corollary 3.12 we know that $p(A) = 2$, that p is a Fermat prime, that $[A_V, P]$ is extra-special with $\Phi([A_V, P])$ centralized by PA_V, and that $\overline{[A_V, P]} \simeq [\bar{A}_V, P]$ is an irreducible $Z_2[P]$-module. It follows that $[A_V, P]$ is the only non-trivial subgroup D of A_V satisfying $D = [D, P]$.

But $[B_W, P] \neq \{1\}$ implies $[B_V, P] \neq \{1\}$. Since $p \neq p(A) = p(B_V)$, we have $[[B_V, P], P] = [B_V, P]$. Therefore $[B_V, P] = [A_V, P]$.

The subgroup $\Phi([A_V, P]) = \Phi([B_V, P])$ is central in PA_V and is a non-trivial subgroup of A_V. Since V is an irreducible $E[PA_V]$-module, this implies that $[V, \Phi([A_V, P])] = V$. It follows that $[W, \Phi([B_V, P])] = W$. Because $[B_V, P] = [A_V, P]$ is extra-special, $\Phi([B_V, P])$ is its center and has prime order $p(B) = 2$. It follows that Ker $([B_V, P]$ on $W) = \{1\}$. I.e., $[B_W, P] = [B_V, P]$ as sections of B. Therefore $[B_W, P]$ is extra-special, with $\Phi([B_W, P])$ centralized by P, and $\overline{[B_W, P]}$ is an irreducible $Z_2[P]$-module.

Since $p(B_W) = 2 \neq p$, we have $[Z(B_W), P, P] = [Z(B_W), P]$. But

$$[Z(B_W), P] \leq Z(B_W) \cap [B_W, P] \leq Z([B_W, P]) = \Phi([B_W, P]).$$

The last group is centralized by P. Therefore $[Z(B_W), P] = \{1\}$. Now we know that $p(B) = p(A) = 2$, that p is a Fermat prime, that P centralizes $Z(B_W)$ but not B_W, and that $[\bar{B}_W, P] \simeq \overline{[B_W, P]}$ is an irreducible $Z_2[P]$-module, i.e. PB_W lies in the exceptional case of (3.9b). This contradicts the hypothesis that W is ample. The contradiction proves that V is ample in all cases, which is the proposition.

Let E and PA be as above. Now, however, we take V to be an arbitrary finite-dimensional $E[PA]$-module. Since the characteristic of E is different from $p(A)$, the restriction V_A of V to an $E[A]$-module is completely reducible. Let $\mathcal{K} = \mathcal{K}(V)$ be the family of all kernels Ker (A on W), where W runs over all irreducible $E[A]$-components of V_A. For each $K \in \mathcal{K}$, let $V_A(K)$ be the sum of all those irreducible $E[A]$-submodules W of V such that Ker (A on $W) = K$. Then the complete reducibility of V_A implies that

(3.14a) *$V_A(K)$ is a non-trivial $E[A]$-submodule of V, for each $K \in \mathcal{K}$,*
(3.14b) $V_A = \oplus \sum_{K \in \mathcal{K}} V_A(K)$.

If K is any normal subgroup of A, let $K(P) = \bigcap_{\pi \in P} K^\pi$. Then $K(P)$ is the largest normal subgroup of PA contained in K. We define $\mathcal{K}_{\text{ample}} = \mathcal{K}_{\text{ample}}(V)$ to be the set of all $K \in \mathcal{K}$ such that (3.9) holds with $A/K(P)$ in place of A_V. Finally, we set

$$V_{\text{ample}} = \oplus \sum_{K \in \mathcal{K}_{\text{ample}}} V_A(K). \tag{3.15}$$

Then we have

PROPOSITION 3.16. *V_{ample} is an $E[PA]$-submodule of V whose irreducible $E[PA]$-components are precisely the ample irreducible $E[PA]$-components of V.*

Proof. Since V is an $E[PA]$-module we have

$$V_A(K^\pi) = V_A(K)\cdot\pi \quad \textit{for all} \quad K \in \mathcal{K}, \pi \in P. \tag{3.17}$$

In particular, $\mathcal{K}$ is a P-invariant family of subgroups of A. From the definition of $\mathcal{K}_{\text{ample}}$ it is clear that it is a P-invariant subfamily of $\mathcal{K}$. So (3.15) and (3.17) imply that V_{ample} is an $E[PA]$-submodule of V.

Let U be any irreducible $E[PA]$-component of V, and W be an irreducible $E[A]$-component of U. If $K = \text{Ker}\,(A \text{ on } W)$, then Clifford's theory (see Theorem V, 17.3 of [4]) says that

$$\text{Ker}\,(A \text{ on } U) = \bigcap_{\pi \in P} K^{\pi} = K(P).$$

Since $K \in \mathcal{K}$, we conclude that U is ample if and only if $K \in \mathcal{K}_{\text{ample}}$.

If U is a component of V_{ample}, then (3.15) implies that $K \in \mathcal{K}_{\text{ample}}$. So U is ample. If U is a component of V/V_{ample}, then (3.14) and (3.15) imply that $K \in \mathcal{K} - \mathcal{K}_{\text{ample}}$. So U is not ample. Therefore the ample irreducible $E[PA]$-components of V are precisely the irreducible $E[PA]$-components of V_{ample}, which proves the proposition.

4. Finding one ample representation

The following situation occurs repeatedly in Fitting chains on which our group P acts:

(4.1a) *PB is the semi-direct product of a group P of prime order p acting on a group $B \in \mathfrak{A}$.*

(4.1b) *PBA is the semi-direct product of PB acting on a group $A \in \mathfrak{A}$.*

(4.1c) *V is an irreducible $Z_q[PA]$-module, for some prime q.*

(4.1d) *p, $p(B)$ and q are all different from $p(A)$.*

(4.1e) $\text{Ker}\,(\Phi(A) \text{ on } V) = \{1\}$.

We shall prove the following consequence of (4.1) under weaker hypotheses because of future applications.

PROPOSITION 4.2. *Let the semi-direct PA of a group P acting on a group $A \in \mathfrak{A}$ itself act on a group V. If* $\text{Ker}\,(\Phi(A) \text{ on } V) = \{1\}$, *then the natural epimorphism of $A/Z(A)$ onto $A_V/Z(A_V)$ is a P-isomorphism.*

Proof. Obviously this epimorphism preserves the actions of P. So we need only show it to be an isomorphism, i.e., that $Z(A)$ is the inverse image of $Z(A_V)$. Suppose that $\sigma \in A - Z(A)$. Then there exists $\tau \in A$ such that $[\sigma, \tau] \neq 1$. Since $[\sigma, \tau] \in \Phi(A)$, our hypotheses say that the image of $[\sigma, \tau]$ in A_V is not 1. Hence the image of σ does not lie in $Z(A_V)$. It follows that $Z(A)$ contains the inverse image of $Z(A_V)$. The opposite inclusion is obvious. So the proposition is true.

In our case the value of Proposition 4.2 is that $[A/Z(A)]^{+}$ is a $Z_{p(A)}[PB]$-module, while $[A_V/Z(A_V)]^{+}$ is only a $Z_{p(A)}[P]$-module. We exploit this fact to prove

PROPOSITION 4.3. *Suppose that* (4.1) *holds and V is not an ample $Z_q[PA]$-module. Then $[B, P]$ centralizes $A/Z(A)$ unless the following exceptional conditions all occur:*

(4.4a) $p(A) = 2$.

(4.4b) *p is a Fermat prime.*

(4.4c) $|\Phi(A)| = 2$.

(4.4d) $U = [[A/Z(A)]^+, [B, P]]$ *is an irreducible* $Z_2[PB]$*-module.*

(4.4e) $[U, P]$ *is an irreducible* $Z_2[P]$*-module.*

(4.4f) *The function* $g\colon \sigma Z(A) \times \tau Z(A) \to [\sigma, \tau]$ *is a well-defined, PB-invariant, non-singular, alternating bilinear map of* $U \times U$ *into* $\Phi(A)^+$.

Proof. Since $p \neq p(A)$ (by (4.1d)) and V is not ample, both (3.9a) and (3.9b) must fail. So there are two possibilities: either P centralizes A_V or the exceptional case in (3.9b) occurs.

If P centralizes A_V, then it centralizes $A_V/Z(A_V)$ and hence centralizes $A/Z(A)$, by Proposition 4.2. But $A/Z(A)$ is a PB-group. So $[B, P]$ must centralize it, and the proposition is true in this case.

Assume that we are in the exceptional case of (3.9b) and that [B, P] does not centralize $A/Z(A)$. Then (4.4a, b) hold and $[\bar{A}_V, P]$ is an irreducible $Z_2[P]$-module. Since $A_V/Z(A_V)$ is a natural epimorphic image of $\bar{A}_V$ (by (1.4b)), we conclude that $[[A_V/Z(A_V)]^+, P]$ is either $\{0\}$ or an irreducible $Z_2[P]$-module. By Proposition 4.2 the same holds for $[[A/Z(A)]^+, P]$. Therefore $[U, P]$ is either $\{0\}$ or an irreducible $Z_2[P]$-module. But $[U, P]$ cannot be $\{0\}$, since $U \neq \{0\}$ by assumption and

$$[U, [B, P]] = U, \tag{4.5}$$

which follows from the definition of U and the fact that $p(B) \neq p(A)$ (by (4.1d)). Therefore (4.4e) holds.

Hypothesis (4.1d) says that $p(A)$ does not divide $|PB|$. Therefore U is a completely reducible $Z_2[PB]$-module. If U is reducible, then $U = U_1 \oplus U_2$, where U_1, U_2 are non-trivial $Z_2[PB]$-submodules. Clearly (4.5) implies $[U_i, [B, P]] = U_i$ and hence $[U_i, P] \neq \{0\}$, for $i = 1, 2$. So $[U, P] = [U_1, P] \oplus [U_2, P]$ is reducible, contradicting (4.4e). Therefore (4.4d) holds.

Since we are in a case of (3.9b), Corollary 3.12 tells us that $[A_V, P]$ is extra special with $\Phi([A_V, P]) = \Phi(A_V)$. Because $p(A) = 2$, this implies that $|\Phi(A_V)| = 2$. By (4.1e) the natural epimorphism of $\Phi(A)$ onto $\Phi(A_V)$ is an isomorphism. Therefore (4.4c) holds.

By (1.4b), the function g of (4.4f) is a well-defined, non-singular, alternating bilinear map of $[A/Z(A)]^+ \times [A/Z(A)]^+$ into $\Phi(A)^+$. It is obviously PB-invariant. From (4.4c) we conclude that PB centralizes $\Phi(A)$. Since $p(B) \neq 2$, this implies that

$$[A/Z(A)]^+ = U \oplus C_{[A/Z(A)]^+}([B, P]),$$

where these two subspaces are g-perpendicular. It follows that the restriction of g to $U \times U$ is non-singular, which proves (4.4f) and completes the proof of the proposition.

Now we investigate the exceptional case in Proposition 4.3. I.e., we assume that (4.1) and (4.4) hold. We choose a finite algebraic extension F

of Z_2 so that F is a splitting field for all subgroups of PB. Then the extension $F \otimes_{Z_2} U$ of U to an $F[PB]$-module has the decomposition

$$F \otimes U = U_1 \oplus \cdots \oplus U_t, \tag{4.6}$$

where $U_1, \cdots, U_t$ are absolutely irreducible $F[PB]$-submodules. From (4.5) and the equation corresponding to (3.11), we know that

(4.7a) $[U_i, [B, P]] = U_i$ *for* $i = 1, \cdots, t$,
(4.7b) $[U_i, P] \neq \{0\}$, *for* $i = 1, \cdots, t$,
(4.7c) $B_{U_i} = B_U$, *for* $i = 1, \cdots, t$ *(as factor groups of* B*)*.

The first step in the investigation is

LEMMA 4.8. $p = p(B)$.

Proof. Suppose that $p \neq p(B)$. Let $i = 1, \cdots, t$. By (4.7a), P does not centralize B_{U_i}. If it does not centralize $Z(B_{U_i})$, then U_i is induced from some $F[B]$-module by Proposition 3.2. Hence U_i contains a regular $F[P]$-submodule. If P does centralize $Z(B_{U_i})$, then Proposition 3.4 says that U_i is isomorphic to an outer Kronecker product $W \otimes_F Y$ of an $F[P[B_{U_i}, P]]$-module W and an $F(C_{B_{U_i}}(P))$-module Y. The exceptional case in (3.5) does not hold here, since $p(B) \neq 2$. Therefore W contains a regular $F[P]$-submodule, which implies that U_i does also.

The above argument shows that each U_i, $i = 1, \cdots, t$, contains a regular $F[P]$-submodule. It follows from (4.6) that the multiplicity of any non-trivial irreducible $F[P]$-module Z as a component of $F \otimes U$ is at least t. But condition (4.4e) forces the multiplicity of Z as a component of $[F \otimes U, P]$ to be at most one. Since Z is non-trivial, these two multiplicities are equal. Therefore $t = 1$.

The function g of (4.4f) has a natural extension to a PB-invariant, non-singular, alternating, F-bilinear map g' of $(F \otimes U) \times (F \otimes U)$ into $F \otimes \Phi(A)^+$. By (4.4c), $F \otimes \Phi(A)^+$ is F-isomorphic to F as a trivial $F[PB]$-module. So the non-singularity of g' gives us an $F[PB]$-isomorphism of $F \otimes U = U_1$ onto its dual module $\mathrm{Hom}_F(F \otimes U, F) = \mathrm{Hom}_F(U_1, F)$. This is impossible since U_1 is a non-trivial irreducible $F[PB]$-module and $|PB|$ is odd (by (4.1d) and (4.4a)). This contradiction proves the lemma.

Let e be the smallest positive integer such that $2^e \equiv 1 \pmod p$. Then we have

LEMMA 4.9. $t = e$ *and* $\dim_F [U_i, P] = 1$, *for* $i = 1, \cdots, e$.

Proof. Since P is cyclic of order p, every non-trivial irreducible $Z_2[P]$-module has dimension e. So (4.4e) and (4.7b) imply

$$e = \dim_F [F \otimes U, P] = \dim_F [U_1, P] + \cdots + \dim_F [U_t, P] \geq t.$$

Furthermore, equality holds if and only if the lemma is true. Hence we need only show that $t \geq e$.

Lemma 4.8 tells us that PB_U is a p-group. Its normal subgroup B_U is non-trivial by (4.5). So there must exist a subgroup P_1 of order p satisfying $P_1 \leq B_U \cap Z(PB_U)$. The group P_1 acts faithfully on U. Hence there is some non-trivial irreducible $Z_2[P_1]$-submodule W of U. Since P_1 is also cyclic of order p, the $F[P_1]$-submodule $F \otimes W$ is the direct sum $W_1 \oplus \cdots \oplus W_e$ of e distinct irreducible $F[P_1]$-submodules W_i. Because P_1 is central in PB_U, distinct W_i must be submodules of distinct absolutely irreducible $F[BP]$-modules U_i. Therefore $t \geq e$, which proves the lemma.

The condition that $\dim_F[U_i, P] = 1$ is very stringent. E.g., it implies

LEMMA 4.10. *P centralizes every P-invariant abelian subgroup of B_U.*

Proof. Let D be a P-invariant abelian subgroup of B_U such that $[D, P] \neq \{1\}$. By (4.7c), $[D, P]$ acts faithfully on U_1. So there must be some irreducible $F[PD]$-submodule W of U_1 such that $[D_W, P] \neq \{1\}$. Because $D = Z(D)$ is abelian, Proposition 3.2 tells us that W is induced from an irreducible $F[D]$-module. Hence W contains a regular $F[P]$-submodule. By Lemma 4.9 this implies

$$1 = \dim_F [U_1, P] \geq \dim_F [W, P] \geq \dim_F [F[P], P] = p - 1.$$

Therefore $p = 2$, which contradicts (4.1d) and (4.4a). This proves the lemma.

Some judicious choices of abelian subgroups of B_U give us a string of consequences from Lemma 4.10.

LEMMA 4.11. *B_U is extra-special, with $[\Phi(B_U), P] = \{1\}$.*

Proof. Lemma 4.10 forces P to centralize $Z(B_U)$. So $Z(B_U) \leq Z(PB_U)$. Because $Z(B_U)$ acts faithfully on the irreducible $Z_2[PB_U]$-module U, this implies that $Z(B_U)$ is cyclic. We know from (4.1d) and (4.4a) that $p(B)$ is odd. Hence (4.1a) and (1.4d) say that B_U has exponent $p(B)$. Therefore $| Z(B_U) | = p(B)$.

Since (4.4d) holds, P cannot centralize B_U. So Lemma 4.10 says that $B_U' \neq \{1\}$. From the inclusion

$$\{1\} < B_U' \leq \Phi(B_U) \leq Z(B_U)$$

(by (1.4b)) and the fact that $| Z(B_U) | = p(B)$, we conclude that $B_U' = \Phi(B_U) = Z(B_U)$ is cyclic of order $p(B)$. I.e., B_U is extra-special. Furthermore, P centralizes $\Phi(B_U) = Z(B_U)$. So the lemma is true.

Fix a generator π of the cyclic group P.

LEMMA 4.12. *$[\bar{B}_U, P] = \bar{B}_U(\pi - 1)$ is f_{B_U}-perpendicular to $C_{\bar{B}_U}(P)$. Hence $[B_U, P]$ centralizes $C_{B_U}(P)$.*

Proof. Since $P = \langle \pi \rangle$ is cyclic and $\bar{B}_U$ is a vector space, we know that $[\bar{B}_U, P] = \bar{B}_U(\pi - 1)$. Suppose that $\sigma \in \bar{B}_U$ and $\tau \in C_{\bar{B}_U}(P)$. Then

$$f_{B_U}(\sigma(\pi - 1), \tau) = f_{B_U}(\sigma, \tau(\pi^{-1} - 1)),$$

since P centralizes $\Phi(B_U)$ (by Lemma 4.11) and leaves f_{B_U} invariant. But $\tau \in C_{\bar{B}_U}(P)$ implies $\tau(\pi^{-1} - 1) = 0$ and hence $f_{B_U}(\sigma, \tau(\pi^{-1} - 1)) = 0$. Therefore $\bar{B}_U(\pi - 1)$ and $C_{\bar{B}_U}(P)$ are f_{B_U}-perpendicular.

The images in $\bar{B}_U$ of $[B_U, P]$ and $C_{B_U}(P)$ are certainly contained in $[\bar{B}_U, P]$ and $C_{\bar{B}_U}(P)$, respectively. So the above facts and (1.6) imply that $[B_U, P]$ centralizes $C_{B_U}(P)$. This proves the lemma.

LEMMA 4.13. $[B_U, P, P] = \{1\}$.

Proof. By Lemma 4.8 there must be some positive integer n such that $[B_U, P]^n = \{1\}$. Let n be the least such integer. Assume that $n \geq 3$. Then we may choose an element σ in $[B_U, P]^{n-2} - C_{B_U}(P)$. We have

$$\sigma^{\pi^i} \in [B_U, P]^{n-2} \leq [B_U, P] \quad \textit{for} \quad i = 1, \cdots, p,$$

and

$$[\sigma^{\pi^i}, \pi^j] \in [B_U, P]^{n-1} \leq C_{B_U}(P) \quad \textit{for} \quad i, j = 1, \cdots, p.$$

So Lemma 4.12 tells us that σ^{π^i} commutes with $[\sigma^{\pi^i}, \pi^j]$, for all $i, j, = 1, \cdots, p$. Hence σ^{π^i} commutes with $\sigma^{\pi^j} = \sigma^{\pi^i}[\sigma^{\pi^i}, \pi^{j-i}]$, for all $i, j = 1, \cdots, p$. We conclude that $\langle \sigma^{\pi}, \sigma^{\pi^2}, \cdots, \sigma^{\pi^p} \rangle$ is a P-invariant abelian subgroup of B_U. Lemma 4.10 says that it is centralized by P. But P does not centralize the element $\sigma^{\pi^p} = \sigma$ of this subgroup. Therefore $n \leq 2$, which is the lemma.

LEMMA 4.14. $\dim_{Z_p}[\bar{B}_U, P] \leq 2$.

Proof. The endomorphism $\sigma \to \sigma(\pi - 1)$ of $\bar{B}_U$ defines a Z_p-isomorphism of $W = \bar{B}_U/C_{\bar{B}_U}(P)$ onto $[\bar{B}_U, P]$. It follows from Lemma 4.12 that the function g given by

$$g(\sigma + C_{\bar{B}_U}(P), \tau + C_{\bar{B}_U}(P)) = f_{B_U}(\sigma, \tau(\pi - 1)) \quad \textit{for} \quad \sigma, \tau \in \bar{B}_U,$$

is a well-defined, Z_p-bilinear map of $W \times W$ into $\Phi(B_U)^+$.

If $\sigma, \tau \in \bar{B}_U$, we compute

$$\begin{aligned} g(\sigma + C_{\bar{B}_U}(P), \tau + C_{\bar{B}_U}(P)) &= f_{B_U}(\sigma, \tau(\pi - 1)) \\ &= f_{B_U}(\sigma(\pi^{-1} - 1), \tau) \quad (\textit{by Lemma } 4.11) \\ &= -f_{B_U}(\tau, \sigma(\pi^{-1} - 1)) \; (f_{B_U} \textit{ is alternating}) \\ &= f_{B_U}(\tau, \sigma(1 - \pi^{-1})) \qquad (f_{B_U} \textit{ is bilinear}) \\ &= f_{B_U}(\tau, (\sigma\pi^{-1})(\pi - 1)) \\ &= g(\tau + C_{\bar{B}_U}(P), (\sigma\pi^{-1}) + C_{\bar{B}_U}(P)) \\ &= g(\tau + C_{\bar{B}_U}(P), \sigma + C_{\bar{B}_U}(P)), \end{aligned}$$

since $\sigma\pi^{-1} - \sigma \in [\bar{B}_U, P] \leq C_{\bar{B}_U}(P)$ by Lemma 4.13. So g is symmetric. But $\Phi(B_U)^+$ is Z_p-isomorphic to Z_p by Lemma 4.11. Therefore g is just an ordinary quadratic form on the vector space W over Z_p.

Suppose that $\dim_{Z_p}(W) \geq 3$. Since Z_p is a finite field of odd characteristic, there must exist some element $w \neq 0$ in W so that $g(w, w) = 0$ (see page 144 of [1]). Let $\sigma \in \bar{B}_U$ satisfy $w = \sigma + C_{\bar{B}_U}(P)$. Then $Z_p\sigma + Z_p\sigma(\pi - 1)$ is totally isotropic with respect to f_{B_U}, since $f_{B_U}(\sigma, \sigma(\pi - 1)) = g(w, w) = 0$. This subspace is P-invariant by Lemma 4.13 and is not centralized by P since $\sigma + C_{\bar{B}_U}(P) = w \neq 0$. Therefore its inverse image is a P-invariant abelian subgroup of B_U which is not centralized by P. This contradicts Lemma 4.10. Hence $\dim_{Z_p}(W) \leq 2$, which is equivalent to the lemma by the first line of the proof.

We must reach back to Lemma 4.9 to prove

LEMMA 4.15. $p = 3$.

Proof. Since (4.4d) holds, P does not centralize B_U. So we may choose $\sigma \in B_U$ such that $[\pi, \sigma] \neq 1$. If σ centralizes $[\pi, \sigma]$, then Lemma 4.13 implies that $\langle \sigma, [\pi, \sigma] \rangle$ is a P-invariant abelian subgroup of B_U. It is not centralized by P since $[\pi, \sigma] \neq 1$. This contradicts Lemma 4.10. Therefore $[[\pi, \sigma], \sigma] = [\pi, \sigma, \sigma] \neq 1$. From Lemmas 4.11 and 4.13 we now conclude that $D = \langle \sigma, [\pi, \sigma] \rangle$ is a P-invariant extra-special subgroup of order p^3 in B_U with $\Phi(D) = \Phi(B_U)$.

The group P centralizes $\Phi(D) = \langle [\pi, \sigma, \sigma] \rangle$ by Lemma 4.11 and centralizes $[\pi, \sigma]$ by Lemma 4.13. If follows that $E = \langle \pi, [\pi, \sigma], [\pi, \sigma, \sigma] \rangle$ is an abelian subgroup of order p^3 in PD. Since $| PD | = p^4$, E is normal in PD.

