ON A CONJECTURE OF MILNOR

BY

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Introduction

Let X denote a connected space dominated by a finite CW-complex and let $\pi = \pi_1(X)$. In his paper *Finiteness conditions for CW-complexes* [6], C. T. C. Wall associates to X an element $\sigma(X) \in \tilde{K}_0\ (\mathbf{Z}(\pi))$, called the obstruction to finiteness of X, whose vanishing is necessary and sufficient for X to be of the homotopy type of a finite complex. Also Wall shows that given any finitely presentable group π and any $\sigma \in \tilde{K}_0\ (\mathbf{Z}(\pi))$ there exists a CW- complex X dominated by a finite complex and satisfying $\pi_1(X) \approx \pi; \sigma(X) = \sigma$.

Any compact topological manifold is a retract of a finite complex. As such to each connected compact manifold M is associated an element $\sigma(M) \in \tilde{K}_0(\mathbf{Z}(\pi))$ (where $\pi = \pi_1 M$) which is the obstruction to finiteness of M. It is a conjecture of Milnor that for a connected, closed manifold M^n of dimension n the relation $\sigma(M) = (-1)^n\overline{\sigma(M)}$ holds, where bar denotes the involution in $\tilde{K}_0(\mathbf{Z}(\pi))$ arising from the involution $\sum m_i x_i \rightarrow \sum m_i x_i^{-1}$ of $\mathbf{Z}(\pi)$. Thus if this conjecture is proved it will follow that not every element in $\tilde{K}_0(\mathbf{Z}(\pi))$ can be realised as the Wall obstruction of a closed manifold, where π is an arbitrary finitely presentable group. Siebenmann in his thesis [4] proves this equality with the additional assumption that $M \times \mathbf{R}^k$ carries a differentiable structure for some integer k. The object of this paper is to prove this equality for all closed, orientable manifolds.

The author has learned that the formula $\sigma(M) = (-1)^n\overline{\sigma(M)}$ for a closed orientable manifold M of dimension n has been proved independently by Milnor and Wall and that Milnor's proof will appear in a forthcoming paper of Wall entitled *Poincaré complexes I* to appear in the Annals of Mathematics shortly.[1] This proof is purely algebraic whereas the author's proof is more geometric. Also the author has learned that the result on the Wall obstruction for sphere bundles obtained in this paper has also been obtained by S. Gersten. But none of these has appeared in print at the time of acceptance of this paper.

The idea of the proof can briefly be explained as replacing $M \times \mathbf{R}^k$ in Siebenmann's proof by the total space of an orientable topological $\mathbf{R}^k$ bundle (i.e. a microbundle) over M carrying a differentiable structure. In the course of his proof Siebenmann uses the fact that the Wall obstruction for $M \times S^{k-1}$ is zero when $k - 1$ is odd. For our proof we have to study the Wall obstruction of a sphere bundle over a connected CW-complex X dominated by a finite complex. We have information only in the case of S^k bundles when $k \geqq 2$ (Theorem 3.3). For an S^1 bundle over X we do not have any information.

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[1] This paper of Wall's has already appeared in vol. 86 (1967), pp. 213–245.

1. Notations and conventions

We write $f : A \simeq A'$ to denote that f is a homotopy equivalence of the space A with the space A'. If $f : (A, B) \to (A', B')$ is a map of pairs of topological spaces (i.e. to say $B \subset A$; $B' \subset A'$ and $f(B) \subset B'$) satisfying $f : A \simeq A'$ and $f \mid B : B \simeq B'$ then we call f a pseudo homotopy equivalence or shortly a p.h. equivalence of (A, B) in (A', B'). As usual by a CW-pair (X, Y) we mean a CW complex X together with a subcomplex Y of X. We denote by $\mathfrak{F}$ (Respectively $\mathfrak{C}$) the family of CW complexes dominated by a finite complex (respectively having in each dimension a finite number of cells). By a fibre bundle we mean a locally trivial fibre space. Thus by a sphere bundle of fibre dimension k or shortly an S^k-bundle over B we mean a locally trivial fibre space over B with fibre S^k.

Let C, C' be chain complexes over a ring Λ and $\varphi : C \to C'$ a chain map. By "abuse of language" we will refer to φ as a chain equivalence if it induces isomorphisms of homology modules. If $\varphi : (C, D) \to (C', D')$ is a chain map of pairs of chain complexes (i.e. to say D and D' are subcomplexes of C and C' respectively and $\varphi(D) \subset D'$) we call φ a chain equivalence of pairs if $\varphi : C \to C'$ and $\varphi : D \to D'$ are chain equivalences. In this case the induced chain map $\varphi : C/D \to C'/D'$ is also a chain equivalence.

2. Sphere bundles and cohomology extensions

The main reference for the results stated here is Section 7, Chap. 5 of [5] dealing with the homology of fibre bundles. The homology and cohomology groups we use are the singular ones and when no coefficients are mentioned we mean integer coefficients.