The group $\Phi(D) = \langle [\pi, \sigma, \sigma] \rangle$ is clearly the center of PD. Since $\Phi(D)$ acts faithfully on U_1 (by (4.7c)), there must be an irreducible $F[PD]$-submodule W of U_1 on which $\Phi(D)$ acts non-trivially. From Lemma 4.9 we know that $\dim_F [W, P]$ is 0 or 1. So the restriction ψ_P to P of the modular character ψ(see [5] for definitions) of the $F[PD]$-module W has the form:

$$\psi_P = \lambda + (\psi(1) - 1)\cdot 1, \tag{4.16}$$

for some linear character λ of P.

Because $| PD | = p^4$ is odd, the modular character ψ is an ordinary irreducible character of PD (see [5]). The non-triviality of $\Phi(D)$ on W implies that $\psi_{\Phi(D)} = \psi(1)\cdot\nu$, for some non-trivial ordinary linear character ν of $\Phi(D)$. Since ν is faithful and $\Phi(D) = \langle [\pi, \sigma, \sigma] \rangle \leq [E, \langle \sigma \rangle]$, no extension μ of ν to a linear character of E can be fixed by $\langle \sigma \rangle$. It follows that $\psi = \mu^{PD}$, for some such extension μ. Therefore

$$\psi_E = \mu + \mu^{\sigma} + \cdots + \mu^{\sigma^{p-1}}.$$

But $P \leq E$. Hence

$$\psi_P = \mu_P + (\mu^{\sigma})_P + \cdots + (\mu^{\sigma^{p-1}})_P .$$

Comparing this with (4.16), we see that $(\mu^{\sigma^i})_P$ must be trivial for all but one $i = 0, 1, \cdots, p - 1$. We may assume that the exceptional value of i

(if any exists) is $i = 0$, so that $\mu^{\sigma^i}(\pi) = 1$, for $i = 1, \cdots, p-1$. Hence

$$1 = \mu^{\sigma^i}(\pi) = \mu(\pi^{\sigma^{-i}}) = \mu(\pi[\pi, \sigma]^{-i}[\pi, \sigma, \sigma]^{C(i,2)})$$
$$= \mu(\pi)\mu([\pi, \sigma])^{-i}([\pi, \sigma, \sigma])^{C(i,2)} \quad for \quad i = 1, \cdots, p-1,$$

where, of course, $C(i, 2)$ is the binomial symbol $i(i-1)/2$. Taking $i = 1$ in this we get $\mu([\pi, \sigma]) = \mu(\pi)$. Taking $i = 2$, we then get $\mu([\pi, \sigma, \sigma]) = \mu(\pi)$. If $p > 3$, we may take $i = 3$, getting $1 = \mu(\pi)^{1-3+3} = \mu(\pi) = \mu([\pi, \sigma, \sigma])$. This contradicts the fact that $\mu([\pi, \sigma, \sigma]) = \nu([\pi, \sigma, \sigma]) \neq 1$. So the lemma is true.

We collect the results of the last eight lemmas and one further consequence in

Proposition 4.17. *If both* (4.1) *and* (4.4) *hold, then*
(4.18a) $p = p(B) = 3$,
(4.18b) *U is not an ample $Z_2[PB]$-module,*
(4.18c) *B_U is extra-special,*
(4.18d) $\dim_{Z_3}[\bar{B}_U, P] \leq 2$,
(4.18e) $\dim_{Z_2} C_U(P) \geq 4$.

Proof. Conclusion (4.18a) is Lemmas 4.8 and 4.15.

Conclusion (4.18b) is Lemma 4.13, since (4.18a) holds (compare (3.9c)).
Conclusion (4.18c) is Lemma 4.11.
Conclusion (4.18d) is Lemma 4.14, since $p = 3$.
By (4.18c) and (4.7c), each U_i has dimension at least $p(B) = 3$. Since $U_i = [U_i, P] \oplus C_{U_i}(P)$ and $\dim_F [U_i, P] = 1$, by Lemma 4.9, each $C_{U_i}(P)$ has dimension at least 2. So (4.6) gives

$$\dim_{Z_2} C_U(P) = \dim_F C_{F \otimes U}(P) = \sum_{i=1}^{t} \dim_F C_{U_i}(P) \geq 2t.$$

But $t = e = 2$ by Lemma 4.9, since $p = 3$. Therefore (4.18e) holds and the proposition is true.

We shall apply the above propositions to the situation in which our group P of arbitrary prime order p acts on a Fitting chain $A_1, \cdots, A_t$ with $[A_1, P] \neq \{1\}$. We wish to show that some representation (PA_i on $\bar{A}_{i+1}$) has an ample irreducible component, provided the Fitting chain is long enough. To specify the necessary length, we define an integer i_0 by

(4.19a) $i_0 = 2$ *if p does not divide* $\prod_{i=1}^{t} |A_i|$,
(4.19b) $i_0 = 3$ *if p divides* $\prod_{i=1}^{t} |A_i|$ *and* $p \neq 3$,
(4.19c) $i_0 = 5$ *if p divides* $\prod_{i=1}^{t} |A_i|$ *and* $p = 3$.

Then we have

Theorem 4.20. *If $t > i_0$, then there is some $i = 1, \cdots, i_0$ such that $p(A_i) \neq p$ and (PA_i on $\bar{A}_{i+1}$) has an ample irreducible component.*

Proof. Define the groups $B_1, \cdots, B_t$ inductively by

$$B_1 = A_1, \qquad B_{i+1} = [A_{i+1}, [B_i, P]], \quad for \quad i = 1, \cdots, t-1.$$

We first prove that

(4.21a) *B_i is a P-invariant subgroup of A_i, for $i = 1, \cdots, t$,*
(4.21b) *B_i is PB_{i-1}-invariant, for $i = 2, \cdots, t$,*
(4.21c) *$[B_i, P] \neq \{1\}$, for $i = 1, \cdots, t$,*
(4.21d) *$B_i \in \mathcal{Q}$, for $i = 1, \cdots, t$.*

Statement (4.21a) is obvious from the definition of the B_i. It implies that $[B_{i-1}, P]$ is a normal subgroup of PB_{i-1}, for $i = 2, \cdots, t$. Therefore

$$B_i = [A_i, [B_{i-1}, P]]$$

is PB_{i-1}-invariant, which is statement (4.21b).

Statement (4.21c) is proved by induction on i. For $i = 1$ it is true since $[A_1, P] \neq \{1\}$ by hypothesis. Suppose that $i > 1$ and that $[B_{i-1}, P] \neq \{1\}$. By (2.2d), $[B_{i-1}, P]$ acts faithfully on A_i. Therefore

$$B_i = [A_i, [B_{i-1}, P]] \neq \{1\}.$$

Since $p([B_{i-1}, P]) = p(A_{i-1}) \neq p(A_i)$ (by (2.2b)), we have $\{1\} \neq B_i = [B_i, [B_{i-1}, P]]$. If P centralizes B_i, then so does $[B_{i-1}, P]$, (by (4.21b)) which contradicts the preceding statement. Therefore $[B_i, P] \neq \{1\}$ and (4.21c) holds. This implies that $B_i \neq \{1\}$, for $i = 1, \cdots, t$. So (4.21d) follows from (1.5) and (2.2a).

Suppose we can prove that

(4.22) *there is some $i = 1, \cdots, i_0$ such that $p \neq p(A_i)$ and (PB_i on $\bar{A}_{i+1}$) has an ample irreducible component.*

Then the theorem will be true. To see this, let W be an ample irreducible component of (PB_i on $\bar{A}_{i+1}$). Then there must be some irreducible component V of (PA_i on $\bar{A}_{i+1}$) such that W is PB_i-isomorphic to an irreducible component of (PB_i on V). Proposition 3.13, applied to $Z_{p(A_{i+1})}$, P, A_i, B_i, V and W, tells us that V is ample. So the theorem will follow from (4.22).

From now on we assume that (4.22) is false, i.e., that no irreducible component of any (PB_i on $\bar{A}_{i+1}$) is ample, for any $i = 1, \cdots, i_0$ such that $p(A_i) \neq p$.

Suppose that $2 \leq i \leq i_0$ and $p(A_i) \neq p$. Since $[B_i, P]$ acts faithfully on A_{i+1} (by (2.2d)) and $p(A_{i+1}) \neq p(A_i) = p([B_i, P])$ (by (2.2b)), it acts faithfully on $\bar{A}_{i+1}$ (see Theorem III, 3.18 of [4]). So (4.21c) implies that $Y_{i+1} = [\bar{A}_{i+1}, [B_i, P]]$ is a non-zero $Z_{p(A_{i+1})}[PB_i]$-submodule of $\bar{A}_{i+1}$. Furthermore, $[Y_{i+1}, [B_i, P]] = Y_{i+1}$, since $p([B_i, P]) \neq p(A_{i+1})$. Therefore there is some irreducible component W_{i+1} of (PB_i on Y_{i+1}) and any such W_{i+1} satisfies $[W_{i+1}, [B_i, P]] = W_{i+1}$.

Let $K_i = \mathrm{Ker}(\Phi(B_i)$ on $W_{i+1})$. Since $K_i \leq \Phi(B_i) \leq \Phi(A_i)$, it is centralized by $B_{i-1} \leq A_{i-1}$ (by (2.2c)). Therefore PB_{i-1} acts on B_i/K_i. Now we see that P, $B = B_{i-1}$, $A = B_i/K_i$, W_{i+1} and $q = p(A_{i+1})$ satisfy (4.1).

Suppose that $(B_i)_{W_{i+1}}$ is abelian. Since $[W_{i+1}, [B_i, P]] = W_{i+1}$, we have $[Z((B_i)_{W_{i+1}}), P] = [(B_i)_{W_{i+1}}, P] \neq \{1\}$. But $p(B_i) = p(A_i) \neq p$. So (3.9a) holds and W_{i+1} is ample, contradicting our assumptions. Therefore $Z((B_i)_{W_{i+1}}) < (B_i)_{W_{i+1}}$. By Proposition 4.2 this implies that $Z(A) < A$.

By construction $[B_i, [B_{i-1}, P]] = B_i$. Therefore $[A/Z(A), [B, P]] = A/Z(A)$. Since $A/Z(A) \neq \{1\}$ and W_{i+1} is not ample, Proposition 4.3 says that (4.4) holds. So Proposition 4.17 tells us that (4.18) holds. In particular, $3 = p = p(B) = p(A_{i-1})$.

Suppose that (4.19a) holds. Then $p(A_2) \neq p$. The above argument shows that $p(A_1) = p$, contradicting (4.19a). So (4.22) cannot be false, and the theorem is true in this case.

Suppose that (4.19b) holds. By (2.2b) there is some $i = 2, 3$ such that $p(A_i) \neq p$. The above argument shows that $p = 3$, contradicting (4.19b). So the theorem is true in this case.

We must be in the case (4.19c). There is some $i = 4, 5$ such that $p(A_i) \neq p$. The above argument shows that $p(A_{i-1}) = p = 3$ and $p(A_i) = 2$. Furthermore, since (4.18) holds, we have an irreducible $Z_2[PB_{i-1}]$-module U_i so that $(B_{i-1})_{U_i}$ is extra-special and $\dim_{Z_3}[\overline{(\bar{B}_{i-1})_{U_i}}, P] \leq 2$. Let

$$K_{i-1} = \mathrm{Ker}(\Phi(B_{i-1}) \quad \text{on} \quad U_i).$$

By (2.2c), B_{i-2} centralizes K_{i-1}. Therefore PB_{i-2} acts on B_{i-1}/K_{i-1}. Evidently P, B_{i-2}, B_{i-1}/K_{i-1}, and U_i satisfy (4.1a, b, c, e). So Proposition 4.2 says that

$$[(B_{i-1}/K_{i-1})/Z(B_{i-1}/K_{i-1})]^{+} = Y_{i-1}$$

is $Z_3[P]$-isomorphic to $[(B_{i-1})_{U_i}/Z((B_{i-1})_{U_i})]^{+}$. The latter group is just $\overline{(B_{i-1})_{U_i}}$, since $(B_{i-1})_{U_i}$ is extra-special. Hence $Y_{i-1} \neq \{0\}$ and $\dim_{Z_3}[Y_{i-1}, P] \leq 2$.

Because PB_{i-2} acts on B_{i-1}/K_{i-1}, it acts on Y_{i-1}. By (2.2b) and the definition of B_{i-1}, we have $[B_{i-1}, [B_{i-2}, P]] = B_{i-1}$. It follows that

$$[Y_{i-1}, [B_{i-2}, P]] = Y_{i-1}.$$

Since $Y_{i-1} \neq \{0\}$, there is some irreducible component W_{i-1} of (PB_{i-2} on Y_{i-1}). Clearly $[Y_{i-1}, [B_{i-2}, P]] = Y_{i-1}$ and $\dim_{Z_3}[Y_{i-1}, P] \leq 2$ imply that

(4.23a) $[W_{i-1}, [B_{i-2}, P]] = W_{i-1}$.
(4.23b) $\dim_{Z_3}[W_{i-1}, P] \leq 2$.

Since W_{i-1} is a section of A_{i-1}, it follows from (2.2c) and (4.23a) that it is isomorphic to some irreducible component of (PB_{i-2} on $\bar{A}_{i-1}$). So (PB_{i-2} on W_{i-1}) is not ample.

Now we repeat our earlier argument with $i - 2$ in place of i. It tells us that P, B_{i-3}, B_{i-2}/K_{i-2} and W_{i-1} satisfy (4.1), (4.4), and (4.18), where $K_{i-2} = \mathrm{Ker}(\Phi(B_{i-2})$ on $W_{i-1})$. The definition (4.4d) of the $Z_2[PB_{i-3}]$-

module U_{i-2}, the definition of B_{i-2}, and (2.2b) imply that

$$U_{i-2} = [(B_{i-2}/K_{i-2})/Z(B_{i-2}/K_{i-2})]^+.$$

From Proposition 4.2 we conclude that U_{i-2} is P-isomorphic to

$$[(B_{i-2})_{W_{i-1}}/Z((B_{i-2})_{W_{i-1}})]^+ = \tilde{B}_{i-2}.$$

Therefore (4.18e) gives

$$\dim_{Z_2} C_{\tilde{B}_{i-2}}(P) \geq 4. \tag{4.24}$$

Let F be a finite algebraic extension field of Z_3 which is a splitting field for every subgroup of PB_{i-2}. Let X be a irreducible $F[PB_{i-2}]$-submodule of the extension $F \otimes W_{i-1}$ of W_{i-1} to an $F[PB_{i-2}]$-module. Then $(B_{i-2})_X = (B_{i-2})_{W_{i-1}}$ as in (4.7c). Because (PB_{i-2} on W_{i-1}) lies in the exceptional case of (3.9b), Proposition 3.4 applies to P, B_{i-2}, F and X. It tells us that X is the outer Kronecker product $X = S \otimes T$ of an irreducible $F[P[(B_{i-2})_X, P]]$-module S and an irreducible $F[C_{(B_{i-2})_X}(P)]$-module T. Since $Z((B_{i-2})_X) \leq C = C_{(B_{i-2})_X}(P)$, and $(B_{i-2})_X$ is the central product of $[(B_{i-2})_X, P]$ and C, we know that $Z((B_{i-2})_X) = Z(C)$. This acts faithfully on T since it acts faithfully on X. It follows from (4.24) and Theorem (III.2) of [2] that

$$\dim_F T = [C:Z(C)]^{1/2} = | C_{\tilde{B}_{i-2}}(P) |^{1/2} \geq (2^4)^{1/2} = 2^2 = 4.$$

Since P acts faithfully on $[(B_{i-2})_X, P]$ and the latter acts faithfully on S, we know that P acts faithfully on S. So $\dim_F [S, P] \geq 1$. It follows from (4.23b) that

$$2 \geq \dim_F [F \otimes W_{i-1}, P] \geq \dim_F [X, P] = \dim_F [S, P] \cdot \dim_F T \geq 4.$$

This contradiction proves the theorem.

5. Finding enough ample representations

We now turn to the problem of going from the ample components of (PA_{i-1} on $\bar{A}_i$) to those of (PA_i on $\bar{A}_{i+1}$) in our P-invariant Fitting chain $A_1, \cdots, A_t$. The critical case is the following situation:

(5.1a) *PB is the semi-direct product of a group P of prime order p acting on a group $B \in \mathfrak{A}$.*

(5.1b) *D is a subgroup of $C_B(P)$.*

(5.1c) *PBA is the semi-direct product of PB acting on a group $A \in \mathfrak{A}$.*

(5.1d) *V is a finite-dimensional $Z_q[PA]$-module, for some prime q.*

(5.1e) *p, $p(B)$ and q are all different from $p(A)$.*

(5.1f) $[\Phi(A), B] = \{1\}$.

(5.1g) *Each irreducible component of (PB on $\bar{A}$) is ample.*

(5.1h) *The representation (A on V) is faithful and weakly B-invariant.*

(5.1i) *If $p = p(B)$ then $D \leq [B, P]^{p-1}$.*

One immediate consequence of these hypotheses is

PROPOSITION 5.2. $A' = \Phi(A)$.

Proof. By (5.1g) and (3.9), B acts non-trivially on each irreducible component U of (PB on $\bar{A}$). Hence $[U, B] = U$. It follows that $[\bar{A}, B] = \bar{A}$. Since $\bar{A}$ is naturally isomorphic to $(A/A')/\Phi(A/A')$, we conclude that $[A/A', B] = A/A'$. The map $\sigma \to \sigma^{p(A)}$ is a B-invariant epimorphism of the abelian group A/A' onto $\Phi(A/A') = \Phi(A)/A'$. Therefore $[\Phi(A)/A', B] = \Phi(A)/A'$. By (5.1f), this implies that $\Phi(A)/A' = \{1\}$. So the proposition holds.

We let $\mathcal{K} = \mathcal{K}(V)$ and $\mathcal{K}_{\text{ample}} = \mathcal{K}_{\text{ample}}(V)$ be the families of (3.14) and (3.15). Define $\mathcal{L}$ to be the subfamily of all $K \in \mathcal{K}$ such that $K \geq \Phi(A)$.

PROPOSITION 5.3. *Both $\mathcal{K}$ and $\mathcal{L}$ are PB-invariant families of normal subgroups of A. Furthermore, $\mathcal{K}$-$\mathcal{L} \subseteq \mathcal{K}_{\text{ample}}$.*

Proof. Since V is a PA-module, $\mathcal{K}$ is P-invariant by (3.17). The weak B-invariance of (A on V) (by (5.1h)) clearly implies that $\mathcal{K}$ is B-invariant. So $\mathcal{K}$ is PB-invariant. Because $\Phi(A)$ is a characteristic subgroup of A, it is PB-invariant. So $\mathcal{L}$ is a PB-invariant subfamily of $\mathcal{K}$, which finishes the proof of the first statement of the proposition.

Suppose that $K \in \mathcal{K}$-$\mathcal{L}$. Then there exists some irreducible component W of (A on V) such that $K = \text{Ker}(A \text{ on } W)$. There must be some irreducible component X of (PA on V) such that W is isomorphic to a component of (A on X). Clifford's theory (see Theorem V, 17.3 of [4]) tells us that

$$\text{Ker}(A \text{ on } X) = \bigcap_{\pi \in P} K^{\pi} = K(P).$$

Therefore $K \in \mathcal{K}_{\text{ample}}$ if and only if P and $A_X = A/K(P)$ satisfy (3.9), i.e., if and only if (PA on X) is ample.

Let $N = \text{Ker}(\Phi(A) \text{ on } X)$. Then N is a P-invariant normal subgroup of A. By (5.1f) it is also B-invariant. So (5.1e) implies that P, B, A/N and X satisfy (4.1).

Proposition 4.2 says that $Y = [(A/N)/Z(A/N)]^{+}$ is isomorphic to $[A_X/Z(A_X)]^{+}$. Since $K \notin \mathcal{L}$, we have $K \cap \Phi(A) < \Phi(A)$. By Proposition 5.2, this implies that $K \cap A' < A'$. Therefore

$$K(P) \cap A' \leq K \cap A' < A' \quad \text{and} \quad A_X' \simeq A'/K(P) \cap A' \neq \{1\}.$$

So A_X is non-abelian. Hence $[A_X/Z(A_X)]^{+} \neq \{0\}$. We conclude from this and (1.4b) that Y is a non-trivial $Z_{p(A)}[PB]$-factor module of $\bar{A}$. In particular, (PB on Y) has at least one irreducible component and, by (5.1g), each such component is ample.

Suppose that (PA on X) is not ample. Then neither is ($P(A/N)$ on X). By Proposition 4.3, either $[B, P]$ centralizes Y or (4.4) holds. But (PB on Y) has an ample irreducible component, which, by (3.9), cannot be centralized by $[B, P]$. Hence (4.4) holds. In particular, (4.4d) says that U is an irreducible component of (PB on Y). So (PB on U) is ample. This contradicts (4.18b). Therefore X is ample and the proposition is true.

Let L be the subgroup of A defined by

$$L = \bigcap_{K \in \mathcal{K} - \mathcal{L}} K. \tag{5.5}$$

If $\mathcal{K}$-$\mathcal{L}$ is empty, this intersection is taken to be A.

PROPOSITION 5.6. *L is PB-invariant normal subgroup of A such that $L \cap \Phi(A) = \{1\}$. Hence L is elementary abelian and the natural $Z_{p(A)}[PB]$-homomorphism φ of L^+ into $\bar{A}$ is a monomorphism.*

Proof. By the definition of $\mathcal{K}$, each $K \in \mathcal{K}$ is a normal subgroup of A. Proposition 5.3 implies that $\mathcal{K}$-$\mathcal{L}$ is a PB-invariant subfamily of $\mathcal{K}$. So L is a PB-invariant normal subgroup of A by (5.5).

Let σ be a non-trivial element of $\Phi(A)$. By (5.1h) σ acts non-trivially on V. Since $q \neq p(A)$ (by (5.1e)), the representation (A on V) is completely reducible. So σ must act non-trivially on some irreducible component W of (A on V). Hence $K = \operatorname{Ker}(A \text{ on } W) \in \mathcal{K}$ and $\sigma \notin K$. It follows that $\Phi(A) \nleq K$, i.e., that $K \in \mathcal{K}$-$\mathcal{L}$. Therefore $\sigma \notin L \leq K$ (by (5.5)). This proves that $L \cap \Phi(A) = \{1\}$. The other statements follow directly from this.
Define families $\mathfrak{M}$, $\mathfrak{N}$ of subgroups of L and a subgroup N of L by

(5.7a) $\mathfrak{M} = \{K \cap L \mid K \in \mathcal{L}, L \nleq K\}$,
(5.7b) $\mathfrak{N} = \{M \in \mathfrak{M} \mid [L, P] \nleq M\}$,
(5.7c) $N = \cap_{M \in \mathfrak{N}} M$.

Let V_{ample} be the $Z_q\,[PA]$-submodule of V defined by (3.15) and Q be the subgroup $\operatorname{Ker}(A \text{ on } V_{\text{ample}})$ of A.

PROPOSITION 5.8. *$\mathfrak{M}$ is a PB-invariant family of maximal subgroups of L satisfying*

$$\cap_{M \in \mathfrak{M}} M = \{1\}. \tag{5.9}$$

The subfamily $\mathfrak{N}$ and the subgroups N and Q are $P \times D$-invariant. Furthermore,

$$Q \leq N \leq L. \tag{5.10}$$

Proof. Proposition 5.3 says that $\mathcal{L}$ is PB-invariant. Proposition 5.6 says that L is PB-invariant. This and (5.7a) imply that $\mathfrak{M}$ is PB-invariant.

Suppose that $M \in \mathfrak{M}$. Then there exists $K \in \mathcal{L}$ such that $L \nleq K$ and $M = K \cap L$. Since $\mathcal{L} \subseteq \mathcal{K}$, there is some irreducible component W of (A on V) such that $K = \operatorname{Ker}(A \text{ on } W)$. Because $K \in \mathcal{L}$, we have $\Phi(A) \leq K$. So the elementary abelian group $A/\Phi(A)$ acts irreducibly on W. This implies that $A_W = [A/\Phi(A)]_W$ is cyclic, and hence has order 1 or $p(A)$. Therefore

$$[L:M] = [L:K \cap L] \leq [A:K] \leq p(A).$$

But L is a $p(A)$-group and $L \nleq K$. So $[L:K \cap L] = p(A)$. Hence any $M \in \mathfrak{M}$ is a maximal subgroup of L.

Since $q \neq p(A)$ (by (5.1e)), the representation (A on V) is fully decomposable. It follows that

$$\begin{aligned} 1 &= \mathrm{Ker}(A \text{ on } V) && (by\ (5.1h)) \\ &= \textstyle\bigcap_{K \in \mathcal{K}} K = [\bigcap_{K \in \mathcal{L}-\mathcal{K}} K] \cap [\bigcap_{K \in \mathcal{L}} K] \\ &= L \cap [\textstyle\bigcap_{K \in \mathcal{L}} K] && (by\ (5.5)) \\ &= \textstyle\bigcap_{K \in \mathcal{L}} [L \cap K] \\ &= \textstyle\bigcap_{K \in L . L \nleq K} [L \cap K] \\ &= \textstyle\bigcap_{M \in \mathfrak{M}} M && (by\ (5.7a)) \end{aligned}$$

So (5.9) holds.

By (5.1a, b), $PD = P \times D \leq PB$. Since $\mathfrak{M}$, L and P are $P \times D$-invariant, so are $\mathfrak{N}$ and N (by (5.7b, c)). It follows from the definition of $\mathcal{K}_{\text{ample}}$ that it is $P \times D$-invariant. Since Q is the intersection of the members of $\mathcal{K}_{\text{ample}}$ (by (3.15)), it is $P \times D$-invariant.

Suppose that $K \in \mathcal{L}\text{-}\mathcal{K}_{\text{ample}}$. Then $K \geq \Phi(A)$. Hence $K(P) = \bigcap_{\pi \in P} K^{\pi} \geq \Phi(A)$. So $A/K(P)$ is abelian. Since $K \notin \mathcal{K}_{\text{ample}}$, P and $A/K(P)$ cannot satisfy (3.9). Therefore (5.1e) and (3.9a) imply that

$$[A/K(P), P] = [Z(A/K(P)), P] = \{1\}.$$

Hence $[A/K, P] = \{1\}$. It follows that $[L, P] \leq [A, P] \cap L \leq K \cap L$. From this and (5.7b) we conclude that $K \cap L \notin \mathfrak{N}$. So we have

$$\mathfrak{N} \subseteq \{K \cap L \mid K \in \mathcal{L} \cap \mathcal{K}_{\text{ample}}\}.$$

This implies

$$\begin{aligned} N &= \textstyle\bigcap_{M \in \mathfrak{N}} M && (by\ (5.7c)) \\ &\geq \textstyle\bigcap_{K \in \mathcal{L} \cap \mathcal{K}_{\text{ample}}} (K \cap L) \\ &\geq [\textstyle\bigcap_{K \in \mathcal{L} \cap \mathcal{K}_{\text{ample}}} K] \cap [\bigcap_{K \in \mathcal{K}-\mathcal{L}} K] && (by\ (5.5)) \\ &= \textstyle\bigcap_{K \in \mathcal{K}_{\text{ample}}} K && (by\ Proposition\ 5.3) \\ &= Q && (by\ (3.15)). \end{aligned}$$

Therefore (5.10) holds, which completes the proof of the proposition.

Define the section C of A by

$$(5.11) \qquad C = [C_A(P)]_{V_{\text{ample}}} = C_A(P)/C_A(P) \cap Q.$$

Since D centralizes P (by (5.1b)), the subgroup $C_A(P)$ is D-invariant. By Proposition 5.8, Q is D-invariant. Hence

(5.12) *C is D-invariant.*

We are interested in conditions which will guarantee the following property:

(5.13) *$C \neq \{1\}$ and (D on $\bar{A}$) is weakly equivalent to (D on $\bar{C}$).*

One set of such conditions is given by

PROPOSITION 5.14. *If* $L = \{1\}$, *then* (5.13) *holds. If* $C_L(P) > C_N(P)$ *and* (D on $C_L(P)^+$) *is weakly equivalent to* (D on $C_L(P)^+/C_N(P)^+$), *then* (5.13) *holds.*

Proof. It follows from (5.1g) and Proposition 3.10 that $C_A(P) \neq \{1\}$ (see Theorem I, 18.6 of [4]). Therefore $L = \{1\}$ gives $Q = \{1\}$ (by (5.10)) and $C \neq \{1\}$ (by (5.11)).

If $C_L(P) > C_N(P)$, then (5.10) implies that $C_L(P) > C_Q(P)$. By (5.11), $C_L(P)/C_Q(P)$ is isomorphic to a subgroup of C. Hence $C \neq \{1\}$ in both cases.

Because $L = \{1\}$ trivially implies the condition "(D on $C_L(P)^+$) is weakly equivalent to (D on $C_L(P)^+/C_N(P)^+$)" we are reduced to deducing from this condition that (D on $\bar{A}$) is weakly equivalent to (D on $\bar{C}$).

Since Q is $P \times D$-invariant (by Proposition 5.8), so are A/Q and the natural epimorphism ψ of A onto A/Q. It follows that ψ induces a $Z_{p(A)}[P \times D]$-epimorphism $\bar{\psi}$ of $\bar{A}$ onto $\overline{(A/Q)}$. From Proposition 5.6 and (5.10) we conclude that $\varphi(Q^+) = \operatorname{Ker} \bar{\psi}$. Hence

(5.15) (D on $\bar{\psi}(C_{\bar{A}}(P))$) *is equivalent to* ($D$ on $C_{\bar{A}}(P)/C_{\bar{A}}(P) \cap \varphi(Q^+)$).

We know from (5.1e) that $p(A)$ does not divide $|P|$. It follows (see Theorem I, 18.6 of [4]) that $C_{\bar{A}}(P)$ is the natural image in $\bar{A}$ of $C_A(P)$. By (5.11), this implies that $\bar{\psi}(C_{\bar{A}}(P))$ is the natural image of $\bar{C}$ in $\overline{(A/Q)}$. The kernel of the natural map of $\bar{C}$ onto $\bar{\psi}(C_{\bar{A}}(P))$ is

$$[(C \cap \Phi(A_{V_{\text{ample}}}))/\Phi(C)]^+,$$

which D centralizes by (5.1f). From this we conclude that (D on $\bar{C}$) is weakly equivalent to (D on $\bar{\psi}(C_{\bar{A}}(P))$). In view of (5.15), this gives

(5.16) (D on $\bar{C}$) *is weakly equivalent to* (D on $C_{\bar{A}}(P)/C_{\bar{A}}(P) \cap \varphi(Q^+)$).

Since $p(A)$ does not divide $|PB|$ (by (5.1e)), there are irreducible $Z_{p(A)}[PB]$-submodules $U_1, \cdots, U_r$ of $\bar{A}$ such that $\bar{A} = \oplus \sum_{i=1}^{r} U_i$. By (5.1g), each U_i is an ample $Z_{p(A)}[PB]$-module. If $p \neq p(B)$, then $D \leq C_B(P)$ by (5.1b). If $p = p(B)$, then $D \leq C_{[B,P]^{p-1}}(P)$ by (5.1b, i). So Proposition 3.10 and (1.3) tell us that (D on U_i) is weakly equivalent to (D on $C_{U_i}(P)$), for $i = 1, \cdots, r$. Since $\bar{A} = \oplus \sum_{i=1}^{r} U_i$ and $C_{\bar{A}}(P) = \oplus \sum_{i=1}^{r} C_{U_i}(P)$, it follows that

(5.17) (D on $\bar{A}$) *is weakly equivalent to* (D on $C_{\bar{A}}(P)$).

By assumption (D on $C_L(P)^+$) is weakly equivalent to

$$(D \text{ on } C_L(P)^+/C_N(P)^+).$$

It follows from this and (5.10) that (D on $C_L(P)^+$) is weakly equivalent to

$$(D \text{ on } C_L(P)^+/C_Q(P)^+).$$

Since φ is an isomorphism, this says that $(D \text{ on } C_{\bar{A}}(P) \cap \varphi(L^+))$ is weakly equivalent to

$$(D \text{ on } [C_{\bar{A}}(P) \cap \varphi(L^+)]/[C_{\bar{A}}(P) \cap \varphi(Q^+)]).$$

Clearly this implies that $(D \text{ on } C_{\bar{A}}(P))$ is weakly equivalent to

$$(D \text{ on } C_{\bar{A}}(P)/[C_{\bar{A}}(P) \cap \varphi(Q^+)]).$$

Combined with (5.16) and (5.17), this shows that $(D \text{ on } \bar{A})$ is weakly equivalent to $(D \text{ on } \bar{C})$, which completes the proof of the proposition.

To establish the hypotheses of Proposition 5.14 it is convenient to pass to the dual $Z_{p(A)}[PB]$-module $U = \mathrm{Hom}_{Z_{p(A)}}(L^+, Z_{p(A)})$ the family $\mathcal{I}$ of all perpendicular subspaces $M^{\perp} = \{u \in U \mid u(M^+) = \{0\}\}$ to the members M of the family $\mathfrak{M}$, the subfamily $\mathcal{J}$ of all $M^{\perp}$, $M \in \mathfrak{N}$, and the subgroup $J = N^{\perp}$. These satisfy

PROPOSITION 5.18. *Every irreducible component of $(PB$ on $U)$ is ample. $\mathcal{I}$ is a PB-invariant family of non-trivial $Z_{p(A)}$-subspaces of U. Furthermore*

(5.19a) $U = \sum_{I \in \mathcal{I}} I,$
(5.19b) $\mathcal{J} = \{I \in \mathcal{I} \mid [I, P] \neq \{0\}\},$
(5.19c) $J = \sum_{I \in \mathcal{J}} I.$

If $U = \{0\}$ or if

(5.20) *$C_J(P) \neq \{0\}$ and $(D$ on $C_U(P))$ is weakly equivalent to $(D$ on $C_J(P))$,*

then (5.13) *holds.*

Proof. Any irreducible component W of $(PB \text{ on } U)$ is obviously $Z_{p(A)}[PB]$-isomorphic to the dual of an irreducible component Y of $(PB \text{ on } L^+)$. By Proposition 5.6, Y is $Z_{p(A)}[PB]$-isomorphic to an irreducible component of $(PB \text{ on } \bar{A})$. So (5.1g) implies that $(PB \text{ on } Y)$ is ample. Since $B_Y = B_W$, it follows from this and (3.9) that $(PB \text{ on } W)$ is ample, which proves the first statement of the proposition.

The second statement and (5.19a) come directly from the first statement of Proposition 5.8 and (5.9) by duality. Equations (5.19b, c) come from (5.7b, c) by duality.

Since $p(A) \neq p$ (by (5.1e)) we have

$$U = C_U(P) \oplus [U, P],$$

where $[U, P] = C_L(P)^{\perp}$. Similarly

$$J = C_J(P) \oplus [J, P]$$

with $[J, P] = J \cap [U, P] = (C_L(P)N)^{\perp} = (C_L(P) \times [N, P])^{\perp}$. *Since* $J = N^{\perp} = (C_N(P)) \times [N, P]^{\perp}$, it follows that

(5.21a) (D on $C_U(P)$) *is equivalent to the dual of* (D on $C_L(P)^+$),
(5.21b) (D on $C_J(P)$) *is equivalent to the dual of* (D on $C_L(P)^+/C_N(P)^+$).

If $U = \{0\}$, then $L = \{1\}$ and (5.13) holds by Proposition 5.14. If $U \neq \{0\}$ and (5.20) is true, then (5.21) gives $C_L(P) > C_N(P)$ and the weak equivalence of (D on $C_L(P)^+$) with (D on $C_L(P)^+/C_N(P)^+$). So Proposition 5.14 also gives (5.13) in this case, and the proof is complete.

When $L \neq \{1\}$, Proposition 5.18 and (5.1) say that P, B, D, U, $\mathcal{I}$ satisfy:

(5.22a) *PB is the semi-direct product of a group P of prime order p acting on a group $B \in \mathfrak{A}$.*
(5.22b) *D is a subgroup of $C_B(P)$.*
(5.22c) *U is a non-zero finite-dimensional $Z_r[PB]$-module, for some prime $r \neq p, p(B)$.*
(5.22d) *$\mathcal{I}$ is a PB-invariant family of non-zero Z_r-subspaces of U.*
(5.22e) $U = \sum_{I \in \mathcal{I}} I$.

Now we consider arbitrary P, B, D, U, $\mathcal{I}$ satisfying (5.22). We define $\mathcal{J}$ and J by (5.19b, c). From (5.22b, d) it is clear that

(5.23a) *$\mathcal{J}$ is a $P \times D$-invariant subfamily of $\mathcal{I}$,*
(5.23b) *J is a $Z_r[P \times D]$-submodule of U.*

Of course, we are looking for situations in which (5.20) holds.

By (5.22a, c), U is a completely reducible $Z_r[PB]$-module. Let $U_1, \cdots, U_s$ be irreducible $Z_r[PB]$-submodules of U so that

(5.24) $$U = \oplus \sum_{i=1}^{s} U_i.$$

Fix $i = 1, \cdots, s$. Then the projection π_i of U onto U_i determined by the decomposition (5.24) is a $Z_r[BP]$-epimorphism. Define $\mathcal{I}_i$ to be the family

$$\{\pi_i(I) \mid I \in \mathcal{I}, \pi_i(\mathrm{I}) \neq \{0\}\}.$$

Then we have

LEMMA 5.25. *P, B, D, U_i, $\mathcal{I}_i$ satisfy* (5.22), *for each $i = 1, \cdots, s$. If P, B, D, U_i, $\mathcal{I}_i$ satisfy* (5.20), *for all $i = 1, \cdots, s$, then P, B, D, U, $\mathcal{I}$ satisfy* (5.20).

PROOF. Fix $i = 1, \cdots, s$. Conditions (5.22a, b, c) are satisfied by P, B, D, U_i, $\mathcal{I}_i$ by hypothesis. Condition (5.22d) for them comes from the original (5.22d) and the PB-invariance of π_i. The original (5.22e) gives:

$$U_i = \pi_i(U) = \sum_{I \in \mathcal{I}} \pi_i(I) = \sum_{I \in \mathcal{I}, \pi_i(I) \neq \{0\}} \pi_i(I) = \sum_{I_i \in \mathcal{I}_i} I_i.$$

So the first statement of the proposition holds.

For each $i = 1, \cdots, s$, we define $\mathcal{J}_i$ and J_i by (5.19b, c) with $\mathcal{I}_i$ in place of $\mathcal{I}$. If $I_i \in \mathcal{J}_i$ then $I_i \in \mathcal{I}_i$ and $[I_i, P] \neq \{0\}$. By definition of $\mathcal{I}_i$, there is some $I \in \mathcal{I}$ such that $I_i = \pi_i(I)$. Because π_i is P-invariant we have $\{0\} \neq [I_i, P] =$

$\pi_i([I, P])$. Therefore $[I, P] \neq \{0\}$, i.e. $I \in \mathcal{J}$. It follows that

(5.26) $J_i = \sum_{I_i \in \mathcal{J}_i} I_i \leq \sum_{I \in \mathcal{J}} \pi_i(I) = \pi_i(J)$, *for* $i = 1, \cdots, s$.

Suppose that P, B, D, U_i, $\mathcal{I}_i$ satisfy (5.20), for all $i = 1, \cdots, s$. By (5.22c) we must have $s \geq 1$. So $C_{J_1}(P) \neq \{0\}$. Since $(P \text{ on } J)$ is completely reducible (by (5.22c)), it follows from this and (5.26) that $C_J(P) \neq \{0\}$.

Let W be any non-trivial irreducible component of $(D \text{ on } C_U(P))$. By (5.24) there is some $i = 1, \cdots, s$ such that W is $Z_r[D]$-isomorphic to an irreducible component of $(D \text{ on } C_{U_i}(P))$. Since (5.20) holds for $P, B, D, U_i, \mathcal{I}_i$, the module W is $Z_r[D]$-isomorphic to an irreducible component of $(D \text{ on } C_{J_i}(P))$. Then (5.26) and the complete reducibility of $(P \text{ on } J)$ imply that W is $Z_r[D]$-isomorphic to an irreducible component of $(D \text{ on } C_J(P))$. Obviously any irreducible component of $(D \text{ on } C_J(P))$ is an irreducible component of $(D \text{ on } C_U(P))$. Therefore $(D \text{ on } C_U(P))$ is weakly equivalent to $(D \text{ on } C_J(P))$, which proves the lemma.

Now we study the elements I of $\mathcal{I} - \mathcal{J}$ and their translates.

LEMMA 5.27. *Let* (5.22) *hold. If* $I \in \mathcal{I} - \mathcal{J}$, *then* $C_B(I)$ *is a* P*-invariant subgroup of* B. *Furthermore, if* $\sigma \in B$, *then* $I\sigma \in \mathcal{I} - \mathcal{J}$ *if and only if* $\sigma^{\pi-1} \in C_B(I)$ *for all* $\pi \in P$.

PROOF. By (5.19b), P centralizes I. Therefore $C_B(I)$ is P-invariant.

If $\sigma \in B$, then $I\sigma \in \mathcal{I}$ by (5.22d). By (5.19b), $I\sigma \in \mathcal{I} - \mathcal{J}$ if and only if P centralizes $I\sigma$, i.e., if and only if $y\sigma\pi = y\sigma$, for all $y \in I$, $\pi \in P$. Since $I \in \mathcal{I} - \mathcal{J}$, we know that $y = y\pi^{-1}$. Hence $I\sigma \in \mathcal{I} - \mathcal{J}$ if and only if $y\pi^{-1}\sigma\pi = y\sigma$, for all $y \in I$, $\pi \in P$, i.e., if and only if $y\sigma^{\pi-1} = y$, for all $y \in I$, $\pi \in P$. So the lemma is true.

The following lemma is the key to proving (5.20):

LEMMA 5.28. *Let* (5.22) *hold. Suppose that* U_D *is a primary* $Z_r[D]$*-module. Assume that there is some* $I \in \mathcal{I} - \mathcal{J}$, *and some* $\sigma \in B$ *satisfying*

(5.29) *for each* $\pi \in P - \{1\}$, *there exists* $\rho \in P - \{1\}$ *such that* $\sigma^{(\pi-1)(\rho-1)} \notin C_B(I)$.

Then (5.20) *holds and* $C_J(P) \neq \{0\}$.

PROOF. By (5.22d) we may choose some $y \in I - \{0\}$. Our hypotheses give us an element $\sigma \in B$ satisfying (5.29). We define $u \in U$ by

$$u = y\sigma(\textstyle\sum_{\pi \in P} \pi).$$

Clearly u is centralized by P.

By (5.19b), P centralizes I. Therefore $y\sigma\pi = y\pi^{-1}\sigma\pi = y\sigma^{\pi}$, for all $\pi \in P$. If $\sigma^{\pi-1} \in C_B(I)$, for any $\pi \in P - \{1\}$, then the P-invariance of $C_B(I)$ (by Lemma 5.27) implies that $\sigma^{(\pi-1)(\rho-1)} \in C_B(I)$, for all $\rho \in P - \{1\}$. This contradicts (5.29). So $\sigma^{\pi-1} \notin C_B(I)$, for all $\pi \in P - \{1\}$. Since P is abelian and $C_B(I)$ is P-invariant, we have $\sigma^{\rho(\pi-1)} = (\sigma^{\pi-1})^{\rho} \notin C_B(I)$, for all $\rho \in P$, $\pi \in P - \{1\}$. Because $P \neq \{1\}$, this and Lemma 5.27 imply that $I\sigma^{\rho} \in \mathcal{J}$, for all $\rho \in P$. Hence $u = \sum_{\pi \in P} y\sigma^{\pi} \in J$ (by (5.19c)). If $u \neq 0$, we conclude that $C_J(P) \neq \{0\}$.

Suppose that $u = 0$. Then

$$0 = u\sigma^{-1} = y + \sum_{\pi \in P - \{1\}} y\sigma^{\pi - 1}$$

Condition (5.29) and Lemma 5.27 tell us that $I\sigma^{\pi-1} \in \mathcal{J}$, for all $\pi \in P - \{1\}$. Hence

$$y = -\sum_{\pi \in P - \{1\}} y\sigma^{\pi-1} \in J.$$

But y is non-zero and centralized by P. Therefore $C_J(P) \neq \{0\}$ in all cases.

Since (D on U) is primary, any two non-trivial $Z_r[D]$-submodules of U are weakly $Z_r[D]$-equivalent. Clearly $C_J(P) \neq \{0\}$ implies $C_U(P) \neq \{0\}$. So $C_J(P)$, $C_U(P)$ are two non-zero $Z_r[D]$-submodules of U and the lemma is true.

To obtain some information about $C_B(I)$ we use the following technical lemma:

LEMMA 5.30. *Suppose that* (5.22a, c) *holds and that* (PB on U) *is irreducible. Let E be a P-invariant subgroup of $Z(B)$ and Y be any non-trivial $Z_r[PE]$-submodule of U. Then* Ker (E on U) = Ker (E on Y).

Proof. Clifford's theory (see Theorem V, 17.3 of [4]) and (5.22a) give us a primary $Z_r[B]$-submodule W of U such that

$$U = \sum_{\pi \in P} W\pi. \tag{5.31}$$

Let X be an irreducible $Z_r[E]$-submodule of W. Since E is central in B and W is $Z_r[B]$-primary, the module W_E is $Z_r[E]$-primary. Because P normalizes E, each $W\pi, \pi \in P$, is $Z_r[E]$-primary with $X\pi$ as an irreducible $Z_r[E]$-submodule. It follows from this and (5.31) that every irreducible component of (E on U) is $Z_r[E]$-isomorphic to $X\pi$, for some $\pi \in P$.

Since $Y \neq \{0\}$, it has an irreducible $Z_r[E]$-component which, by the above argument, must be isomorphic to $X\pi$, for some $\pi \in P$. By Y is a $Z_r[PE]$-submodule. Therefore it must contain irreducible $Z_r[E]$-components isomorphic to $X\pi\pi_0$, for all $\pi_0 \in P$. By the above argument it can contain no other irreducible $Z_r[E]$-components. Since (E on Y) is completely reducible (by (5.22c)), we conclude that:

$$\text{Ker } (E \text{ on } Y) = \bigcap_{\pi \in Q} \text{Ker } (E \text{ on } X\pi).$$

Obviously this expression is independent of Y, which proves the lemma.

We use this to prove

LEMMA 5.32. *Suppose that* (5.22) *holds and that* (PB on U) *is irreducible. Then* $C_B(I) \cap Z(B) =$ Ker ($Z(B)$ on U), *for any* $I \in \mathcal{I} - \mathcal{J}$.

Proof. By Lemma 5.27, $C_B(I)$ is P-invariant. Hence $E = C_B(I) \cap Z(B)$ is a P-invariant subgroup of $Z(B)$ which centralizes I. It follows from this and (5.22d) that I is a non-trivial $Z_r[PE]$-submodule of U such that $E = \operatorname{Ker}(E \text{ on } I)$. Since $(PB \text{ on } U)$ is irreducible, Lemma 5.30 implies that $E = \operatorname{Ker}(E \text{ on } U) \leq \operatorname{Ker}(Z(B) \text{ on } U)$. Since $\operatorname{Ker}(Z(B) \text{ on } U)$ is obviously contained in $E = C_B(I) \cap Z(B)$, this proves the lemma.

Now we can establish (5.20) in a substantial case:

LEMMA 5.33. *If* (5.22) *holds,* $(PB \text{ on } U)$ *is irreducible, and*

$$[Z(B_U), P, P] \neq \{1\},$$

then (5.20) *is true.*

Proof. Obviously B_U is non-trivial. Therefore it lies in $\mathfrak{A}$ (by (1.5)). It follows easily that P, B_U, D_U, U and $\mathcal{I}$ satisfy (5.22). Clearly (5.20) holds for $P, B, D, U, \mathcal{I}$ if and only if it holds for $P, B_U, D_U, U, \mathcal{I}$. So we may replace B, D by B_U, D_U, respectively, and assume that $(B \text{ on } U)$ is faithful.

Let $B_1 = Z(B)\cdot D$. This is a P-invariant non-trivial subgroup of B. So it lies in $\mathfrak{A}$. It follows that $P, B_1, D, U, \mathcal{I}$ satisfy (5.22). Of course, $(PB_1 \text{ on } U)$ need not be irreducible. Let $U_1, \cdots, U_s$ be irreducible $Z_r[PB_1]$-submodules of U so that (5.24) holds. Lemma 5.25 tells us that we need only verify (5.20) for $P, B_1, D, U_i, \mathcal{I}_i$, for each $i = 1, \cdots, s$.