Let $(E, \dot{E})$ be a fibre bundle pair over B with fibre pair (D^k, S^{k-1}), with projection pair $(p, \dot{p})$, k being an integer $\geqq 1$. In otherwords $(E, \dot{E})$ is a pair of topological spaces with $\dot{E} \subset E$ and $p : E \to B$ is a map with the property that $\exists$ an open covering $\{U_\alpha\}_{\alpha \in I}$ of B and for each α a homeomorphism

$$\theta_\alpha : U_\alpha \times (D^k, S^{k-1}) \to (p^{-1}(U_\alpha), p^{-1}(U_\alpha) \cap \dot{E})$$

of pairs such that the composition $p \circ \theta_\alpha : U_\alpha \times D^k \to U_\alpha$ is the projection onto the first factor. The map $\dot{p} : \dot{E} \to B$ is the restriction of p to $\dot{E}$. For any $x \in B$ the pair

$$(D^k_x, S^{k-1}_x) = (p^{-1}(x), p^{-1}(x) \cap \dot{E})$$

is the fibre pair over x. We now recall the definition of a cohomology extension.

DEFINITION 2.1. By a cohomology extension for the bundle pair $(E, \dot{E})$ we mean a cohomology class $U \in H^k(E, \dot{E})$ such that for each $x \in B$ the inclusion $(D^k_x, S^{k-1}_x) \subset (E, \dot{E})$ carries U onto a generator of $H^k(D^k_x, S^{k-1}_x)$. When such a cohomology extension exists the pair $(E, \dot{E})$ is said to be an orientable bundle pair.

If ξ is a $(k - 1)$ sphere bundle with total space $\dot{E}_\xi$ and projection $\dot{p}_\xi : \dot{E}_\xi \to B$, the mapping cylinder E_ξ of $\dot{p}_\xi$ with the canonical retraction $p_\xi : E_\xi \to B$ gives a bundle pair $(E_\xi, \dot{E}_\xi)$ with fibre (D^k, S^{k-1}). The sphere bundle ξ is said to be orientable if the bundle pair $(E_\xi, \dot{E}_\xi)$ is orientable.

For the rest of this section X denotes a connected space which is of the homotopy type of a CW complex; π denotes the fundamental group of X and $\tilde{X}$ the universal covering of X. As usual π is considered as a group of operators on $\tilde{X}$. Let $\alpha : \tilde{X} \to X$ denote the covering projection.

Let now $(E, \dot{E})$ be an orientable bundle pair over X with fiber pair (D^k, S^{k-1}) and projection pair $(p, \dot{p})$. Choose a cohomology extension $U \in H^k(E, \dot{E})$. Let $\gamma \in CS^k(E, \dot{E})$ be a fixed cocyle representing U. (We denote the singular chain and cochain complexes of $(E, \dot{E})$ by $CS(E, \dot{E})$ and $CS^*(E, \dot{E})$ respectively.) Let $\tilde{p}:\tilde{E} \to \tilde{X}$ be the pull back of the bundle $p:E \to X$ by means of the map $\alpha:\tilde{X} \to X$. Thus $\tilde{E}$ is the subspace of $\tilde{X} \times E$ consisting of elements $(\tilde{x}, e)$ such that $\alpha(\tilde{x}) = p(e)$. The map $\tilde{p}:\tilde{E} \to \tilde{X}$ is given by $p(\tilde{x}, e) = \tilde{x}$. The group π acts in an obvious way on $\tilde{E}$, namely $a.(\tilde{x}, e) = (a.\tilde{x}, e)$ for all $a \in \pi$ and $(\tilde{x}, e) \in \tilde{E}$. The map $\beta:\tilde{E} \to E$ given by $\beta(\tilde{x}, e) = e$ is clearly a bundle map covering the map $\alpha:\tilde{X} \to X$. Also it is clear that $\beta:\tilde{E} \to E$ is the universal covering of E. Let $\tilde{E}^{\bullet} = \beta^{-1}(\dot{E})$. Then

$$\tilde{p}^{\cdot} = \tilde{p} \mid \tilde{E}^{\bullet}:\tilde{E}^{\bullet} \to X$$

is precisely the pull back of the bundle $\dot{p}:\dot{E} \to X$ by $\alpha:\tilde{X} \to X$ and $(\tilde{E}, \tilde{E}^{\bullet})$ is a bundle pair over X with fibre pair (D^k, S^{k-1}). It is clear that $V = \beta^*(U) \in H^k(\tilde{E}, \tilde{E}^{\bullet})$ is a cohomology extension for the bundle pair $(\tilde{E}, \tilde{E}^{\bullet})$ and that $\tilde{\gamma} = \beta^*(\gamma) \in CS^k(\tilde{E}, \tilde{E}^{\bullet})$ is a cocycle representing V.