Let ρ be a generator of the cyclic group P. Since $Z(B)$ is abelian, we have $\{1\} \neq [Z(B), P, P] = Z(B)^{(\rho-1)^2}$. So we may choose an element $\sigma \in Z(B)$ such that $\sigma^{(\rho-1)^2} \neq 1$. Because $p = |P|$ is a prime, this condition implies that

$$\sigma^{(\pi-1)(\rho-1)} \neq 1, \quad \textit{for all } \pi \in P - \{1\}. \tag{5.34}$$

Fix $i = 1, \cdots s$. The group PB_1 is the central product of $PZ(B)$ and D (by (5.22b)). Since U_i is an irreducible $Z_r[PB_1]$-module, its restriction $(U_i)_D$ must be a primary $Z_r[D]$-module. Lemma 5.30 with $E = Z(B)$ and $Y = U_i$ tells us that $Z(B)$ acts faithfully on U_i.

Suppose that $\mathcal{J}_i = \mathcal{I}_i$. Then $J_i = U_i$. So (5.20) for $P, B_1, D, U_i, \mathcal{I}_i$ reduces to the condition $C_{U_i}(P) \neq \{0\}$. Since $[Z(B), P] \neq \{1\}$ and $Z(B)$ acts faithfully on U_i, we have $[Z(B_1)_{U_i}, P] \supseteq [z(B)_{U_i}, P] \supset \{1\}$. As in the proof of Proposition 3.10, it follows from this and Corollary 3.3 that $C_{U_i}(P) \neq \{0\}$. Hence (5.20) for P, B_1, D, U_i, I_i holds in this case.

Suppose that $\mathcal{J}_i \subset \mathcal{I}_i$. Choose $I \in \mathcal{I}_i - \mathcal{J}_i$. Since $Z(B)$ acts faithfully on U_i, Lemma 5.32 implies that $C_B(I) \cap Z(B) = \{1\}$. But $\sigma \in Z(B)$. So none of the elements on the left in (5.34) can lie in $C_B(I)$. Now Lemma 5.28 proves (5.20) for P, B_1, D, U_i, I_i in this case.

We conclude that (5.20) holds for $P, B_1, D, U_i, \mathcal{I}_i$ in all cases. As noted above, this is enough to prove the lemma.

We now have enough information to handle the case in which $p \neq p\,,B)$.

PROPOSITION 5.35 *Assume that* (5.22) *holds with* $p \neq p(B)$. *If each irreducible component of* $(PB \text{ on } U)$ *is ample, then* (5.20) *is true.*

Proof. In view of Lemma 5.25, it suffices to prove this proposition under the additional hypothesis that (PB on U) is irreducible. As in the first paragraph of the proof of Lemma 5.33, we may also replace B, D by B_U, D_U and assume that (B on U) is faithful.

If $[Z(B), P] \neq \{1\}$, then $p \neq p(B)$ implies that $[Z(B), P, P] = [Z(B), P] \neq \{1\}$. Since ($B$ on U) is faithful and (PB on U) is irreducible, Lemma 5.33 gives (5.20).

Now suppose that $[Z(B), P] = \{1\}$. By hypothesis, $[B, P] \neq \{1\}$. So Corollary 3.12 tells us that PB is the central product of $P[B, P]$ and $C_B(P)$. In view of (1.3) and (5.22b) it suffices to prove (5.20) under the additional hypothesis that $D = C_B(P)$. Then U_D is a primary $Z_r[D]$-module.

If $\mathcal{J} = \mathfrak{I}$, then $J = U$. Since (PB on U) is ample and irreducible, Proposition 3.10 gives $C_J(P) = C_U(P) \neq \{0\}$, which proves (5.20). Hence we may assume that $\mathcal{J} \subset \mathfrak{I}$.

Let I be an element of $\mathfrak{I} - \mathcal{J}$. By Lemma 5.32, $C_B(I) \cap Z(B) = \{1\}$. It follows that $C_{[B,P]}(I) \cap Z([B, P]) = \{1\}$. So $C_{[B,P]}(I)$ is a proper P-invariant subgroup of $[B, P]$. Let B_1 be a maximal P-invariant subgroup of $[B, P]$ containing $C_{[B,P]}(I)$. Then $[[B, P]/B_1]^+$ is an irreducible $Z_{p(B)}[P]$-module such that

$$[[[B, P]/B_1]^+, P] = [[B, P]/B_1]^+.$$

Since P is cyclic of prime order p, it follows that $\bar{\sigma}(\pi - 1) \neq 0$, for all $\bar{\sigma} \in [[B, P]/B_1]^+ - \{0\}$ and all $\pi \in P - \{1\}$. Choose $\sigma \in [B, P]$ so that its image $\bar{\sigma}$ in $[[B, P]/B_1]^+$ is non-zero. Then none of $\sigma^{(\pi-1)(\rho-1)}$, where π, $\rho \in P - \{1\}$, has a nonzero image in $[[B, P]/B_1]^+$. In particular, none of them can lie in $C_B(I)$. So σ satisfies (5.29). Now the hypotheses of Lemma 5.28 are all satisfied. So that lemma completes the proof of this proposition.

For the case $p = p(B)$, we need a few routine technical lemmas.

LEMMA 5.36. *Suppose that* (5.22a) *holds with* $p = p(B)$. *Then*

(5.37) $\quad [[B, P]^i, [B, P]^j] \leq [\Phi(B), P]^{\mathrm{Max}(i+j+1-p,0)}$, *for all* $i, j \geq 0$.

Proof. Suppose that $i = 0$. If $j \leq p - 1$, then Max $(i + j + 1 - p, 0) = 0$ and (5.37) holds. If $j \geq p$, then $[\bar{B}, P]^j = \{0\}$ (since $p = p(B)$), and (5.37) follows from (1.6). So (5.37) is true for $i = 0$.

Suppose that $i > 0$ and that (5.37) is true for all smaller values of i. Let σ be an element of $[B, P]^{i-1}$, τ be an element of $[B, P]^j$, and π be an element of P. Let $\bar{\sigma}$, $\bar{\tau}$ be the images of σ, τ, respectively, in $\bar{B}$. From (1.6) and the bilinearity and P-invariance of f_B we compute

$$\begin{aligned}
[[\sigma, \pi], \tau]^+ &= f_B(\bar{\sigma}(\pi - 1), \bar{\tau}) \\
&= f_B(\bar{\sigma}\pi, \bar{\tau}) - f_B(\bar{\sigma}, \bar{\tau}) \\
&= f_B(\bar{\sigma}, \bar{\tau}\pi^{-1})\pi - f_B(\bar{\sigma}, \bar{\tau})
\end{aligned}$$

$$= f_B(\bar\sigma, \bar\tau\pi^{-1})(\pi - 1) + f_B(\bar\sigma, \bar\tau(\pi^{-1} - 1))$$
$$= [[\sigma, \tau^{\pi^{-1}}]^+, \pi] + [\sigma, [\tau, \pi^{-1}]]^+.$$

By induction the first term lies in

$$[[\Phi(B)^+, P]^{\mathrm{Max}(i+j-p,0)}, P] \leq [\Phi(B)^+, P]^{\mathrm{Max}(i+j+1-p,0)}$$

and the second lies in

$$[\Phi(B)^+, P]^{\mathrm{Max}(i+j+1-p,0)}.$$

Therefore $[[\sigma, \pi], \tau] \in [\Phi(B), P]^{\mathrm{Max}(i+j+1-p,0)}$, for all $\sigma \in [B, P]^{i-1}$, $\tau \in [B, P]^j$, $j \geq 0$. Evidently this proves (5.37) for i and finishes the inductive proof of the lemma.

LEMMA 5.38. *Suppose that* (5.22a) *holds, that* $p = p(B) \geq 5$, *and that* $[\Phi(B), P]^2 = \{1\}$. *For any,* $\rho, \pi \in P - \{1\}$, *the map*

$$\lambda_{\rho,\pi} : \sigma \to \sigma^{(\rho-1)(\pi-1)}$$

is a P-epimorphism of $[B, P]^{p-3}$ *onto* $[B, P]^{p-1}$. *If* B_1 *is a P-invariant subgroup of* $[B, P]^{p-3}$, *then the image* $\lambda_{\rho,\pi}(B_1)$ *is independent of the choice of* $\rho, \pi \in P - \{1\}$.

Proof. First we show that $\lambda_{\rho,\pi}$ is a homomorphism. If $\sigma, \tau \in [B, P]^{p-3}$, we compute

$$\begin{aligned}(\sigma\tau)^{(\rho-1)(\pi-1)} &= [[(\sigma\tau)^\rho(\sigma\tau)^{-1}]^{\pi-1} \\ &= [\sigma^\rho\tau^{\rho-1}\sigma^{-1}]^{\pi-1} \\ &= (\sigma^{\rho-1}\tau^{\rho-1}[\tau^{\rho-1}, \sigma^{-1}])^{\pi-1} \\ &= (\sigma^{\rho-1}\tau^{\rho-1})^\pi[\tau^{\rho-1}, \sigma^{-1}]^{\pi-1}(\sigma^{\rho-1}\tau^{\rho-1})^{-1}.\end{aligned}$$

Since $\tau^{\rho-1} \in [B, P]^{p-2}$ and $\sigma^{-1} \in [B, P]^{p-3}$, it follows from (5.37) that their commutator lies in

$$[\Phi(B), P]^{(p-2)+(p-3)+1-p} = [\Phi(B), P]^{p-4}.$$

This is contained in $[\Phi(B), P]$, since $p \geq 5$. From $[\Phi(B), P]^2 = \{1\}$, we conclude that $[\tau^{\rho-1}, \sigma^{-1}]^{\pi-1} = 1$. So this term may be dropped from the above expression, giving

$$\begin{aligned}(\sigma\tau)^{(\rho-1)(\pi-1)} &= (\sigma^{\rho-1}\tau^{\rho-1})^\pi(\sigma^{\rho-1}\tau^{\rho-1})^{-1} \\ &= \sigma^{(\rho-1)\pi}\tau^{(\rho-1)(\pi-1)}(\sigma^{\rho-1})^{-1}.\end{aligned}$$

Now $(\sigma^{\rho-1})^{-1} \in [B, P]^{p-2}$ and $\tau^{(\rho-1)(\pi-1)} \in [B, P]^{p-1}$. So (5.37) says that their commutator lies in

$$[\Phi(B), P]^{(p-2)+(p-1)+1-p} = [\Phi(B), P]^{p-2}.$$

This is $\{1\}$, since $p \geq 5$ and $[\Phi(B), P]^2 = \{1\}$. Therefore the terms $\tau^{(\rho-1)(\pi-1)}$

and $(\sigma^{\rho-1})^{-1}$ commute in the above expression and we have

$$(\sigma\tau)^{(\rho-1)(\pi-1)} = \sigma^{(\rho-1)(\pi-1)}\tau^{(\rho-1)(\pi-1)},$$

which proves that $\lambda_{\rho,\pi}$ is a homomorphism.

Since P is abelian, $\lambda_{\rho,\pi}$ is P-invariant. Clearly it sends $[B, P]^{p-3}$ into $[B, P]^{p-1}$. Before showing that it is onto (and hence proving the first statement of the proposition), we prove the last statement of the proposition.

Fix π, $\rho \in P - \{1\}$. If i $= 1, \cdots, p - 2$, then π^i and π^{i+1} lie in $P - \{1\}$, since $p = |P|$ is a prime. For $\sigma \in B_1$, we compute:

$$\begin{aligned}\lambda_{\rho,\pi^{i+1}}(\sigma) &= (\sigma^{\rho-1})^{\pi^{i+1}-1} = (\sigma^{\rho-1})^{\pi^{i+1}-\pi+\pi-1}\\ &= \sigma^{\pi(\rho-1)(\pi^i-1)}\sigma^{(\rho-1)(\pi-1)} = \lambda_{\rho,\pi^i}(\sigma^{\pi})\lambda_{\rho,\pi}(\sigma).\end{aligned}$$

Using the P-invariance of B_1, we conclude by induction on i that

$$\lambda_{\rho,\pi^i}(B_1) \leq \lambda_{\rho,\pi}(B_1), \quad \textit{for all} \quad i = 1, \cdots, p - 1.$$

Since $|P| = p$ is a prime, there exists, for each $i = 1, \cdots, p - 1$, some $j = 1, \cdots, p - 1$ such that $\pi = \pi^{ij}$. The above inclusion for π^i, j in place of π, i respectively is just:

$$\lambda_{\rho,\pi}(B_1) \leq \lambda_{\rho,\pi^i}(B_1), \quad \textit{for all} \quad i = 1, \cdots, p - 1,$$

Combined with the original inclusion and the fact that P is cyclic, this proves that $\lambda_{\rho,\pi}(B_1)$ is independent of the choice of $\pi \in P - \{1\}$.

Now we vary ρ. For $i = 1, \cdots, p - 2$ we have

$$\begin{aligned}\lambda_{\rho^{i+1},\pi}(\sigma) &= (\sigma^{\rho^{i+1}-1})^{\pi-1} = [\sigma^{\rho(\rho^i-1)}\sigma^{\rho-1}]^{\pi-1}\\ &= \sigma^{\rho(\rho^1-1)\pi}\sigma^{(\rho-1)(\pi-1)}[\sigma^{\rho(\rho^i-1)}]^{-1}.\end{aligned}$$

We know that $\sigma^{(\rho-1)(\pi-1)} \in [B, P]^{p-1}$ and $[\sigma^{\rho(\rho^i-1)}]^{-1} \in [B, P]^{p-2}$. By (5.37), their commutator lies in

$$[\Phi(B), P]^{(p-1)+(p-2)+1-p} = [\Phi(B), P]^{p-2} = \{1\},$$

since $p \geq 5$ and $[\Phi(B), P]^2 = \{1\}$. So they commute and the above expression becomes

$$\begin{aligned}\lambda_{\rho^{i+1},\pi}(\sigma) &= \sigma^{\rho(\rho^i-1)(\pi-1)}\sigma^{(p-1)(\pi-1)}\\ &= \lambda_{\rho^i,\pi}(\sigma^{\rho})\lambda_{\rho,\pi}(\sigma), \qquad \textit{for all} \quad i = 1, \cdots, p - 2.\end{aligned}$$

As in the above case of π, this is enough to show that $\lambda_{\rho,\pi}(B_1)$ is independent of the choice of $\rho \in P - \{1\}$. So the last statement of the lemma is true.

Obviously $[B, P]^{p-1}$ is generated by its subgroups $\lambda_{\rho,\pi}([B, P]^{p-3})$, where ρ, $\pi \in P - \{1\}$. By the preceding argument all these subgroups are equal to each other and hence to $[B, P]^{p-1}$. Therefore $\lambda_{\rho,\pi}([B, P]^{p-3}) = [B, P]^{p-1}$, for all ρ, $\pi \in P - \{1\}$, which finishes the proof of the lemma.

Now we can handle the case $p = p(B) \geq 5$.

PROPOSITION 5.39. *Suppose that* (5.22) *holds with* $p = p(B) \geq 5$. *If each irreducible component of* (PB *on* U) *is ample and* $D \leq [B, P]^{p-1}$, *then* (5.20) *is true.*

Proof. In view of Lemma 5.25, we may assume that (PB on U) is irreducible. Since (PB on U) is ample, (3.9c) implies that $B_U \neq \{1\}$. So $B_U \in \mathfrak{A}$ (by (1.5)), and we may replace, B, D by B_U, D_U without disturbing our hypotheses, assumptions, or conclusions. I.e., we may assume that (B on U) is faithful.

Since (PB on U) is faithful, irreducible, and ample, (3.9c) implies that $[B,P]^{p-1} \neq \{1\}$. In view of (1.1), we may assume that $D = C_{[B,P]^{p-1}}(P) \neq \{1\}$.

If $[\Phi(B), P, P] \neq \{1\}$, then $[Z(B), P, P] \neq \{1\}$ by (1.4b). Since (B on U) is faithful and (PB on U) is irreducible, Lemma 5.33 gives (5.20) in this case. So we may assume that $[\Phi(B), P, P] = \{1\}$.

Let ρ be any element of $P - \{1\}$. Our assumptions and Lemma 5.38 tell us that $\lambda_{\rho,\rho} : \sigma \to \sigma^{(\rho-1)^2}$ is a P-epimorphism of $[B, P]^{p-3}$ onto $[B, P]^{p-1}$. Since (PB on U) is ample, (3.9c) implies that $[B, P]^{p-1} \neq \{1\}$. Hence $[B, P]^{p-3} \neq \{1\}$. It follows that P, $[B, P]^{p-3}$, D, U, $\mathfrak{I}$ satisfy (5.22) and that we only need prove (5.20) for this quintuple.

Decompose U as in (5.24) into a direct sum of irreducible $Z_r[P[B, P]^{p-3}$-submodules U_i where $i = 1, \cdots, s$. We first consider such an i for which $\mathfrak{J}_i \subset \mathfrak{I}_i$ and $[B_{U_i}, P]^{p-1} \neq \{1\}$.

It follows from (5.37) that

$$[[B, P]^{p-3}, [B, P]^{p-1}] \leq [\Phi(B), P]^{(p-3)+(p-1)+1-p} = [\Phi(B), P]^{p-3}.$$

This is $\{1\}$, since $p \geq 5$ and $[\Phi(B), P]^2 = \{1\}$. Hence $[B, P]^{p-1}$ is central in $[B, P]^{p-3}$. If $I \in \mathfrak{I}_i - \mathfrak{J}_i$, we conclude from Lemma 5.32 that

$$C_{[B,P]^{p-1}}(I) = \text{Ker } ([B, P]^{p-1} \text{ on } U_i).$$

By the choice of U_i, the last group is not equal to $[B, P]^{p-1}$. Hence there exists some $\sigma \in [B, P]^{p-3}$ such that

$$\lambda_{\rho,\rho}(\sigma) \in [B, P]^{p-1} - C_{[B,P]^{p-1}}(I).$$

It follows from the last statement of Lemma 5.38 that the inverse image in $[B, P]^{p-3}$ of $C_{[B,P]^{p-1}}(I)$ under $\lambda_{\pi,\rho}$ is independent of the choice of $\pi \in P - \{1\}$. Therefore

$$\lambda_{\pi,\rho}(\sigma) = \sigma^{(\pi-1)(\rho-1)} \notin C_{[B,P]^{p-3}}(I),$$

for all $\pi \in P - \{1\}$, i.e., σ, I satisfy (5.29).

Because D is a subgroup of $[B, P]^{p-1}$, it is central in $[B, P]^{p-3}$. Because D centralizes P, it is central in $P[B, P]^{p-3}$. Since ($P[B, P]^{p-3}$ on U_i) is irreducible, this implies that (D on U_i) is primary. So Lemma 5.28 says that (5.20) holds for P, $[B, P]^{p-3}$, D, U_i, $\mathfrak{I}_i$.

If $\mathcal{J}_i = \mathcal{I}_i$, for some $i = 1, \cdots, s$, then $J_i = U_i$. So $(D \text{ on } C_{J_i}(P)) = (D \text{ on } C_{U_i}(P))$. If $[B_{U_i}, P]^{p-1} = \{1\}$, for some $i = 1, \cdots, s$, then D centralizes U_i. So $(D \text{ on } C_{J_i}(P))$ is trivially weakly equivalent to $(D \text{ on } C_{U_i}(P))$. This and the above arguments tell us that $(D \text{ on } C_{J_i}(P))$ is weakly equivalent to $(D \text{ on } C_{U_i}(P))$, for all $i = 1, \cdots, s$. As in Lemma 5.25, we conclude that $(D \text{ on } C_J(P))$ is weakly equivalent to $(D \text{ on } C_U(P))$.

Since $(PB \text{ on } U)$ is ample and irreducible, Proposition 3.10 implies that $(D \text{ on } U)$ is weakly equivalent to $(D \text{ on } C_U(P))$, and hence to $(D \text{ on } C_J(P))$. But $D = D_U \neq \{1\}$. Therefore D acts non-trivially on $C_J(P)$, which implies $C_J(P) \neq \{0\}$ and completes the proof of the proposition.

We collect the results of this section in

THEOREM 5.40. *If* (5.1) *holds with either* $p \neq p(B)$ *or* $p = p(B) \geq 5$, *then,* (5.13) *is true.*

Proof. Define U, $\mathcal{I}$ as in Proposition 5.18. By that proposition we may assume that $U \neq \{0\}$. Then we only need prove (5.20).

Evidently P, B, D, U, $\mathcal{I}$ satisfy (5.22). Furthermore, each irreducible component of $(PB \text{ on } U)$ is ample (by Proposition 5.18). If $p \neq p(B)$, then Proposition 5.35 proves (5.20). If $p = p(B) \geq 5$, then $D \leq [B, P]^{p-1}$ by (5.1i) and Proposition 5.39 proves (5.20). Therefore the theorem is true.

6. The case $p = 3$

When $p = p(B) = 3$, Proposition 5.29 does not hold and the arguments of the last section do not suffice. However, in this case our Fitting chain is augmented. So we consider the more complicated situation in which:

(6.1a) *PE is the semi-direct product of a group P of order* 3 *acting on a non-trivial group E of prime power order.*

(6.1b) *PEB is the semi-direct product of PE acting on a group $B \in \mathfrak{A}$.*

(6.1c) *F is a subgroup of $C_E(P)$.*

(6.1d) *$PFBA$ is the semi-direct product of PFB acting on a group $A \in \mathfrak{A}$.*

(6.1e) *V is a finite dimensional $Z_q[PA]$-module, for some prime q.*

(6.1f) $p(E) \neq p(B) = 3 \neq p(A) \neq q$.

(6.1g) $[\Phi(B), E] = \{1\}$.

(6.1h) *$\bar{B}$ is a completely reducible $Z_3[PE]$-module.*

(6.1i) $[\Phi(A), B] = \{1\}$.

(6.1j) *Each irreducible component of $(PB$ on $\bar{A})$ is ample.*

(6.1k) *The representation $(A$ on $V)$ is faithful and weakly FB-invariant.*

We define

$$D = [[B, P]^2, F]. \tag{6.2}$$

Then we have

PROPOSITION 6.3. *It follows from* (6.1a, b, c, g) *and* $p(B) = 3$ *that* $D \leq C_B(P)$. *If all of* (6.1) *holds, then* P, B, A, D, V *satisfy* (5.1) *with* $p = 3$.

Proof. Suppose that (6.1a, b, c, g) hold and $p(B) = 3$. Since $|P| = 3$, we have $[\bar{B}, P]^3 = \{1\}$. From $B \in \mathfrak{A}$ and (1.4b) we conclude that $[B, P]^3 \leq Z(B)$. It follows that, for each $\pi \in P$, the map $\mu : \sigma \to \sigma^{\pi - 1}$ is a homomorphism of $[B, P]^2$ into $\Phi(B)$. It is clear from (6.1b, c) that μ is F-invariant. By (6.1c, g), F centralizes $\mu([B, P]^2) \leq \Phi(B)$. Therefore $D \leq \operatorname{Ker} \mu$, i.e., P centralizes D. This proves the first statement of the proposition.

If all of (6.1) holds, then (5.1a) comes from (6.1a, b), (5.1b) is the first statement of this proposition, (5.1c) comes from (6.1d), (5.1d) from (6.1e), (5.1e) from (6.1f), (5.1f) from (6.1i), (5.1g) from (6.1j), (5.1h) from (6.1k), and (5.1i) from (6.2) since $p = 3$. So the proposition is true.

Now we may define the families $\mathcal{K}$, $\mathcal{L}$, $\mathfrak{M}$, $\mathfrak{N}$ and the subgroups L, N, Q and C as in §5. Furthermore, we define U, $\mathcal{I}$, $\mathcal{J}$, J as in Proposition 5.18.

PROPOSITION 6.4. *D is an F-invariant subgroup of $C_B(P)$. The families $\mathcal{K}$, $\mathcal{L}$, $\mathfrak{M}$ and the subgroup L are PFB-invariant. Hence U is a $Z_{p(A)}[PFB]$-module and $\mathcal{I}$ is a PFB-invariant family. The family $\mathfrak{N}$ and the subgroups N, C and Q are $P \times FD$-invariant. Hence so are $\mathcal{J}$ and J.*

Proof. The first statement follows directly from (6.1a, b, c), (6.2), and Proposition 6.3.

The F-invariance of the families $\mathcal{K}$, $\mathcal{L}$ follows from their definitions and (6.1d, k). They are PB-invariant by Proposition 5.3. Hence they are PFB-invariant. By (5.5), this implies that L is PFB-invariant. This and (5.7a) give the PFB-invariance of $\mathfrak{M}$ and complete the proof of the second statement.

The third statement follows from the second by duality and the definitions of U and $\mathcal{I}$ preceding Proposition 5.18.

Both F and D centralize P by (6.1c) and Proposition 6.3. It follows from this and (5.7b, c) that $\mathfrak{N}$ and N are $P \times FD$-invariant. Furthermore, it follows that $\mathcal{K}_{\text{ample}}$ is $P \times FD$-invariant. Since Q is the intersection of the members of $\mathcal{K}_{\text{ample}}$ (by (3.15)), it is $P \times FD$-invariant. Clearly $C_A(P)$ is $P \times FD$-invariant. So (5.11) implies that C is $P \times FD$-invariant, which completes the proof of the fourth statement.

The last statement follows from the fourth by duality and the definitions of $\mathcal{J}$ and J. Therefore the proposition holds.

Instead of (5.20) we now try to establish

(6.5) *(F on $D_{C_U(P)}$) is weakly equivalent to (F on $D_{C_J(P)}$).*

This has the following consequence:

PROPOSITION 6.6. *If (6.5) holds, then (F on $D_{\bar{A}}$) is weakly equivalent to (F on $D_{\bar{C}}$).*

Proof. By (6.1f) the representations (D on $\bar{A}$) and (D on $\bar{C}$) are both fully reducible. So (5.16) and (5.17) give

(6.7a) $\operatorname{Ker}(D \text{ on } \bar{A}) = \operatorname{Ker}(D \text{ on } C_{\bar{A}}(P))$,

(6.7b) $\operatorname{Ker}(D \text{ on } \bar{C}) = \operatorname{Ker}(D \text{ on } C_{\bar{A}}(P)/C_{\bar{A}}(P) \cap \varphi(Q^+))$.

In particular, Ker (D on $\bar{A}$) $\leq$ Ker (D on $\bar{C}$). Therefore we only need to show that a given non-trivial irreducible component W of (F on $D_{\bar{A}}$) is $Z_3\,[F]$-isomorphic to an irreducible component W of (F on $D_{\bar{C}}$). To simplify the notation we make the definition:

(6.8) "$W \lesssim D_1$" *means* "D_1 *is an* F-*invariant section of* D *and* W *is* $Z_3\,[F]$-*isomorphic to an irreducible component of* (F on D_1)".

The subgroup L is PFB-invariant by Proposition 6.4. Hence so are the natural monomorphism φ of Proposition 5.6 and the submodule $\varphi(L^+)$ of $\bar{A}$. Since (D on $\bar{A}$) is completely reducible, we conclude from this and (6.7) that

(6.9a) Ker (D on $\bar{A}$)

$$= \operatorname{Ker}\ (D \text{ on } C_{\bar{A}}(P)/\varphi(C_L(P)^+)) \cap \operatorname{Ker}\ (D \text{ on } \varphi(C_L(P)^+)).$$

(6.9b) Ker (D on $\bar{C}$)

$$= \operatorname{Ker}\ (D \text{ on } C_{\bar{A}}(P)/\varphi(C_L(P)^+))$$
$$\cap \operatorname{Ker}\ (D \text{ on } \varphi(C_L(P)^+)/\varphi(C_Q(P)^+)).$$

By hypothesis $W \lesssim D_{\bar{A}} = D/\operatorname{Ker}\ (D \text{ on } \bar{A})$. So (6.9a) implies that either $W \lesssim D/\operatorname{Ker}\ (D \text{ on } C_{\bar{A}}(P)/\varphi(C_L(P)^+))$ or $W \lesssim D/\operatorname{Ker}\ (D \text{ on } \varphi(C_L(P)^+))$. In the former case, $W \lesssim D/\operatorname{Ker}\ (D \text{ on } \bar{C}) = D_{\bar{C}}$ by (6.9b), and we are done. So we may assume that the latter case holds.

Proposition 5.6 says that φ is a monomorphism. Hence

$$\operatorname{Ker}\ (D \text{ on } \varphi(C_L(P)^+)) = \operatorname{Ker}\ (D \text{ on } C_L(P)^+).$$

By (5.21a) this is just Ker (D on $C_U(P)$). Therefore

$$W \lesssim D/\operatorname{Ker}\ (D \text{ on } C_U(P)) = D_{C_U(P)}\,.$$

Now (6.5) implies that $W \lesssim D_{C_J(P)} = D/\operatorname{Ker}\ (D \text{ on } C_J(P))$. From (5.21b) we have

$$\operatorname{Ker}\ (D \text{ on } C_J(P)) = \operatorname{Ker}\ (D \text{ on } C_L(P)^+/C_N(P)^+).$$

This contains Ker (D on $C_L(P)^+/C_Q(P)^+$) by (5.10). Since φ is a monomorphism, the last kernel is just Ker (D on $\varphi(C_L(P)^+)/\varphi(C_Q(P)^+)$). Therefore

$$W \lesssim D/\operatorname{Ker}\ (D \text{ on } \varphi(C_L(P)^+)/\varphi(C_Q(P)^+)).$$

By (6.9b) this implies that $W \lesssim D/\operatorname{Ker}\ (D \text{ on } \bar{C}) = D_{\bar{C}}$, which completes the proof of the proposition.

It is clear from (6.1) and Propositions 5.18 and 6.4 that P, F, E, B, U, $\mathfrak{I}$ satisfy

(6.10a) *PE is the semi-direct product of a group P of order* 3 *acting on a non-trivial group E of prime power order.*

(6.10b) *PEB is a semi-direct product of PE acting on a group $B \in \mathfrak{A}$.*

(6.10c) *F is a subgroup of $C_E(P)$.*

(6.10d) *U is a finite-dimensional $Z_r[PFB]$-module, for some prime r.*
(6.10e) *$\mathfrak{I}$ is a PFB-invariant family of non-trivial Z_r-subspaces of U.*
(6.10f) $p(E) \neq p(B) = 3 \neq r$.
(6.10g) $[\Phi(B), E] = \{1\}$.
(6.10h) *$\bar{B}$ is a completely reducible $Z_3[PE]$-module.*
(6.10i) $U = \sum_{I \in \mathfrak{I}} I$.

Now we consider the most general, P, F, E, B, U, $\mathfrak{I}$ satisfying these conditions. We define D by (6.2), $\mathfrak{J}$ by (5.19b) and J by (5.19c). Since (6.10a, b, c, g) are (6.1a, b, c, g), Proposition 6.3 and the definitions of D, $\mathfrak{J}$, J imply

(6.11a) *D is an F-invariant subgroup of $C_B(P)$,*
(6.11b) *$\mathfrak{J}$ is a $P \times FD$-invariant subfamily of $\mathfrak{I}$,*
(6.11c) *J is a $Z_r[P \times FD]$-submodule of U.*

Of course, we are trying to prove (6.5). We first make some preliminary reductions to the "minimal case".

LEMMA 6.12. *Suppose that* (6.5) *holds whenever we assume, in addition to* (6.10), *that*

(6.13) *$\bar{B}$ is an irreducible $Z_3[PE]$-module with $\bar{B} = [\bar{B}, E]$.*
Then (6.5) *always holds when* (6.10) *does.*

Proof. Let (6.10) hold. By (6.10h) there exist irreducible $Z_3[PE]$-submodules $Y_1, \cdots, Y_t$ of $\bar{B}$ so that:

$$\bar{B} = \oplus \sum_{i=1}^{t} Y_i \quad (as \quad Z_3[PE]\text{-}modules). \tag{6.14}$$

For each $i = 1, \cdots, t$, it follows from the normality of E in PE and the irreducibility of (PE on Y_i) that $[Y_i, E]$ is either $\{0\}$ or Y_i. We choose the notation so that $[Y_i, E] = Y_i$, for $i = 1, \cdots, s$, and $[Y_i, E] = \{0\}$, for $i = s + 1, \cdots, t$. Then

(6.15) *the natural map of $\bar{D}$ into $\bar{B}$ is a $Z_3[F]$-isomorphism of $\bar{D}$ onto $\oplus \sum_{i=1}^{s} [[Y_i, P]^2, F]$.*

Indeed, by (6.2) the image of $\bar{D}$ is $[[\bar{B}, P]^2, F]$, which is

$$\oplus \sum_{i=1}^{t} [[Y_i, P]^2, F]$$

by (6.14). Evidently $[[Y_i, P]^2, F] \leq [Y_i, E] = \{0\}$, for $i = s + 1, \cdots, t$. So the image of $\bar{D}$ is $\oplus \sum_{i=1}^{s} [[Y_i, P]^2, F]$.

Since $p(E) \neq 3 = p(B)$ by (6.10f), it follows from (6.2) that $[D, F] = D$. Hence $[\bar{D}, F] = \bar{D}$ and $C_{\bar{D}}(F) = \{0\}$. The kernel of the map in (6.15) is $[D \cap \Phi(B)/\Phi(D)]^+$, which is contained in $C_{\bar{D}}(F)$ by (6.10g). Therefore the kernel is $\{0\}$ and the map, which is obviously F-invariant, satisfies (6.15).

Let B_{1i} be the inverse image in B of Y_i and $B_i = [B_{1i}, E]$ for $i = 1, \cdots, s$. Obviously each B_i is a PE-invariant subgroup of B. Furthermore

(6.16) *the natural map of* $\bar{B}_i$ *into* $\bar{B}$ *is a* $Z_3\,[PE]$*-isomorphism of* $\bar{B}_i$ *onto* Y_i, *for* $i = 1, \cdots, s$.

Indeed, the image of this map is $[Y_i, E] = Y_i$, by the construction. The kernel is $[B_i \cap \Phi(B)/\Phi(B_i)]^+$ which is contained in $C_{\bar{B}_i}(E)$ by (6.10g). But $p(E) \neq p(B) = 3$ implies that $C_{\bar{B}_i}(E) = \{0\}$. So (6.16) holds.

Fix $i = 1, \cdots, s$. It follows easily from (6.10) and (6.16) that both (6.10) and (6.13) hold with B_i in place of B. Let $D_i = [[B_i, P]^2, F]$. Then our hypotheses tell us that

(6.17) $(F$ on $(D_i)_{C_U(P)})$ *is weakly equivalent to* $(F$ on $(D_i)_{C_J(P)})$, *for* i $= 1, \cdots,$ s. Since $J \leq U$ we have

$$\operatorname{Ker}(D \text{ on } C_J(P)) \geq \operatorname{Ker}(D \text{ on } C_U(P)).$$

So (6.5) will follow once we prove that any non-trivial irreducible component W of $(F$ on $D_{C_U(P)})$ is $Z_3\,[F]$-isomorphic to an irreducible component of $(F$ on $D_{C_J(P)})$. For simplicity we adopt the notation (6.8).

It is clear from their definitions and (6.2) that each D_i, $i = 1, \cdots, s$, is a subgroup of D. So each $D_i\Phi(D)$ is an F-invariant normal subgroup of D. The natural image of D_i in $\bar{B}$ is clearly $[[Y_i, P]^2, F]$, for $i = 1, \cdots, s$. It follows from (6.15) that $\prod_{i=1}^{s} (D_i \Phi(D))$ covers $D/\Phi(D) = \bar{D}$. Hence D is the product of its F-invariant normal subgroups $D_i\Phi(D)$, $i = 1, \cdots, s$. Since

$$W \lesssim D_{C_U(P)} = \prod_{i=1}^{s} (D_i \Phi(D))_{C_U(P)},$$

we conclude that there is some $i = 1, \cdots, s$ such that $W \lesssim (D_i \Phi(D))_{C_U(P)}$. By (1.4b), $\Phi(D)$ is central in $D_i\Phi(D)$. By (6.10g) it is centralized by F. Since $(F$ on $W)$ is non-trivial, we must have $W \lesssim (D_i)_{C_U(P)}$. Then (6.17) tells us that $W \lesssim (D_i)_{C_J(P)}$. This implies that $W \lesssim D_{C_J(P)}$, which proves the lemma.

Having reduced $\bar{B}$, we now simplify U.

LEMMA 6.18. *Suppose that* (6.5) *holds whenever we assume, in addition to* (6.10) *and* (6.13) *that*

(6.19) $(PFB$ on $U)$ *is irreducible.*

Then (6.5) *holds whenever* (6.10) *is satisfied.*

Proof. By Lemma 6.12 it suffices to prove (6.5) under the hypotheses that (6.10) and (6.13) hold.

Let $\mathfrak{U}$ be the family of all irreducible $Z_r\,[PFB]$-factor modules of U. We do not know that $(PFB$ on $U)$ is completely reducible, since $p(E)$ may equal r. However, PB is a normal subgroup of PFB by (6.10a, b, c) and $(PB$ on $U)$ is completely reducible, by (6.10f). It follows easily that any irreducible component of $(PB$ on $U)$ is $Z_r\,[PB]$-isomorphic to an irreducible component of $(PB$ on $Y)$, for some $Y \in \mathfrak{U}$. Using the complete reducibility of $(P \times D$ on $U)$, we see from this that any irreducible component of $(D \text{ on } C_U(P))$ is $Z_r[D]$-isomorphic to an irreducible component of $(D \text{ on } C_Y(P))$, for some $Y \in \mathfrak{U}$. It follows that

$$(6.20) \qquad \operatorname{Ker}(D \text{ on } C_U(P)) = \bigcap_{Y \in \mathfrak{U}} \operatorname{Ker}(D \text{ on } C_Y(P)).$$

Suppose that W is a non-trivial irreducible component of $(F \text{ on } D_{C_U(P)})$. We adopt the notation (6.8). Then we need only show that $W \lesssim D_{C_J(P)}$.

By (6.20) there exists some $Y \in \mathfrak{U}$ such that $W \lesssim D_{C_Y(P)}$. Let $\mathcal{I}_Y$ be the family of all non-zero images in Y of elements $I \in \mathcal{I}$. Define $\mathfrak{J}_Y$ and J_Y by (5.19b, c) with Y, $\mathcal{I}_Y$ in place of U, $\mathcal{I}$, respectively. Then P, E, F, B, Y, $\mathcal{I}_Y$ are easily seen to satisfy (6.10), (6.13), and (6.19). By hypothesis, they then satisfy (6.5). So $W \lesssim D/\operatorname{Ker}(D \text{ on } C_{J_Y(P)})$.

It is obvious from (5.19b) that any $I_Y \in \mathfrak{J}_Y$ is the image in Y of some $I \in \mathfrak{J}$. It follows that J_Y is contained in the image of J. Since $(P \text{ on } J)$ is completely reducible (by (6.10f)), this implies that $C_{J_Y}(P)$ is contained in the image of $C_J(P)$. Therefore

$$\operatorname{Ker}(D \text{ on } C_J(P)) \leq \operatorname{Ker}(D \text{ on } C_{J_Y}(P)).$$

From this and $W \lesssim D/\operatorname{Ker}(D \text{ on } C_{J_Y}(P))$, we conclude that

$$W \lesssim D/\operatorname{Ker}(D \text{ on } C_J(P)) = D_{C_J(P)},$$

which proves the lemma.

We get rid of the easy cases by

LEMMA 6.21. *Suppose that* (6.10) *and* (6.19) *hold. If* $[Z(B_U), P, P] \neq \{1\}$, *then* (6.5) *is true.*

Proof. Evidently P, B, D, U and $\mathcal{I}$ satisfy (5.22). Let U_1 be an irreducible $Z_r[PB]$-submodule of U. Since $(PBF \text{ on } U)$ is irreducible (by (6.19)) and PB is a normal subgroup of PFB, any irreducible component of $(PB \text{ on } U)$ is $Z_r[PB]$-isomorphic to $(U_1)\sigma$, for some $\sigma \in F$.

Clearly $[Z(B_U), P, P]$ is an F-invariant subgroup of B_U. If it centralizes U_1, it therefore centralizes each $(U_1)\sigma$, $\sigma \in F$. Hence it centralizes U, contradicting our hypotheses. So

$$[Z(B_{U_1}), P, P] \geq [Z(B_U), P, P]_{U_1} > \{1\},$$

for all irreducible $Z_r[PB]$-submodules U_1 of U. Now Lemmas 5.25 and 5.33 tell us that (5.20) holds. This and the complete reducibility of $(D \text{ on } U)$ imply that $D_{C_U(P)} = D_{C_J(P)}$. So (6.5) holds and the lemma is proved.

We now prove several lemmas under the following hypotheses:

(6.22a) *Conditions* (6.10), (6.13) *and* (6.19) *all hold.*
(6.22b) $\operatorname{Ker}(\Phi(B) \text{ on } U) = \{1\}$.
(6.22c) $\Phi(B) \neq \{1\}$.
(6.22d) $[\Phi(B), P, P] = \{1\}$.

First we draw some routine conclusions.

LEMMA 6.23. *Let (6.22) hold. Then* $Z(B) = \Phi(B) = B'$, *i.e.*, B *is non-abelian and special. The representation* ($C_{\Phi(B)}(P)$ on U) *is faithful, primary, and fully decomposable. The groups* $C_{\Phi(B)}(P)$ *and* $\Phi(B)/[\Phi(B), P]$ *are both cyclic of order* 3. *Finally,* $D \cap \Phi(B) = \{1\}$.

Proof. By (6.13) we have $[B, E] = B$. So $[B/B', E] = B/B'$. Applying the E-invariant epimorphism $\sigma \to \sigma^3$ of B/B' onto $\Phi(B)/B'$ we get $[\Phi(B)/B', E] = \Phi(B)/B'$. This and (6.10g) imply that $\Phi(B) = B'$.

Since $\Phi(B) \leq Z(B) \leq B$ (by (1.4b)) and (PE on $\bar{B}$) is irreducible (by (6.13)), either $Z(B) = B$ or $Z(B) = \Phi(B)$. But $Z(B) = B$ implies $\{1\} = B' = \Phi(B)$, contradicting (6.22c). Hence $Z(B) = \Phi(B)$, and the first statement is true.

It follows from (1.4b) and (6.10g) that $C_{\Phi(B)}(P)$ is central in PFB. Since (PFB on U) is irreducible (by (6.19)), we conclude that ($C_{\Phi(B)}(P)$ on U) is primary and fully decomposable. It is faithful by (6.22b). So the second statement is true.

Since $\Phi(B)$ is a non-trivial elementary 3-group (by (6.22c) and (1.4c)), the second statement implies that $|C_{\Phi(B)}(P)| = 3$. The other half of the third statement follows from this since P is cyclic.

From Lemma 5.36 we obtain

$$[[B, P]^2, [B, P]^2] \leq [\Phi(B), P]^{2+2+1-3} = [\Phi(B), P]^2.$$

This is $\{1\}$ by (6.22d). Therefore $[B, P]^2$ is an abelian subgroup of B. Since $p(B) = 3$ does not divide $|F|$ (by (6.10f)), we have

$$[B, P]^2 = [[B, P]^2, F] \times C_{[B,P]^2}(F).$$

The first factor is D by (6.2). The second contains $[B, P]^2 \cap \Phi(B)$ by (6.10g). Therefore the last statement is proved and the lemma is true.

The next lemma is merely an aide to the following one.

LEMMA 6.24. *Let (6.22) hold. If* $\delta \in D - \{1\}$ *and* τ *is any non-trivial element of* $\langle \delta, C_{\Phi(B)}(P) \rangle$, *then there is some irreducible component* W *of* ($\langle \delta, C_{\Phi(B)}(P) \rangle$ on U) *such that neither* δ *nor* τ *acts trivially on* W.

Proof. Let β be a generator for $C_{\Phi(B)}(P)$ (which is cyclic by Lemma 6.23). Then β is central in B (by (1.4b)), so $\langle \delta, C_{\Phi(B)}(P) \rangle = \langle \delta, \beta \rangle$ is abelian. It follows from the last statement of Lemma 6.23 that $\delta \notin \langle \beta \rangle$. This and (1.4d) imply that $\langle \delta, \beta \rangle = \langle \delta \rangle \times \langle \beta \rangle$ is elementary of order 9.

Let W_0 be any irreducible component of ($\langle \delta, \beta \rangle$ on U). By Lemma 6.23, ($\langle \beta \rangle$ on U) is faithful, primary and fully reducible. Hence

$$\mathrm{Ker}\ (\langle \delta, \beta \rangle \text{ on } W_0) \cap \langle \beta \rangle = \{1\}.$$

But $\langle \delta, \beta \rangle_{W_0}$ must be cyclic. Therefore Ker ($\langle \delta, \beta \rangle$ on W_0) $= \langle \delta\beta^i \rangle$, for some $i = 0, 1, 2$.

The image $\bar{\delta}$ of δ in $\bar{B}$ lies in $C_{\bar{B}}(P)$ by (6.11a). So the map $g : \bar{\sigma} \to f_B(\bar{\delta}, \bar{\sigma})$ is a $Z_3[P]$-homomorphism of $\bar{B}$ into $\Phi(B)^+$. The last statement of Lemma

6.23 implies that $\tilde{\delta} \neq 0$. Then the first statement of that lemma says that $g(\bar{B}) \neq \{0\}$. Since $|P| = 3$, we conclude that $g(\bar{B}) \cap C_{\Phi(B)}(P)^+ \neq \{0\}$. But $C_{\Phi(B)}(P)^+$ is a one-dimensional subspace of $\Phi(B)^+$, by Lemma 6.23. Hence $C_{\Phi(B)}(P)^+ \leq g(\bar{B})$. Therefore there exists some element $\sigma \in B$ such that $[\delta, \sigma] = \beta$.

For each $j = 0, 1, 2$ we have: $[\delta, \sigma^j] = \beta^j$, $[\beta, \sigma^j] = 1$. So σ^j normalizes $\langle \delta, \beta \rangle$. Therefore $W = W_0 \sigma^j$ is also an irreducible component of $(\langle \delta, \beta \rangle$ on $U)$. Furthermore

$$\begin{aligned} \operatorname{Ker}\,(\langle \delta, \beta \rangle \text{ on } W) &= \operatorname{Ker}\,(\langle \delta, \beta \rangle \text{ on } W_0)^{\sigma^j} \\ &= \langle (\delta\beta^i)^{\sigma^j} \rangle = \langle \delta\beta^i[\delta\beta^i, \sigma^j] \rangle = \langle \delta\beta^{i+j} \rangle. \end{aligned}$$

Both τ and δ are non-trivial elements of $\langle \delta, \beta \rangle$. Hence we may choose $j = 0, 1, 2$ so that neither τ nor δ lies in $\langle \delta\beta^{i+j} \rangle$. Then W satisfies the conditions of the lemma.

Now comes the key step.

Lemma 6.25. *Let* (6.22) *hold. Fix a generator π for P. For any $\delta \in D - \{1\}$, suppose that we can find an element $\bar{\tau} \in \bar{B}$ satisfying*

(6.26a) $\bar{\tau}(\pi - 1)^2 = \tilde{\delta}$

(6.26b) $f_B(\bar{\tau}(\pi - 1)^i, \bar{\tau}(\pi - 1)^j) \in C_{\Phi(B)}(P)^+$, *for all* $i, j = 0, 1, 2$, *where $\tilde{\delta}$ is the image of δ in $\bar{B}$. Then* (6.5) *holds.*

Proof. If $(D$ on $C_J(P))$ is faithful, then $D_{C_J(P)} = D_{C_U(P)} = D$ and (6.5) holds. So we need only prove that each element $\sigma \in D - \{1\}$ acts non-trivially on $C_J(P)$.

Fix $\delta \in D - \{1\}$. Choose $\bar{\tau} \in \bar{B}$ satisfying (6.26). Let B_1 be the inverse image in B of the $Z_3[P]$-submodule $\langle \bar{\tau}, \bar{\tau}(\pi - 1), \bar{\tau}(\pi - 1)^2 \rangle$ of $\bar{B}$. Then B_1 is a non-trivial P-invariant subgroup of B containing δ. So P, B_1, $\langle \delta \rangle$, U and $\mathfrak{g}$ satisfy (5.22).

Next we prove

$$\langle \delta \rangle \times \Phi(B) \leq Z(B_1). \tag{6.27}$$

Obviously $\Phi(B) \leq Z(B) \cap B_1 \leq Z(B_1)$. Since $\delta \in D - \{1\}$, the last statement of Lemma 6.23 implies that $\langle \delta, \Phi(B) \rangle = \langle \delta \rangle \times \Phi(B)$. So we need only show that δ is central in B_1. It clearly suffices to prove that $f_B(\bar{\tau}(\pi - 1)^i, \tilde{\delta}) = 0$, for $i = 0, 1, 2$. If $i \geq 1$, then

$$f_B(\bar{\tau}(\pi - 1)^i, \tilde{\delta}) = f_B(\bar{\tau}, \tilde{\delta})(\pi - 1)^i = f_B(\bar{\tau}, \bar{\tau}(\pi - 1)^2)(\pi - 1)^i = 0$$

by (6.26a, b). If $i = 0$, then (6.26) implies that

$$\begin{aligned} f_B(\bar{\tau}, \tilde{\delta}) &= f_B(\bar{\tau}, \bar{\tau}(\pi - 1)^2) = f_B(\bar{\tau}(\pi^{-1} - 1)^2, \bar{\tau}) = f_B(\bar{\tau}(\pi - 1)^2\pi^{-2}, \bar{\tau}) \\ &= f_B(\tilde{\delta}\pi^{-2}, \bar{\tau}) = f_B(\tilde{\delta}, \bar{\tau}), \end{aligned}$$

since $\tilde{\delta} \in C_{\bar{B}}(P)$. But f_B is alternating and $p(B) = 3$ is odd. So $f_B(\bar{\tau}, \tilde{\delta}) = f_B(\tilde{\delta}, \bar{\tau})$ implies $f_B(\bar{\tau}, \tilde{\delta}) = 0$, which finishes the proof of (6.27).

Let τ be an element of B_1 having the image $\bar{\tau}$ in $\bar{B}$. By (6.26), $\tau^{(\pi-1)^2} \equiv \delta \pmod{\Phi(B)}$. If P centralizes $\tau^{(\pi-1)^2}$, we take $\sigma = \tau$. Otherwise, we take $\sigma = \tau^{\pi-1}$. In either case we have

$$(6.28) \quad \sigma^{(\pi^i-1)(\pi-1)} = (\sigma^{(\pi-1)^2})^i \in \langle \delta, C_{\Phi(B)}(P) \rangle - \{1\}, \textit{for } i = 1, 2.$$

Indeed, (6.27) implies that $C_{\langle\delta\rangle\times\Phi(B)}(P) = \langle\delta\rangle \times C_{\Phi(B)}(P)$. If P centralizes $\tau^{(\pi-1)^2}$ then $\tau^{(\pi-1)^2}\delta^{-1} \in \Phi(B) \cap C_B(P)$ implies (6.28) for $i = 1$. If P does not centralize $\tau^{(\pi-1)^2} = \sigma^{\pi-1}$, then (6.27) gives $1 \neq \sigma^{(\pi-1)^2} \in [\Phi(B), P]$. So (6.28) for $i = 1$ follows from (6.22d) in this case. Therefore (6.28) always holds for $i = 1$.

We compute

$$\sigma^{(\pi^2-1)(\pi-1)} = [\sigma^{\pi^2-\pi+\pi-1}]^{(\pi-1)} = (\sigma^{(\pi-1)\pi}\sigma^{\pi-1})^{(\pi-1)} = \sigma^{(\pi-1)\pi^2}\sigma^{(\pi-1)^2}[\sigma^{(\pi-1)\pi}]^{-1}.$$

But $\sigma^{(\pi-1)^2} \in \langle \tau^{(\pi-1)^2}, \Phi(B)\rangle = \langle\delta\rangle \times \Phi(B) \leq Z(B_1)$, by (6.26a) and (6.27). Therefore $\sigma^{(\pi-1)^2}$ commutes with $[\sigma^{(\pi-1)\pi}]^{-1}$, and we have

$$\sigma^{(\pi^2-1)(\pi-1)} = \sigma^{(\pi-1)^2\pi}\sigma^{(\pi-1)^2} = [\sigma^{(\pi-1)^2}]^2,$$

since $\sigma^{(\pi-1)^2}$ is centralized by π. Therefore (6.28) always holds.

By Lemma 6.24, there is an irreducible component W of

$$(\langle \delta, C_{\Phi(B)}(P)\rangle \text{ on } U)$$

such that neither $\sigma^{(\pi-1)^2}$ nor δ acts trivially on W. We decompose U into irreducible $Z_r[PB_1]$-modules as in (5.24). There must be some $i = 1, \cdots, s$, say $i = 1$, such that W is an irreducible component of $(\langle \delta, C_{\Phi(B)}(P)\rangle$ on $U_1)$. By Lemma 5.25, P, B_1, $\langle\delta\rangle$, U_1 and $\mathfrak{I}_1$ satisfy (5.22). Since $\langle\delta\rangle$ is central in PB_1 (by (6.27) and (6.11a)), $(U_1)_{\langle\delta\rangle}$ is a completely reducible primary $Z_r[\langle\delta\rangle]$-module.

The element $\sigma^{(\pi-1)^2} \in [B_1, P]^2$ acts non-trivially on W. So $[B_1, P]^2$ acts non-trivially on U_1. Hence $(PB_1$ on $U_1)$ is ample and irreducible. Therefore $C_{U_1}(P) \neq \{0\}$ by Proposition 3.10.

We wish to prove that $C_{J_1}(P) \neq \{0\}$. If $\mathfrak{J}_1 = \mathfrak{I}_1$, then $J_1 = U_1$ and this follows from the preceding paragraph. So we may assume that $\mathfrak{J}_1 \subset \mathfrak{I}_1$.

Fix $I \in \mathfrak{I}_1 - \mathfrak{J}_1$. By Lemma 5.32 and (6.27) we have

$$\begin{aligned} C_{B_1}(I) \cap \langle \delta, C_{\Phi(B)}(P)\rangle &= \operatorname{Ker}\,(\langle \delta, C_{\Phi(B)}(P)\rangle \text{ on } U_1) \\ &\leq \operatorname{Ker}\,(\langle \delta, C_{\Phi(B)}(P)\rangle \text{ on } W). \end{aligned}$$

We chose W so that $\sigma^{(\pi-1)^2}$ does not lie in the last group. This and (6.28) imply that σ, I satisfy (5.29) with B_1 in place of B. So Lemma 5.28 tells us that $C_{J_1}(P) \neq \{0\}$.

Because δ acts non-trivially on W, it acts non-trivially on U_1. Since $(U_1)_{\langle\delta\rangle}$ is a completely reducible primary $Z_r[\langle\delta\rangle]$-module, this implies that δ acts non-trivially on the non-trivial $Z_r[\langle\delta\rangle]$-submodule $C_{J_1}(P)$. So δ acts non-trivially on $C_J(P)$, which completes the proof of the lemma.

One possibility in (6.22) is now easy to handle.

LEMMA 6.29. *If* (6.22) *holds with* $[\Phi(B), P] = \{1\}$, *then* (6.5) *is true.*

Proof. Let δ be an element of $D - \{1\}$ and $\bar{\delta}$ be its image in $\bar{B}$. By Lemma 6.25, we need only find an element $\bar{\tau} \in \bar{B}$ satisfying (6.26) for some generator π of P. By (6.2), $\bar{\delta} \in [\bar{B}, P]^2 = \bar{B}(\pi - 1)^2$. So there exists some $\bar{\tau} \in \bar{B}$ satisfying (6.21a). Condition (6.26b) obviously holds, since $\Phi(B)^+ = C_{\Phi(B)}(P)^+$. Therefore the lemma is true.

For the other possibility in (6.22) we need more information about the action of P on $\bar{B}$.

LEMMA 6.30. *Let* (6.22) *hold with* $[\Phi(B), P] \neq \{1\}$. *Then* $\bar{B}$ *is a free* $Z_3[P]$*-module.*

Proof. We may choose a finite algebraic extension field Z_3^e of Z_3 so that Z_3^e is a splitting field for all subgroups of PE. Since Z_3 is a finite field, Z_3^e is a normal separable extension of Z_3. We denote by G the Galois group of Z_3^e over Z_3. Then G operates naturally on the extension $Z_3^e \otimes_{Z_3} \bar{B}$ of $\bar{B}$ to a $Z_3^e[PE]$ module, by $(z \otimes \beta)\sigma = (z\sigma) \otimes \beta$, for all $z \in Z_3^e$, $\beta \in \bar{B}$, $\sigma \in G$. There are absolutely irreducible $Z_3^e[PE]$-submodules $\bar{B}_1, \cdots, \bar{B}_t$ of $Z_3^e \otimes \bar{B}$ so that

(6.31a) $Z_3^e \otimes \bar{B} = \bar{B}_1 \oplus \cdots \oplus \bar{B}_t$ *(as* $Z_3^e[PE]$*-modules),*
(6.31b) *for any* $i, j = 1, \cdots, t$, *there exists* $\sigma \in G$ *such that* $(\bar{B}_i)\sigma$ *is* $Z_3^e[PE]$*-isomorphic to* $\bar{B}_j$.

(See Theorem V, 13.13 of [4].)

The subgroup E is normal of prime index $3 = |P|$ in PE. It follows from Clifford's theory, (Theorem V, 17.3 of [4]) that there are two possibilities: either ($Z_3^e[E]$ on $\bar{B}_1$) is irreducible, or $\bar{B}_1$ is induced from some irreducible $Z_3^e[E]$-submodule.

Suppose that ($Z_3^e[E]$ on $\bar{B}_1$) is irreducible. By (6.31b), ($Z_3^e[E]$ on $\bar{B}_i$) is irreducible, for $i = 1, \cdots, t$. These modules are absolutely irreducible by the choice of Z_3^e. It follows from this and Schur's Lemma that

$$\dim_{Z_3^e}(\bar{B}_i \otimes_{Z_3^e} \bar{B}_j/[\bar{B}_i \otimes \bar{B}_j, E]) \leq 1, \quad \textit{for all } i, j = 1, \cdots, t.$$

Since $|P| = 3$, the only one-dimensional $Z_3^e[P]$-module is the trivial one. Therefore P centralizes $(\bar{B}_i \otimes \bar{B}_j/[\bar{B}_i \otimes \bar{B}_j, E])$, for $i, j = 1, \cdots, t$. This and (6.31a) imply that P centralizes

$$(Z_3^e \otimes \bar{B}) \otimes_{Z_3^e} (Z_3^e \otimes \bar{B})/[(Z_3^e \otimes \bar{B}) \otimes_{Z_3^e} (Z_3^e \otimes \bar{B}), E].$$

Hence P centralizes $\bar{B} \otimes_{Z_3} \bar{B}/[\bar{B} \otimes \bar{B}, E]$ But f_B defines a $Z_3[PE]$-homomorphism g of $\bar{B} \otimes \bar{B}$ into $\Phi(B)^+$. By (6.10g), the kernel of g contains $[\bar{B} \otimes \bar{B}, E]$. Therefore P centralizes the image of g. This image is clearly $(B')^+$, which equals $\Phi(B)^+$, by Lemma 6.23. By hypothesis, $[\Phi(B)^+, P] \neq \{0\}$. The contradiction proves that ($Z_3^e[E]$ on $\bar{B}_1$) cannot be irreducible.

We now know that $\bar{B}_1$ is $Z_3^e[PE]$-isomorphic to Y_1^{PE}, for some irreducible $Z_3^e[E]$-module Y_1. Therefore $(\bar{B}_1)_P$ is a free $Z_3^e[P]$-module. Hence so is $Z_3^e \otimes$

$\bar{B}$, by (6.31). It follows easily that $\bar{B}$ is a free $Z_3[P]$-module. Therefore the lemma is true.

We investigate $f_B(\bar{\sigma}, \bar{\tau})$ (mod $[\Phi(B)^+, P]$) more closely.

LEMMA 6.32. *Let* (6.22) *hold. Then the function*

$$h(\bar{\sigma}, \bar{\tau}) = f_B(\bar{\sigma}, \bar{\tau}(\pi - 1)) + f_B(\bar{\tau}, \bar{\sigma}(\pi - 1)) + [\Phi(B)^+, P]$$

is a symmetric, bilinear map of $\bar{B} \times \bar{B}$ *into* $\Phi(B)^+/[\Phi(B)^+, P]$. *Its radical is* $C_{\bar{B}}(P)$. *And the subspace* $[\bar{B}, P]$ *is h-isotropic.*

Proof. The first statement is obvious from the definition of h.

For the second, notice that the map

$$g(\bar{\sigma}, \bar{\tau}) = f_B(\bar{\sigma}, \bar{\tau}) + [\Phi(B)^+, P]$$

is a PE-invariant (by (6.10g)) alternating bilinear map of $\bar{B} \times \bar{B}$ into $\Phi(B)^+/[\Phi(B)^+, P]$. Since $B' = \Phi(B)$ (by Lemma 6.23), the map g is not trivial. So the radical of g is a PE-invariant proper subspace of $\bar{B}$. By (6.13), this radical must be $\{0\}$, i.e., g is non-singular.

For any $\bar{\sigma}, \bar{\tau} \in \bar{B}$ we compute

$$\begin{aligned} \text{(6.33)} \qquad h(\bar{\sigma}, \bar{\tau}) &= g(\bar{\sigma}, \bar{\tau}(\pi - 1)) + g(\bar{\tau}, \bar{\sigma}(\pi - 1)) \\ &= g(\bar{\sigma}, \bar{\tau}(\pi - 1)) + g(\bar{\tau}(\pi^{-1} - 1), \bar{\sigma}) \\ &= g(\bar{\sigma}, \bar{\tau}(\pi - 1)) - g(\bar{\sigma}, \bar{\tau}(\pi^{-1} - 1)) \\ &= g(\bar{\sigma}, \bar{\tau}(\pi^2 - 1)\pi^{-1}), \end{aligned}$$

using the PE-invariance of G and the fact that PE centralizes $\Phi(B)/[\Phi(B), P]$. Since g is non-singular, we conclude that $\bar{\tau}$ lies in the radical of h if and only if $\bar{\tau}(\pi^2 - 1)\pi^{-1} = 0$, i.e., if and only if $\tau \in C_{\bar{B}}(P)$. This is the second statement of the lemma.

If $\bar{\sigma}, \bar{\tau} \in \bar{B}$, then

$$\begin{aligned} g(\bar{\sigma}(\pi - 1), \bar{\tau}(\pi - 1)^2) &= g(\bar{\sigma}, \bar{\tau}(\pi - 1)^2(\pi^{-1} - 1)) \\ &= -g(\bar{\sigma}, \bar{\tau}(\pi - 1)^3\pi^{-1}) = 0, \end{aligned}$$

since $\bar{\tau}(\pi - 1)^3 = \bar{\tau}(\pi^3 - 1) = \bar{\tau}(1 - 1) = 0$. The third statement of the lemma follows directly from this and (6.33). So the lemma is true.

At last we can prove

LEMMA 6.34. *Let* (6.22) *hold with* $[\Phi(B), P] \neq \{1\}$. *Then* (6.5) *is true.*

Proof. By Lemma 6.30, $\bar{B}$ is a free $Z_3[P]$-module. So there is some integer $n > 0$ such that $\bar{B}$ has dimension $3n$, $[\bar{B}, P]$ has dimension $2n$, and $[\bar{B}, P]^2 = C_{\bar{B}}(P)$ has dimension n. Therefore $\bar{B}_1 = \bar{B}/C_{\bar{B}}(P)$ has dimension $2n$.

Lemma 6.32 says that h induces a non-singular symmetric bilinear form h_1 on $\bar{B}_1 \times \bar{B}_1$ to $\Phi(B)^+/[\Phi(B)^+, P]$. The latter space is one-dimensional, by

Lemma 6.23. So we may apply the ordinary theory of quadratic forms to h_1. The subspace $[\bar{B}, P]/[\bar{B}, P]^2$ is h_1-isotropic, by Lemma 6.32, and has dimension n, which is one half the dimension of $\bar{B}_1$. Therefore there is some complementary h_1-isotropic subspace Y_1 such that

$$\bar{B}_1 = Y_1 \oplus [\bar{B}, P]/[\bar{B}, P]^2$$

(see Theorem 3.8 of [1]). It follows that the inverse image Y of Y_1 in $\bar{B}$ is h-isotropic and satisfies $Y + [\bar{B}, P] = \bar{B}$.

Now let δ be any element of D, and $\tilde{\delta}$ be the image of δ in $\bar{B}$. By (6.2), $\tilde{\delta} \in [\bar{B}, P]^2$. If π is a generator for P, then $[\bar{B}, P]^2 = \bar{B}(\pi - 1)^2 = (Y + [\bar{B}, P])(\pi - 1)^2 = Y(\pi - 1)^2$, since $[\bar{B}, P](\pi - 1)^2 = \bar{B}(\pi - 1)^3 = \{0\}$. So there exists $\bar{\tau} \in Y$ such that $\tilde{\delta} = \bar{\tau}(\pi - 1)^2$, i.e., so that (6.26a) holds. Because Y is h-isotropic, we have

$$0 = h(\bar{\tau}, \bar{\tau}) = 2f_B(\bar{\tau}, \bar{\tau}(\pi - 1)) + [\Phi(B)^+, P].$$

Hence $f_B(\bar{\tau}, \bar{\tau}(\pi - 1)) \in [\Phi(B)^+, P] \leq C_{\Phi(B)}(P)^+$ (by (6.22d)).

We compute

$$\begin{aligned} f_B(\bar{\tau}, \bar{\tau}(\pi - 1)^2) &\equiv f_B(\bar{\tau}(\pi^{-1} - 1)^2, \bar{\tau}) \pmod{[\Phi(B)^+, P]} \\ &\equiv f_B(\bar{\tau}(\pi - 1)^2\pi^{-2}, \bar{\tau}) \pmod{[\Phi(B)^+, P]} \\ &\equiv -f_B(\bar{\tau}, \bar{\tau}(\pi - 1)^2) \pmod{[\Phi(B)^+, P]}, \end{aligned}$$

since f_B is alternating, P-invariant, and bilinear, and $\bar{\tau}(\pi - 1)^2 \in C_B(P)$. Since 3 is odd, we conclude that $f_B(\bar{\tau}, \bar{\tau}(\pi - 1)^2) \in [\Phi(B)^+, P]$.

Finally, $f_B(\bar{\tau}(\pi - 1), \bar{\tau}(\pi - 1)^2) \in [\Phi(B)^+, P]$ by Lemma 5.36. Therefore (6.26b) holds for $0 \leq i < j \leq 2$. Because f_B is alternating, this proves (6.26b) in all cases. So Lemma (6.25) says that (6.5) holds. This finishes the proof of this lemma.

We collect the results of this section in

THEOREM 6.35. *If* (6.1) *holds, then* (F on $D_{\bar{A}}$) *is weakly equivalent to* (F on $D_{\bar{C}}$).

Proof. By Proposition 6.6, it suffices to show that (6.5) holds whenever (6.10) does. Lemma 6.18 says that it is enough to prove (6.5) when (6.10), (6.13) and (6.19) all hold, i.e., when (6.22a) holds.

The subgroup Ker ($\Phi(B)$ on U) is E-invariant (by (6.10g)) and PB-invariant by (6.10d). It follows that P, F, E, B/Ker ($\Phi(B)$ on U), U, $\mathfrak{g}$ also satisfy (6.22a). Clearly $D_{C_U(P)}$ and $D_{C_J(P)}$ are unchanged when we replace B by B/Ker ($\Phi(B)$ on U). So it suffices to prove (6.5) when (6.22a, b) hold.

If $\Phi(B) = \{1\}$, then B is abelian. So

$$D_U \leq [B_U, P, P] = [Z(B_U), P, P].$$

If $D_U \neq \{1\}$, then (6.5) holds by Lemma 6.21. If $D_U = \{1\}$, then (6.5) is trivial. Therefore it suffices to prove (6.5) when (6.22a, b, c) hold.

If $[\Phi(B), P, P] \neq \{1\}$, then (6.22b) implies that

$$\{1\} \neq [\Phi(B_U), P, P] \leq [Z(B_U), P, P].$$

Again Lemma 6.21 proves (6.5). So it suffices to prove (6.5) when (6.22) holds.

If $[\Phi(B), P] = \{1\}$, Lemma 6.29 proves (6.5). If $[\Phi(B), P] \neq \{1\}$, Lemma 6.34 proves (6.5). Therefore (6.5) holds in all cases and the theorem is true.

7. Proofs of the basic theorems

We shall carry out a large part of the proofs of Theorems 2.6, 2.7 and 2.13 simultaneously. For the first two theorems we assume that (2.5) holds. For Theorem 2.13, we assume in addition that $p = 3$ and that $A_1, \cdots, A_t$ has been extended to an augumented Fitting Chain $A_1, \cdots, A_t, \{B_i\}$ on which H also acts.

We may also assume that $t \geq 3$ for Theorem 2.6, $t \geq 4$ for Theorem 2.7, and $t \geq 6$ for Theorem 2.13. In particular, $t > i_0$, where i_0 is defined by (4.19). In view of (2.5c), Theorem 4.20 gives us an integer j satisfying:

(7.1a) $1 \leq j \leq i_0 < t$.
(7.1b) $p(A_j) \neq p$.
(7.1c) $\{0\} \neq \bar{A}_{j+1,\text{ample}}$ *(defined with respect to* $(PA_j$ on $\bar{A}_{j+1}))$.

This integer j will be fixed throughout this section.

It is convenient to add one more term to our chain. Let q be any prime different from $p(A_t)$. Form the semidirect product $HA_{t-1}A_t$. Let A_{t+1} be the regular $Z_q[HA_{t-1}A_t]$-module written multiplicatively. If $p(A_t) = 3$ and we are proving Theorem 2.13, let $B_{t-1} = A_{t-1}$ and η_{t-1} be the identity isomorphism. Since (A_t on $\bar{A}_{t+1}$) is weakly invariant under Aut (A_t), we easily verify that H, P and $A_1, \cdots, A_{t+1}$ or $A_1, \cdots, A_{t+1}, \{B_i\}$ satisfy the hypotheses of our theorems.

We define subspaces S_i of $\bar{A}_i$ and subgroups E_i of A_i by

(7.2a) $S_j = \bar{A}_j$, $E_j = A_j$,
(7.2b) *S_i is the sum of all ample irreducible $Z_{p(A_i)}[PE_{i-1}]$-submodules of $\bar{A}_i$, for $i = j+1, \cdots, t+1$,*
(7.2c) *$E_i = [X, E_{i-1}]$, where X is the inverse image in A_i of S_i, for $i = j+1, \cdots, t+1$.*

Clearly S_i and E_i are P-invariant whenever E_{i-1} is. So it makes sense to form the group PE_{i-1} in (7.2b).

PROPOSITION 7.3. *Let $i = j+1, \cdots, t+1$. Then both S_i and E_i are HE_{i-1}-invariant. The natural map of $\bar{E}_i$ into $\bar{A}_i$ is an HE_{i-1}-isomorphism of $\bar{E}_i$ onto S_i. Hence $\bar{E}_i$ is a direct sum of ample irreducible $Z_{p(A_i)}[PE_{i-1}]$-submodules. Finally, defining $\bar{A}_{i,\text{ample}}$ by* (3.15) *with respect to $(PE_{i-1}$ on $\bar{A}_i)$, we have that $(E_{i-1}$ on $\bar{A}_{i,\text{ample}})$ is weakly equivalent to $(E_{i-1}$ on $\bar{E}_i)$.*

Proof. By (7.2a), S_j and E_j are H-invariant. Since HE_{i-1} acts on $\bar{A}_i$ and P is a normal subgroup of H, it follows from (3.9) that H permutes the ample irreducible $Z_{p(A_i)}[PE_{i-1}]$-submodules of $\bar{A}_i$ among themselves. Hence HE_{i-1} leaves S_i invariant. By (7.2c), it also leaves E_i invariant. This completes the inductive proof of the first statement.

By (3.9), E_{i-1} acts non-trivially on any ample irreducible $Z_{p(A_i)}[PE_{i-1}]$-submodule V of $\bar{A}_i$. Therefore $[V, E_{i-1}] = V$. We conclude that $S_i = [S_i, E_{i-1}]$ is the image in $\bar{A}_i$ of $\bar{E}_i$. Since $p(A_{i-1}) \neq p(A_i)$ (by (2.2b)) we have

$$[E_i, E_{i-1}] = [X, E_{i-1}]^2 = [X, E_{i-1}] = E_i,$$

where X is as in (7.2c). Furthermore, $\bar{E}_i = [\bar{E}_i, E_{i-1}] \oplus C_{\bar{E}_i}(E_{i-1})$. It follows that $C_{\bar{E}_i}(E_{i-1}) = \{0\}$ But the kernel of the natural map of $\bar{E}_i$ into $\bar{A}_i$ is $[E_i \cap \Phi(A_i)/\Phi(E_i)]^+$, which is contained in $C_{\bar{E}_i}(E_{i-1})$ by (2.2c). Therefore this kernel is $\{0\}$ and the second statement is true.

The third statement follows immediately from the second and (7.2b).

Since $p(A_{i-1}) \neq p(A_i)$, the action (E_{i-1} on $\bar{A}_i$) is completely reducible. It follows that any reducible component of (E_{i-1} on $\bar{A}_{i,\text{ample}}$) is E_{i-1}-isomorphic to one of (E_{i-1} on S_i) and hence one of (E_{i-1} on $\bar{E}_i$). The converse is obvious. So the last statement is true, which proves the proposition.

We define subgroups F_i by

(7.4a) $F_j = \{1\}$,
(7.4b) $F_i = C_{E_i}(P)$ *if* $p(A_i) \neq p$ *and* $i = j + 1, \cdots, t + 1$,
(7.4c) $F_{j+1} = C_{[E_{j+1}, P]^{p-1}}(P)$, *if* $p(A_{j+1}) = p$,
(7.4d) $F_i = [[E_i, P]^{p-1}, F_{i-1}]$, *if* $p(A_i) = p$ *and* $i = j + 2, \cdots, t + 1$.

They satisfy

PROPOSITION 7.5. *For each* $k = j, \cdots, t + 1$, *the subgroup* F_i *of* E_i *is normalized by* H *and centralized by* P. *If* $p(A_i) = p$, *then* $F_i \leq [E_i, P]^{p-1}$. If $i \geq j + 1$, *then* F_{i-1} *normalizes* F_i.

Proof. Since H normalizes P and also each E_i (by Proposition 7.3), it follows easily from (7.4) and induction that H normalizes each F_i.

If F_i is defined by (7.4a, b, c), then it is clearly centralized by P. Assume that (7.4d) holds and that P centralizes F_{i-1}. Then F_{i-1} normalizes $[E_i, P]^{p-1}$. Since P is cyclic of order $p = p(A_i)$, we have $[\bar{E}_i, P]^p = \{0\}$. Hence $[E_i, P]^p \leq \Phi(E_i) \leq Z(E_i)$ (by (2.2a) and (1.4b)). For any $\pi \in P$ we easily compute that $\mu : \sigma \rightarrow \sigma^{\pi-1}$ is an F_{i-1}-homomorphism of $[E_i, P]^{p-1}$ into $\Phi(E_i)$. Since F_{i-1} centralizes $\Phi(E_i) \leq \Phi(A_i)$ (by (2.2c)), we must have

$$F_i = [[E_i, P]^{p-1}, F_{i-1}] \leq \operatorname{Ker} \mu.$$

Therefore P centralizes F_i and the first statement is true.

If (7.4c) holds, then clearly $F_i \leq [E_i, P]^{p-1}$. If (7.4d) holds, then F_{i-1}

normalizes $[E_i, P]^{p-1}$ by the above argument. So the second statement is true.

If $i \geq j + 1$, then F_{i-1} centralizes P and normalizes E_i (by Proposition 7.3). The third statement follows directly from this and (7.4). So the proposition is true.

The sections D_i are defined in most cases by

(7.6) $D_i = (F_i)_{\bar{E}_{i+1}}$, *for* $i = j + 1, \cdots, t$, *unless* $p(A_i) = p = 3$.

From Theorem 5.40 we get

PROPOSITION 7.7. *Suppose that* $p(A_i) \neq p$ *and* $E_i \neq \{1\}$, *for some* $i = j + 1, \cdots, t$. *Then* D_i *is* F_{i-1}*-invariant. If either* $p \neq p(A_{i-1})$ *or* $p = p(A_{i-1}) \geq 5$, *then* $D_i \neq \{1\}$ *and* $(F_{i-1}$ on $\bar{E}_i)$ *is weakly equivalent to* $(F_{i-1}$ on $\bar{D}_i)$.

Proof. It is obvious from (7.2) that $E_i \neq \{1\}$ implies $E_{i-1} \neq \{1\}$. From the first statement of Proposition 7.3 we see that P, $B = E_{i-1}$, $A = E_i$ and $V = \bar{A}_{i+1}$ satisfy (5.1a, c, d). Proposition 7.5 says that $D = F_{i-1}$ satisfies (5.1b, i). Condition (5.1e) comes from (2.2b), since $p \neq p(A_i)$. Condition (5.1f) comes from (2.2c), condition (5.1g) from Proposition 7.3, and condition (5.1h) from (2.2d, e).

By (7.4b) the section C of (5.11) is

$$F_i/F_i \cap \operatorname{Ker}\, (F_i \text{ on } \bar{A}_{i+1,\text{ample}}).$$

Proposition 7.3 implies that $(F_i$ on $\bar{A}_{i+1,\text{ample}})$ is weakly equivalent to $(F_i$ on $\bar{E}_{i+1})$. Since $p(A_i) \neq p(A_{i+1})$ (by (2.2b)), we conclude that

$$\operatorname{Ker}\, (F_i \text{ on } \bar{A}_{i+1,\text{ample}}) = \operatorname{Ker}\, (F_i \text{ on } \bar{E}_{i+1}).$$

So (7.6) says that $C = D_i$.

Now (5.12) tells us that D_i is F_{i-1}-invariant. The last statement of the proposition comes from Theorem 5.40. So the proof is complete.

We can now finish the definition of D_i. Suppose that $i = j + 1, \cdots, t - 1$, and $p(A_i) = p = 3$. Then $p(A_{i+1}) \neq p$ by (2.2b). So D_{i+1} is defined by (7.6). If F_i centralizes E_{i+1}, then it centralizes D_{i+1}. If F_i does not centralize E_{i+1}, then $E_{i+1} \neq \{1\}$ and F_i normalizes D_{i+1} by Proposition 7.7. Since F_i normalizes D_{i+1} in both cases, we may define

(7.8a) $D_i = (F_i)_{\bar{D}_{i+1}}$, *if* $i = j + 1, \cdots, t - 1$ *and* $p(A_i) = p = 3$,
(7.8b) $D_t = F_t$, *if* $p(A_t) = p = 3$.

The result corresponding to Proposition 7.7 for $p(A_i) = p$ is

PROPOSITION 7.9. *Suppose that* $p(A_i) = p \geq 3$ *for some* $i = j + 1, \cdots, t$. *If* $i > j + 1$, *we also assume that* $D_{i-1} \neq \{1\}$. *Then* $\operatorname{Ker}\, (F_i$ on $\bar{E}_{i+1}) = \{1\}$, $F_i \neq \{1\}$, *and* $(F_{i-1}$ on $\bar{E}_i)$ *is weakly equivalent to* $(F_{i-1}$ on $\bar{F}_i)$.

Proof. $(E_i$ on $\bar{A}_{i+1})$ is faithful by (2.2d). It follows from this and (3.9c) that $([E_i, P]^{p-1}$ on $\bar{A}_{i+1,\text{ample}})$ is faithful. By Proposition 7.3, $([E_i, P]^{p-1}$ on $\bar{A}_{i+1,\text{ample}})$ is weakly equivalent to $([E_i, P]^{p-1}$ on $\bar{E}_{i+1})$. Since $p(A_i) \neq p(A_{i+1})$, this implies that $([E_i, P]^{p-1}$ on $\bar{E}_{i+1})$ is faithful. Proposition 7.5 says that $F_i \leq [E_i, P]^{p-1}$. Hence Ker $(F_i$ on $\bar{E}_{i+1}) = \{1\}$.

Suppose that $i = j + 1$. Then $\bar{A}_{j+1,\text{ample}} \neq \{0\}$ by (7.1c). It follows from Proposition 7.3 that $\bar{E}_{j+1} \neq \{0\}$. Since $\bar{E}_{j+1}$ is a sum of ample irreducible PE_j-submodules (by Proposition 7.3) and $p(A_j) \neq p$ (by (2.2b)), Proposition 3.10 implies that $[\bar{E}_{j+1}, P]^{p-1} \neq \{0\}$. This and (7.4c) give $F_{j+1} \neq \{1\}$. Since $F_j = \{1\}$ (by (7.4a)), this proves the proposition for $i = j + 1$.

Suppose that $i > j + 1$. Since $\bar{E}_i$ is a sum of ample irreducible PE_{i-1}-modules and $F_{i-1} = C_{\bar{E}_{i-1}}(P)$, Proposition 3.10 says that $(F_{i-1}$ on $\bar{E}_i)$ is weakly equivalent to $(F_{i-1}$ on $[\bar{E}_i, P]^{p-1})$ and hence to

$$(F_{i-1} \text{ on } [[\bar{E}_i, P]^{p-1}, F_{i-1}]).$$

We know that F_{i-1} centralizes $F_i \cap \Phi(E_i)$ by (2.2c). It follows from this and (7.4d) that $(F_{i-1}$ on $\bar{E}_i)$ is weakly equivalent to $(F_{i-1}$ on $\bar{F}_i)$. By (2.2b), $p(A_{i-1}) \neq p(A_i) = p$. Therefore $D_{i-1} = (F_{i-1})_{\bar{E}_i} = (F_{i-1})_{\bar{F}_i}$. Hence $\bar{F}_i \neq \{0\}$, which finishes the proof of the proposition.

In the case of Theorem 2.13 we must also define sections C_i of B_i. They are given by

(7.10) $C_i = C_X(P)$, *where X is the inverse image in B_i of F_i, for all relevant* $i = j + 1, \cdots, t - 2$.

Since η_i is a P-epimorphism of B_i onto A_i and $p(A_i) \neq p = 3$, we conclude from (7.4b) that

(7.11) $\eta_i(C_i) = F_i$, *for all relevant* $i = j + 1, \cdots, t - 2$.

Theorem 6.35 will give us

PROPOSITION 7.12. *Suppose that $p(A_i) = p = 3$ and $D_{i-1} \neq \{1\}$, for some $i = j + 2, \cdots, t - 1$. Then F_{i-1} normalizes D_i, C_{i-1} normalizes D_{i+1}, and $(F_{i-1}$ on $\bar{E}_i)$ is weakly equivalent to $(F_{i-1}$ on $\bar{D}_i)$.*

Proof. Proposition 7.9 tells us that $F_i \neq \{1\}$ and Ker $(F_i$ on $\bar{E}_{i+1}) = \{1\}$. Hence $E_i \neq \{1\}$ and $E_{i+1} \neq \{1\}$. Let E be the inverse image in B_{i-1} of E_{i-1}, B be E_i, F be C_{i-1}, A be E_{i+1} and V be $\bar{A}_{i+2}$. The augmentation tells us that $PC_{i-1} E_i$ acts on A_{i+1}. Since C_{i-1} centralizes P, it must leave S_{i+1} and E_{i+1} invariant (by (3.9) and (7.2)). Hence E_{i+1} is $PC_{i-1} E_i$-invariant, and (6.1a–e) hold. Condition (6.1f) comes from (2.2b), since $p(A_i) = 3$. Conditions (6.1g, i) come from (2.2c) Conditions (6.1h, j) come from Proposition 7.3. Finally, condition (6.1k) comes from (2.2e) and (2.10c).

Evidently (7.11) implies that the group D of (6.2) is the group F_i defined by (7.4d). It follows from Proposition 7.3 that the section C of (5.11) is D_{i+1} (see the proof of Proposition 7.7). So (7.8a) becomes $D_i = D_{\bar{C}}$.

Proposition 6.4 says that C_{i-1} normalizes D_{i+1} and F_i. Hence it normalizes D_i. This and (7.11) imply that F_{i-1} normalizes D_i.

Since F_{i-1} centralizes $\Phi(D_i)$, the action (F_{i-1} on $\bar{D}_i$) is weakly equivalent to (F_{i-1} on D_i). Theorem 6.35 and (7.11) say that (F_{i-1} on D_i) is weakly equivalent to (F_{i-1} on $(F_i)_{\bar{E}_{i+1}}$). By Proposition 7.9, the group $(F_i)_{\bar{E}_{i+1}}$ is just F_i. Since F_{i-1} centralizes $\Phi(F_i)$ (by (2.2c)), (F_{i-1} on F_i) is weakly equivalent to (F_{i-1} on $\bar{F}_i$). This, in turn, is weakly equivalent to (F_{i-1} on $\bar{E}_i$) by Proposition 7.9. So the proposition is true.

We must return to the techniques of §5 to prove

PROPOSITION 7.13. *If $p(A_{j+1}) = p = 3$, then $D_{j+1} \neq \{1\}$.*

Proof. If $j + 1 = t$, this is clear from (7.8b) and Proposition 7.9. So we may assume that $t > j + 1$.

Proposition 7.9 says that $F_{j+1} \neq \{1\}$ and Ker (F_{j+1} on $\bar{E}_{j+2}$) $= \{1\}$. Hence $E_{j+2} \neq \{1\}$. As in the proof of Proposition 7.7, this implies that P, $B = E_{j+1}$, $A = E_{j+2}$, $D = F_{j+1}$ and $V = \bar{A}_{j+3}$ satisfy (5.1) with $p(B) = p = 3$ and $q = p(A_{j+3})$. Furthermore, the section C of (5.11) is D_{j+2}. Since D_{j+1} is given by (7.8a) and F_{j+1} by (7.4c) we are reduced to proving

(7.14) *Suppose that* (5.1) *holds with $p = p(B) = 3$ and $D = C_{[B,P]^2}(P)$. If $D = D_{\bar{A}} \neq \{1\}$, then D does not centralize $\bar{C}$.*

Next we pass to the situation (5.22). Let the hypotheses of (7.14) hold with $D_{\bar{C}} = \{1\}$. Then (5.16) and $p(B) \neq p(A)$ imply that D centralizes $C_{\bar{A}}(P)/C_{\bar{A}}(P) \cap \varphi(Q^+)$. This and (5.10) imply that D centralizes $C_{\bar{A}}(P)/C_{\bar{A}}(P) \cap \varphi(N^+)$. On the other hand $D = D_{\bar{A}}$ is faithfully represented on $\bar{A}$ and hence on $C_{\bar{A}}(P)$ by (5.17). So D acts faithfully on $C_{\bar{A}}(P) \cap \varphi(N^+)$. Using the fact that φ is a monomorphism (by Proposition 5.6), we conclude that D acts faithfully on $C_N(P)$ and centralizes $C_L(P)/C_N(P)$. By (5.21), this implies that D centralizes $C_J(P)$ and acts faithfully on $C_U(P)$. So we are reduced to deriving a contradiction from the situation in which

(7.15a) *conditions* (5.22) *hold,*
(7.15b) $p = p(B) = 3$,
(7.15c) $D = C_{[B,P]^2}(P) \neq \{1\}$,
(7.15d) *(D on $C_U(P)$) is faithful,*
(7.15e) *D centralizes $C_J(P)$.*

We define $U_i, \pi_i, \mathcal{I}_i, \mathfrak{J}_i, J_i$, for $i = 1, \cdots, s$, as in (5.24) and Lemma 5.25. Notice that (5.26), $p \neq r$, and (7.15e) imply

(7.16) *D centralizes $C_{J_i}(P) \leq \pi_i(C_J(P))$, for $i = 1, \cdots, s$.*

The next step is to prove

(7.17) $\sigma^{(\pi-1)^2} \notin Z(B) - \{1\}$, *for all $\sigma \in B$, $\pi \in P$.*

Suppose that $\sigma^{(\pi-1)^2} \in Z(B) - \{1\}$, for some $\sigma \in B$, $\pi \in P$. If $\pi = 1$, then

$\sigma^{(\pi-1)^2} = 1$. So $\pi \neq 1$ and $P = \langle\pi\rangle$. Since $p = p(B)$ there exists an integer $n \geq 0$ such that $\sigma^{(\pi-1)^{2+n}} \neq 1$, and $\sigma^{(\pi-1)^{3+n}} = 1$. Replacing σ by $\sigma^{(\pi-1)^n}$, we may assume that $\sigma^{(\pi-1)^3} = 1$. Then

$$\sigma^{(\pi-1)^2} \in C_{[B,P]^2}(P) = D.$$

Since $\sigma^{(\pi-1)^2} \in D - \{1\}$, condition (7.15d) and (5.24) give us an integer $i = 1, \cdots, s$ such that $\sigma^{(\pi-1)^2}$ does not centralize $C_{U_i}(P)$. Hence

$$\sigma^{(\pi-1)^2} \in Z(B) - \text{Ker } (B \text{ on } U_i).$$

Furthermore, this and (7.16) give $J_i < U_i$. So there is some member I_i of $\mathcal{I}_i - \mathcal{J}_i$. Now Lemma 5.32 tells us that $\sigma^{(\pi-1)^2} \notin C_B(I_i)$. Since $\sigma^{(\pi-1)^2} \in Z(B)$, we easily compute that $\sigma^{(\pi^2-1)(\pi-1)} = [\sigma^{(\pi-1)^2}]^2$. This does not lie in $C_B(I_i)$, since $p(B) = 3$. Hence σ and I_i satisfy (5.29). Obviously U_i is a primary $Z_r[\langle\sigma^{(\pi-1)^2}\rangle]$-module. So Lemma 5.28, applied to $\langle\sigma^{(\pi-1)^2}\rangle$, tells us that $(\langle\sigma^{(\pi-1)^2}\rangle$ on $C_{U_i}(P))$ is weakly equivalent to $(\langle\sigma^{(\pi-1)^2}\rangle$ on $C_{J_i}(P))$. This is impossible since $\sigma^{(\pi-1)^2}$ centralizes $C_{J_i}(P)$ but not $C_{U_i}(P)$ and $r \neq p(B)$. Therefore (7.17) holds

From (7.17) we will conclude that

(7.18) $D = [B, P]^2$ *is generated by all* $\sigma^{(\pi-1)^2}$, $\sigma \in B$, $\pi \in P$.

Suppose that $\sigma \in B$, $\pi \in P$, and $\sigma^{(\pi-1)^2} \in [B, P]^2 - D$. Then $\pi \neq 1$ and $P = \langle\pi\rangle$. Hence $\sigma^{(\pi-1)^3} \neq 1$ (by (7.15c)). But $[\bar{B}, P]^3 = \{0\}$, since $p = 3 = p(B)$. Therefore

$$\sigma^{(\pi-1)^3} \in \Phi(B) - \{1\} \subseteq Z(B) - \{1\}$$

(by (1.4b)), which violates (7.17). We conclude that D contains $\sigma^{(\pi-1)^2}$, for all $\sigma \in B$, $\pi \in P$.

The subgroup $[B, P]^2$ is generated by the elements $\sigma^{(\pi-1)^2}$ and $\sigma^{(\pi-1)(\pi^2-1)}$, $\sigma \in B$, $\pi \in P$. The first elements lie in D. For the second we compute

$$\sigma^{(\pi-1)(\pi^2-1)} = \sigma^{(\pi-1)^2\pi}\sigma^{(\pi-1)^2} = [\sigma^{(\pi-1)^2}]^2 \in D,$$

since $\sigma^{(\pi-1)^2} \in D$ is centralized by P. Hence $[B, P]^2 \leq D$. This and (7.15c) give (7.18).

Next we show that

(7.19) $D \cap [\langle\sigma^{(\pi-1)^2}\rangle, B] = \{1\}$, *for all* $\sigma \in B, \pi \in P$.

Suppose that δ is a non-trivial member of $D \cap [\langle\sigma^{(\pi-1)^2}\rangle, B]$, for some $\sigma \in B$, $\pi \in P$. Clearly $\langle\pi\rangle = P$. By (7.15d) and (5.24), we may choose some $i = 1, \cdots, s$ so that δ acts non-trivially on $C_{U_i}(P)$.

Let σ_i, δ_i be the images of σ, δ, respectively in $B_i = B_{U_i}$. Since $\delta \in D \cap B'$ it follows from (7.15c) and (1.4b) that $\langle\delta_i\rangle \leq Z(PB_i) \cap \Phi(B_i)$. Because B_i acts faithfully on the irreducible $Z_r[PB_i]$-module U_i, the subgroup $Z(PB_i) \cap B_i$ is cyclic. From this, $\delta_i \neq 1$, and (1.4c) we conclude that

(7.20) $$\langle\delta_i\rangle = Z(PB_i) \cap \Phi(B_i) = C_{\Phi(B_i)}(P).$$

Since $\delta_i \in [\langle\sigma_i^{(\pi-1)^2}\rangle, B_i]$, there exists an element $\tau \in B_i$ so that $[\sigma_i^{(\pi-1)^2}, \tau] = \delta_i$. Hence $\sigma_i^{(\pi-1)^2} \notin \Phi(B_i)$. It follows from this, (1.4d) and (7.20) that $Y = \langle\sigma_i^{(\pi-1)^2}\rangle \times \langle\delta_i\rangle$ is an elementary abelian τ-invariant subgroup of order 9 in B_i. Clearly

$$\sigma_i^{(\pi^2-1)(\pi-1)} \equiv [\sigma_i^{(\pi-1)^2}]^2 \pmod{\Phi(B_i)}.$$

So $\sigma_i^{(\pi^2-1)(\pi-1)} \neq 1$. Furthermore, (7.15c), (7.18) and (7.20) give

$$\sigma_i^{(\pi^2-1)(\pi-1)}[\sigma_i^{(\pi-1)^2}]^{-2} \in C_{\Phi(B_i)}(P) = \langle\delta_i\rangle.$$

Therefore both $\sigma_i^{(\pi-1)^2}$ and $\sigma_i^{(\pi^2-1)(\pi-1)}$ are non-trivial elements of Y.

Let W be an irreducible $Z_r[Y]$-submodule of U_i. Since δ_i is central in PB_i and (PB_i on U_i) is irreducible, we know that ($\langle\delta_i\rangle$ on U_i) is primary, completely reducible, and non-trivial. Hence ($\langle\delta_i\rangle$ on W) is non-trivial and Ker (Y on W) $= \langle\sigma_i^{(\pi-1)^2}\delta_i^e\rangle$, for some $e = 0, 1, 2$. Because τ normalizes Y, the translate $W\tau^k$ is also an irreducible $Z_r[Y]$-submodule of U_i, for each $k = 0, 1, 2$. But

$$\begin{aligned}\text{Ker } (Y \text{ on } W\tau^k) &= [\text{Ker } (Y \text{ on } W)]^{\tau^k} \\ &= \langle\sigma_i^{(\pi-1)^2}\delta_i^e[\sigma_i^{(\pi-1)^2}\delta_i^e, \tau^k]\rangle = \langle\sigma_i^{(\pi-1)^2}\delta_i^{e+k}\rangle.\end{aligned}$$

Therefore we may choose W so that neither $\sigma_i^{(\pi-1)^2}$ nor $\sigma_i^{(\pi^2-1)(\pi-1)}$ lies in Ker (Y on W).

By (7.15c), (3.9), and the fact that $\delta_i \neq 1$, the action (PB_1 on U_i) is ample. Proposition 3.10 says that we may take $W \leq C_{U_i}(P)$. The complete reducibility of (Y on $C_{U_i}(P)$) gives us a $Z_r[Y]$-submodule X of $C_{U_i}(P)$ so that

$$C_{U_i}(P) = W \oplus X \quad (\textit{as } Z_r[Y]\textit{-modules.})$$

Since ($\langle\delta_i\rangle$ on U_i) is primary, completely reducible, and non-trivial, (7.16) implies that $C_{J_i}(P) = \{0\}$. By (5.22e) and (5.19), this gives

$$C_{U_i}(P) = C_{J_i}(P) + \textstyle\sum_{I_i \in \mathcal{I}_i - \mathcal{J}_i} I_i = \sum_{I_i \in \mathcal{I}_i - \mathcal{J}_i} I_i.$$

Therefore there is some $I_i \in \mathcal{I}_i - \mathcal{J}_i$ whose projection $(I_i + X) \cap W$ is non-trivial. We conclude that $C_Y(I_i) \leq$ Ker (Y on W). In particular, neither $\sigma_i^{(\pi-1)^2}$ nor $\sigma_i^{(\pi^2-1)(\pi-1)}$ lies in $C_Y(I_i)$. Hence σ_i, I_i satisfy (5.29).

Lemma 5.28, applied to $\langle\delta_i\rangle$, tells us that $C_{J_i}(P) \neq \{0\}$, contradicting a statement above. Therefore (7.19) holds.

Now we finish the proof of Proposition 7.13. By (7.15c) and (7.18) there exists some $\sigma \in B$, $\pi \in P$ such that $\sigma^{(\pi-1)^2} \neq 1$. We compute

$$\begin{aligned}\sigma^{(\pi^2-1)(\pi-1)} &= \sigma^{(\pi^2-\pi)\pi}\sigma^{(\pi-1)^2}[\sigma^{\pi^2-\pi}]^{-1} \\ &= \sigma^{(\pi-1)^2\pi}\sigma^{(\pi-1)^2}[\sigma^{(\pi-1)^2}, (\sigma^{\pi^2-\pi})^{-1}].\end{aligned}$$

It follows from this and (7.18) that

$$[\sigma^{(\pi-1)^2}, (\sigma^{\pi^2-\pi})^{-1}] \in D \cap [\langle\sigma^{(\pi-1)^2}\rangle, B].$$

So (7.19) gives

$$(7.21) \qquad [\sigma^{(\pi-1)^2}, \sigma^{\pi-1}] = [\sigma^{(\pi-1)^2}, (\sigma^{\pi^2-\pi})^{-1}\cdot\sigma^{(\pi-1)^2}]^{-1} = 1.$$

Next we compute

$$\begin{aligned}(\sigma^2)^{(\pi-1)^2} &= (\sigma^{\pi}\sigma^{\pi-1}\sigma^{-1})^{(\pi-1)} = ((\sigma^{\pi-1})^2[\sigma^{\pi-1}, \sigma^{-1}])^{(\pi-1)} \\ &= \sigma^{(\pi-1)\pi}\sigma^{(\pi-1)^2}(\sigma^{\pi-1})^{-1}[\sigma^{\pi-1}, \sigma^{-1}]^{(\pi-1)} \\ &= \sigma^{(\pi-1)^2}\sigma^{(\pi-1)^2}[\sigma^{\pi-1}, \sigma^{-1}]^{(\pi-1)} \qquad (by\ (7.21)).\end{aligned}$$

This and (7.18) give $[\sigma^{\pi-1}, \sigma^{-1}]^{(\pi-1)} \in D$. But

$$\begin{aligned}[\sigma^{\pi-1}, \sigma^{-1}]^{(\pi-1)} &= [\sigma^{(\pi-1)\pi}, \sigma^{-\pi}][\sigma^{\pi-1}, \sigma^{-1}]^{-1} \\ &= [\sigma^{(\pi-1)^2}, \sigma^{-\pi}][\sigma^{\pi-1}, \sigma^{-\pi+1}] \\ &= [\sigma^{(\pi-1)^2}, \sigma^{-\pi}].\end{aligned}$$

Therefore (7.19) gives

$$1 = [\sigma^{(\pi-1)^2}, \sigma^{-\pi}]^{-1} = [\sigma^{(\pi-1)^2\pi^{-1}}, \sigma] = [\sigma^{(\pi-1)^2}, \sigma].$$

We conclude from this and (7.21) that $\sigma^{(\pi-1)^2}$ lies in the center of the group

$$B_1 = \langle \sigma, \sigma^{\pi-1}, \sigma^{(\pi-1)^2}, \Phi(B)\rangle.$$

Obviously $\sigma^{(\pi-1)^2} \in D_1 = C_{[B_1,P]^2}(P) \leq D$. Therefore P, B_1, D_1, U and $\mathfrak{g}$ also satisfy (7.15). But $\sigma^{(\pi-1)^2} \in Z(B_1) - \{1\}$ violates (7.17). This final contradiction proves that (7.15) is impossible and that Proposition 7.13 is true.

Proofs of Theorems 2.6 *and* 2.7. In these cases either $p(A_i) \neq p$ or $p(A_i) = p \geq 5$, for all $i = 1, \cdots, t$. So D_i is always defined by (7.6).

We know from (7.1c) and Proposition 7.3 that $E_{j+1} \neq \{1\}$. If $p(A_{j+1}) \neq p$, then Proposition 7.7 implies that $D_{j+1} \neq \{1\}$. If $p(A_{j+1}) = p$, then $D_{j+1} = F_{j+1} \neq \{1\}$ by Proposition 7.9. So $D_{j+1} \neq \{1\}$ in both cases.

Suppose that $D_{i-1} \neq \{1\}$, for some $i = j+2, \cdots, t$. Then $E_i \neq \{1\}$ by (7.6). If $p(A_i) \neq p$, then $D_i \neq \{1\}$ by Proposition 7.7. If $p(A_i) = p$, then $D_i = F_i \neq \{1\}$ by Proposition 7.9. So $D_i \neq \{1\}$ in all cases.

(7.22a) $D_i \neq \{1\}$, *for all* $i = j+1, \cdots, t$.
(7.22b) $E_i \neq \{1\}$, *for all* $i = j+1, \cdots, t$.

Now Proposition 7.9 implies that $D_i = F_i$ whenever $p(A_i) =$ p. From this, Proposition 7.5 and Proposition 7.7 we have

(7.23) F_{i-1} *normalizes* D_i, *for* $i = j+2, \cdots, t$.

In view of (7.22), Propositions 7.7 and 7.9 also tell us that

(7.24) (F_{i-1} *on* $\bar{E}_i$) *is weakly equivalent to* (F_{i-1} *on* $\bar{D}_i$), *for* $i = j+2, \cdots, t$.

We shall use Proposition 2.3 to show that $D_{j+1}, \cdots, D_t$ is a Fitting subchain of $A_{j+1}, \cdots, A_t$. We must verify (2.4).

Condition (2.4a) comes from (7.22a).

If $i = j + 2, \cdots, t$, then F_{i-1} normalizes D_i by (7.23). Since $p(A_{i-1}) \neq p(A_i)$, it follows from (7.24) that

$$\operatorname{Ker}(F_{i-1} \text{ on } \bar{E}_i) = \operatorname{Ker}(F_{i-1} \text{ on } \bar{D}_i) = \operatorname{Ker}(F_{i-1} \text{ on } D_i).$$

This and (7.6) imply that D_{i-1} normalizes D_i and $\operatorname{Ker}(D_{i-1} \text{ on } D_i) = \{1\}$, which are (2.4b, c).

Let $i = j + 3, \cdots, t$. By (2.2e), $(E_{i-1} \text{ on } \bar{A}_i)$ is weakly F_{i-2}-invariant. Since F_{i-2} centralizes P (by Proposition 7.5), it follows easily from (3.9) and (3.15) that $(E_{i-1} \text{ on } \bar{A}_{i,\text{ample}})$ is weakly F_{i-2}-invariant. By Proposition 7.3, $(E_{i-1} \text{ on } \bar{E}_i)$ is weakly F_{i-2}-invariant. Hence $(F_{i-1} \text{ on } \bar{E}_i)$ is weakly F_{i-2}-invariant. By (7.24), $(F_{i-1} \text{ on } \bar{D}_i)$ is weakly F_{i-2}-invariant. Therefore $(D_{i-1} \text{ on } \bar{D}_i)$ is weakly D_{i-2}-invariant, which is (2.4d).

Now $D_{j+1}, \cdots, D_t$ is a Fitting subchain of $A_{j+1}, \cdots, A_t$ by Proposition 2.3. Proposition 7.5 and (7.6) imply that P centralizes each D_i. Since H normalizes each F_i (by Proposition 7.5) and E_{i+1} (by Proposition 7.3), it normalizes each D_i (by (7.6)). From (7.1a) and (4.19a, b) we see that $j + 1 \leq 3$ in the case of Theorem 2.6 and $j + 1 \leq 4$ in the case of Theorem 2.7. So $D_3, \cdots, D_t$ (respectively $D_4, \cdots, D_t$) satisfy the conditions of Theorem 2.6 (Theorem 2.7). This completes the proofs of these theorems.

Proof of Theorem 2.13. In this case $p = 3$.

We know from (7.1c) and Proposition 7.3 that $E_{j+1} \neq \{1\}$. If $p(A_{j+1}) \neq p$, then (7.1b) and Proposition 7.7 give $D_{j+1} \neq \{1\}$. If $p(A_{j+1}) = p = 3$, then $D_{j+1} \neq \{1\}$ by Proposition 7.13. So $D_{j+1} \neq \{1\}$ in both cases.

Suppose that $D_{i-1} \neq \{1\}$, for some $i = j + 2, \cdots, t$. Then $E_i \neq \{1\}$ by (7.6) and (7.8). If $p(A_{i-1}) \neq p \neq p(A_i)$, then $D_i \neq \{1\}$ by Proposition 7.7. If $p(A_{i-1}) = p$, then (7.8a) and $D_{i-1} \neq \{1\}$ imply that $D_i \neq \{1\}$. If $p(A_i) = p$ and $i = t$, then $D_t = F_t \neq \{1\}$ by (7.8b) and Proposition 7.9. If $p(A_i) = p$ and $i < t$, then (7.6) and Proposition 7.12 give

$$\{1\} \neq D_{i-1} = (F_{i-1})_{\bar{E}_i} = (F_{i-1})_{\bar{D}_i}.$$

Hence $D_i \neq \{1\}$ in all cases, so that (7.22) holds.

Now Propositions 7.7, 7.9, and 7.12 tell us that (7.23) holds. In place of (7.24), they now give

(7.25) *$(F_{i-1}$ on $\bar{E}_i)$ is weakly equivalent to $(F_{i-1}$ on $\bar{D}_i)$, for all $i = j + 2, \cdots, t$ such that $p(A_{i-1}) \neq 3$.*

In addition, Proposition 7.12 says that

(7.26) *C_i normalizes D_{i+2}, for each relevant index $i = j + 1, \cdots, t - 2$.*

We shall use Proposition 2.11 to show that $D_{j+1}, \cdots, D_t, \{C_i\}$ is an augmented Fitting subchain of $A_{j+1}, \cdots, A_t, \{B_i\}$. We must verify (2.12).

Condition (2.12a) comes from (7.22a).

If $i = j + 2, \cdots, t$ and $p(A_{i-1}) \neq 3$, then (2.12b, e) come from (7.25) as (2.4b, c) came from (7.24). If $p(A_{i-1}) = 3$, they come from the definition (7.8) and the remarks preceding it.

Condition (2.11c) comes from (7.11) and (7.6).

Condition (2.12d) is (7.26).

Condition (2.12f) comes from (7.25) as (2.4d) came from (7.24).

Let $i = j + 1, \cdots, t - 3$ be a relevant index. It follows from (7.11) and Proposition 7.3 that C_i normalizes E_{i+1}. Since C_i centralizes P (by (7.10)) it must permute the ample irreducible $Z_{p(A_{i+2})}[PE_{i+1}]$-submodules of $\bar{A}_{i+2}$ among themselves. So it leaves S_{i+2} and E_{i+2} invariant. It follows from (2.10c) that (E_{i+2} on $\bar{A}_{i+3}$) is weakly C_i-invariant. Since C_i centralizes P, it follows that (E_{i+2} on $\bar{A}_{i+3,\text{ample}}$) is weakly C_i-invariant. By Proposition 7.3, this implies that (E_{i+2} on $\bar{E}_{i+3}$) is weakly C_i-invariant. Hence so is (F_{i+2} on $\bar{E}_{i+3}$). Since $p(A_{i+2}) \neq p(A_{i+1}) = 3$, it follows from (7.25) that (F_{i+2} on $\bar{D}_{i+3}$) is weakly C_i-invariant. This proves (2.12g).

Now $D_{j+1}, \cdots, D_t, \{C_i\}$ is an augmented Fitting subchain of $A_{j+1}, \cdots, A_t$, $\{B_i\}$ by Proposition 2.11. Proposition 7.5, (7.6), and (7.8) imply that P centralizes each D_i. By (7.10), P centralizes each C_i. Since H normalizes each F_i (by Proposition 7.5) and E_i (by Proposition 7.3), it follows easily from (7.6) and (7.8) that it normalizes each D_i. Since H normalizes each B_i and P, it normalizes each C_i (by (7.10)). From (7.1a) and (4.19c) we see that $j + 1 \leq 6$. Therefore $D_6, D_7, \cdots, D_t, \{C_i\}$ satisfy the conditions of Theorem 2.13. This completes the proof of that theorem.

8. Thompson's conjecture

We first prove Thompson's conjecture in the special case of solvable groups G whose Carter subgroups have a normal complement. We use the following lemma, which was mentioned to me by R. Carter:

LEMMA 8.1. *Let G be a finite solvable group whose Carter subgroups have a normal complement K. Let H be a Carter subgroup of G and L be an H-invariant subgroup of K. Then H normalizes some Sylow system of L.*

Proof. Since H is a Carter subgroup of G and $H \leq HL$, it is a Carter subgroup of HL (see Lemma VI, 7.9 and Theorem VI, 12.2 of [4]). So there is a system normalizer N of HL contained in H (see Theorem VI, 12.8 of [4]). Let C/D be any chief section of H. Then $CL/DL \simeq C/D$ is a chief section of HL, since L is a complement to H in HL. Clearly the nilpotence of H makes CL/DL a central chief section of HL. So it is covered by N (see Theorem 11.10 of Chapter VI of [4]). Hence N covers every chief section C/D of H. Therefore $N = H$. If $\{S_\pi\}$ is a Sylow system of HL normalized by $N = H$, then $\{L \cap S_\pi\}$ is a Sylow system of L normalized by H. So the lemma is true.

To construct H-invariant Fitting chains we use

LEMMA 8.2. *Let K be a finite solvable group and H be a group acting on K and* *leaving fixed some Sylow system of K. Then there exist sections $A_i = C_i/D_i$ of K, for $i = 1, \cdots, h = h(K)$, satisfying:*

(8.3a) $A_i \in \mathfrak{A}$, *for* $i = 1, \cdots, h$.
(8.3b) A_i *is* H*-invariant, for* $i = 1, \cdots, h$.
(8.3c) $p(A_i) \neq p(A_{i+1})$, *for* $i = 1, \cdots, h-1$.
(8.3d) C_i *normalizes* A_j, *for* $1 \leq i \leq j \leq h$.
(8.3e) $D_i = \operatorname{Ker}(C_i \text{ on } A_{i+1})$, *for* $i = 1, \cdots, h-1$.
(8.3f) $D_h = \{1\}$.
(8.3g) $(H \cdot \prod_{i<h} C_i \text{ on } \bar{A}_h)$ *is irreducible.*
(8.3h) $[\Phi(A_{i+1}), C_i] = \{1\}$, *for* $i = 1, \cdots, h-1$.
(8.3i) $C_h \leq F(K)$.

Proof. We use induction on h. If $h = 0$, there is nothing to prove. If $h = 1$, let C_1 be any minimal H-invariant subgroup of K and $D_1 = \{1\}$. The relevant conditions (8.3) are immediately verified in this case.

Now assume that $h > 1$ and that the lemma is true for all smaller values of h. Since $F(K)$ is a characteristic subgroup of K, the group H acts on $K^* = K/F(K)$. The images in K^* of the groups forming an H-invariant Sylow system of K are obviously the members of an H-invariant Sylow system of K^*. For each $x = \text{a, b}, \cdots, \text{i}$, let $(8.3\text{x})^*$ be (8.3x) with K, A_j, C_j, D_j, h replaced by $K^*, A_j^*, C_j^*, D_j^*, h-1$, respectively, for all indices j. Clearly $h(K^*) = h - 1$. So induction gives us sections $A_i^* = C_i^*/D_i^*$ of K^*, for $i = 1, \cdots, h-1$, satisfying $(8.3)^*$.

Let S be a $p(A_{h-1}^*)$-Sylow subgroup of K belonging to a Sylow system fixed by H. Then $S \cap F_2(K)$ is an H-invariant $p(A_{h-1}^*)$-Sylow subgroup of $F_2(K)$, and $N = N_K(S \cap F_2(K))$ is an H-invariant subgroup of K. Considering $S \cap F_2(K)$ as a $p(A_{h-1}^*)$-Sylow subgroup of the normal subgroup $(S \cap F_2(K))F(K)$ of K, we see by the Frattini argument that $NF(K) = K$. We denote by φ the natural epimorphism of N onto $K/F(K) = K^*$.

For each $i = 1, \cdots, h-2$, we define C_i and D_i to be the inverse images under φ of C_i^*, D_i^*, respectively. Since φ defines an H-isomorphism of $N/N \cap F(K)$ onto K^*, we see from $(8.3)^*$ that those parts of condition (8.3) involving only those C_i, D_i and $A_i = C_i/D_i$ with $i \leq h-2$ are all satisfied.

The image $\varphi(S \cap F_2(K))$ is the $p(A_{h-1}^*)$-Sylow subgroup of $F(K^*) = F_2(K)/F(K)$. From $(8.3\text{f}, \text{i})^*$ we see that $C_{h-1}^* \leq \varphi(S \cap F_2(K))$. Let C_{h-1} be the inverse image in $S \cap F_2(K)$ of C_{h-1}^* under φ. Since N is the normalizer of $S \cap F_2(K)$, it follows from $(8.3\text{b, d})^*$ that $H \cdot \prod_{i<h-1} C_i$ normalizes C_{h-1}.

Because $A_{h-1}^* \in \mathfrak{A}$, we have $A_{h-1}^* \neq \{1\}$ (by (1.4a)). It follows that

$$C_{h-1} > S \cap F_2(K) \cap \operatorname{Ker} \varphi = S \cap F(K).$$

So there must exist some prime $p \neq p(A_{h-1}^*)$ such that C_{h-1} does not centralize the p-Sylow subgroup T of $F(K)$. Then $H \cdot \prod_{i<h} C_i$ normalizes T and $[T, C_{h-1}] \neq \{1\}$. The Hall-Higman theory (see Theorem III, 13.5 of [4]) gives us an $H \cdot \prod_{i<h} C_i$-invariant special subgroup C_h of T such that $(H \cdot \prod_{i<h} C_i$

on $\bar{C}_h$) is irreducible, (C_{h-1} on C_h) is non-trivial, and $[\Phi(C_h), C_{h-1}] = \{1\}$. If p is odd, we even have $\exp(C_h) = p$. Hence $C_h \in \mathfrak{A}$.

Now define D_{h-1} by $D_{h-1} = \text{Ker}(C_{h-1} \text{ on } C_h)$. Since $F(K)$ is nilpotent and $p \neq p(A^*_{h-1})$, the subgroup $S \cap F(K)$ centralizes T. So $D_{h-1} \geq S \cap F(K)$. But $S \cap F(K)$ is the kernel of the natural epimorphism of C_{h-1} onto A^*_{h-1} (by (8.3f)*). The image E of D_{h-1} in A^*_{h-1} is evidently $H \cdot \prod_{i<h-1} C^*_i$-invariant and not equal to A^*_{h-1}. By (8.3g)* we must have $D_{h-1} \leq \Phi(A^*_{h-1})$. Defining A_{h-1} to be C_{h-1}/D_{h-1}, we see that φ induces a natural isomorphism of $\bar{A}_{h-1}$ on to $\bar{A}^*_{h-1}$.

Condition (8.3a) for $i = h - 1$ comes from $1 < A_{h-1} \simeq A^*_{h-1}/E$ and (1.5). Condition (8.3b) for $i = h - 1$ comes from the construction of C_{h-1} and D_{h-1}. Condition (8.3c) for $i = h - 2$ comes from (8.3c)*, since $p(A_{h-2}) = p(A^*_{h-2})$ and $p(A_{h-1}) = p(A^*_{h-1})$. Condition (8.3d) for $j = h - 1$ comes from the construction of C_{h-1}, D_{h-1}. Condition (8.3e) for $i = h - 2$ comes from (8.3e)*, since $p(A_{h-2}) \neq p(A_{h-1})$ implies

$$\text{Ker}(A_{h-2} \text{ on } A_{h-1}) = \text{Ker}(A_{h-2} \text{ on } \bar{A}_{h-1})$$

and

$$\text{Ker}(A^*_{h-2} \text{ on } \bar{A}^*_{h-1}) = \text{Ker}(A^*_{h-2} \text{ on } A^*_{h-1}) = \{1\},$$

and φ induces an isomorphism of $\text{Ker}(A_{h-2} \text{ on } \bar{A}_{h-1})$ onto $\text{Ker}(A^*_{h-2} \text{ on } \bar{A}^*_{h-1})$. Finally, condition (8.3h) for $i = h - 2$ comes from (8.3h)*.

Set $D_h = \{1\}$ and $A_h = C_h/D_h$. The constructions of A_h and D_{h-1} give those conditions (8.3) involving A_h, C_h or D_h with no difficulty and complete the inductive proof of the lemma.

Now we can prove the special case of Thompson's conjecture.

THEOREM 8.4. *Let G be a finite solvable group whose Carter subgroups have normal complement K. If H is a Carter subgroup of G, then* $h(K) \leq 5(2^{l(H)} - 1)$.

Proof. By Lemma 8.1, H normalizes a Sylow system of K. So Lemma 8.2 gives us a chain $A_1, \cdots, A_h$, $h = h(K)$, of sections of K satisfying (8.3). By (8.3d, e), A_i normalizes A_{i+1}, for $i = 1, \cdots, h - 1$. We claim that $A_1, \cdots, A_h$ with these actions is a Fitting chain, i.e., that it satisfies (2.2). Indeed, property (2.2a) comes from (8.3a), property (2.2b) from (8.3c), property (2.2c) from (8.3h) and property (2.2d) from (8.3e). Since C_i normalizes both A_{i+1} and A_{i+2} (by (8.3d)), the action (A_{i+1} on $\bar{A}_{i+2}$) is C_i-invariant and therefore A_i-invariant. So (2.2e) holds, and $A_1, \cdots, A_h$ is an H-invariant Fitting chain (by (8.3b)).

Let $i = 1, 2, \cdots, t - 2$ be a relevant index. By Lemma 8.1, H leaves invariant some $p(A_i)$-Sylow subgroup B_i of C_i. Let η_i be the natural epimorphism of B_i onto $A_i = C_i/D_i$. By (8.3d), B_i normalizes

$$A_{i+1}, A_{i+2}, A_{i+3}, \cdots, A_h.$$

This gives as a natural action of B_i on A_{i+2}. Clearly B_i satisfies (2.10a,b).

Since $B_i A_{i+2}$ acts on A_{i+3}, condition (2.10c) is also satisfied if $i \leq h - 3$. The H-invariance of B_i implies that $A_1, \cdots, A_h, \{B_i\}$ is an H-invariant augmented Fitting chain.

Because H is a Carter subgroup of G and $H \cap K = \{1\}$, it centralizes no nontrivial section of K. Furthermore, H is nilpotent. So Theorem 2.14 tells us that $h \leq 5(2^{l(H)} - 1)$, which is this theorem.

At last we have

THEOREM 8.5. *Let H be a Carter subgroup of a finite solvable group G. Then* $h(G) \leq 10(2^{l(H)} - 1) - 4l(H)$.

Proof. By induction on $l = l(H)$. If $l(H) = 0$, then $H = \{1\}$ and $G = \{1\}$. So $h(G) = 0 = 10(2^0 - 1) - 4\cdot 0$, and the theorem is true in this case.

Now assume that $l > 0$, and that the theorem is true for all smaller values of $l(H)$.

Fix a Carter subgroup H of G. The Fitting series satisfies

$$\{1\} = F_0(G) < F_1(G) < \cdots < F_h(G) = G,$$

where $h = h(G)$. So there exists an integer $k \geq 0$ such that

(8.6a) $F_k(G) \cap H = \{1\}$.
(8.6b) $F_{k+1}(G) \cap H \neq \{1\}$.

Let $G_1 = G/F_{k+1}(G)$. The image H_1 of H in G_1 is a Carter subgroup of G_1 (see Lemma VI, 12.3 of [4]). By (8.6b) we have $l(H_1) < l(H)$.So induction gives

$$\begin{aligned} h(G_1) = h - k - 1 &\leq 10(2^{l(H_1)} - 1) - 4l(H_1) \\ &\leq 10(2^{l-1} - 1) - 4(l - 1). \end{aligned} \tag{8.7}$$

The subgroup $G_2 = H \cdot F_k(G)$ contains the Carter subgroup H of G. So H is a Carter subgroup for G_2 (see Lemma VI, 7.9 and Theorem VI, 12.2 of [4]). Clearly (8.6a) says that $F_k(G)$ is a normal complement to H in G_2. From Theorem 8.4 we conclude that $h(F_k(G)) = k \leq 5(2^l - 1)$. Adding this to (8.7) we get

$$\begin{aligned} h &= 1 + k + (h - k - 1) \\ &\leq 1 + 5(2^l - 1) + 10(2^{l-1} - 1) - 4(l - 1) = 10(2^l - 1) - 4l. \end{aligned}$$

So the theorem is true.

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