The operations of π carry $\tilde{E}^{\bullet}$ into itself and hence the singular chain complex $CS(\tilde{E}, \tilde{E}^{\bullet})$ acquires the structure of a $\mathbf{Z}(\pi)$ chain complex.

LEMMA 2.2. *The map* $\tilde{\gamma} \cap : CS_m(\tilde{E}, \tilde{E}^{\cdot}) \to CS_{m-k}(\tilde{E}, \tilde{E}^{\cdot})$ *carrying any* $c \in CS_m(\tilde{E}, \tilde{E}^{\cdot})$ *into the cap product* $\tilde{\gamma} \cap c$ *is a homomorphism of* $\mathbf{Z}(\pi)$*-modules.*

Proof. Let $f:\Delta_m \to E$ be any singular simplex. Let $f_{(0,1,\cdots,k)}$ denote the *front* k-dimensional face of f and $f_{(k,\cdots,m)}$ the *rear* $(m - k)$-dimensional face of f. Then by the definition of the cap-product

$$\tilde{\gamma} \cap f = (-1)^{k(m-k)} \tilde{\gamma}(f_{(0,\cdots,k)}) f_{(k,\cdots,m)}$$

Now, for any $a \in \pi$ it is clear that

$$(a.f)_{(0,\cdots,k)} = a.(f_{(0,\cdots,k)}) \quad \text{and} \quad (a.f)_{(k,\cdots,m)} = a.(f_{(k,\cdots,m)}).$$

Also

$$\tilde{\gamma}(a.f_{(0,\cdots,k)}) = \tilde{\gamma}(a.(f_{(0,\cdots,k)}) = \gamma\{\beta_*(a.(f_{(0,\cdots,k)}))\}.$$

Since $\beta(a.\tilde{e}) = \beta(\tilde{e})$ for every $\tilde{e} \in \tilde{E}$ we see that

$$\gamma\{\beta_*(a.(f_{(0,\cdots,k)}))\} = \gamma\{\beta_*(f_{(0,\cdots,k)})\} = \tilde{\gamma}\{f_{(0,\cdots,k)}\}$$

Hence

$$\begin{aligned}\tilde{\gamma} \cap (a.f) &= (-1)^{k(m-k)}\tilde{\gamma}((a.f)_{(0,\cdots,k)})(a.f)_{(k,\cdots,m)} \\ &= (-1)^{k(m-k)}\tilde{\gamma}(f_{(0,\cdots,k)})\{a.(f_{(k,\cdots,m)})\} \\ &= a.\{\tilde{\gamma} \cap f\}\end{aligned}$$

This proves Lemma 2.2.

Let H be the chain complex over $\mathbf{Z}$ defined by $H_k = \mathbf{Z}$ and $H_i = 0$ for $i \neq k$. The tensor product $CS(\tilde{X}) \otimes_{\mathbf{Z}} H$ is considered as a chain complex over $\mathbf{Z}(\pi)$ in the obvious way, namely the π-operators on $CS(\tilde{X})$ as usual and trivial π-operators on H.

LEMMA 2.3. *The map $\tau: CS(\tilde{E}, \tilde{E}^{\bullet}) \to CS(\tilde{X}) \otimes_{\mathbf{Z}} H$ defined by*

$$\tau(c) = CS(\tilde{p})(\tilde{\gamma} \cap c) \otimes 1$$

is a $\mathbf{Z}(\pi)$-homomorphism of chain complexes inducing isomorphisms in homology.

Proof. Clearly $\tilde{p}: \tilde{E} \to \tilde{X}$ commutes with operators of π. That τ is a $\mathbf{Z}(\pi)$-homomorphism follows from Lemma 2.2 and the above observation. That it is a chain map inducing isomorphisms in homology is proved in Sec. 7, Chap. 5 of [5].

3. The Wall obstruction for sphere bundles

LEMMA 3.1. *Suppose K is a finite simplicial complex and $p: E \to K$ a locally trivial fibre space with fibre a finite simplicial complex L. Then E is dominated by a finite complex.*

Proof. Clearly E is a compact metric space satisfying the second countability axiom and is further local ANR. By Theorem 3.2 of [3] it is an ANR. Any compact ANR is dominated by a finite complex.

LEMMA 3.2. *Let $X \in \mathfrak{F}$ and $p: E \to X$ a locally trivial fibre space with fibre a finite simplical complex. Then E is dominated by a finite complex.*

Proof. X is dominated by a finite simplicial complex. Let

$$X \xrightarrow{f} K \xrightarrow{g} X$$

be maps such that $g \circ f \sim \mathrm{Id}_X$ with K a finite simplicial complex. If E' denotes the total space of the pull back of the bundle $p: E \to X$ by means of the map $g: K \to X$, it is easy to show that E is dominated by E'. By Lemma 3.1 E' is dominated by a finite complex and hence E too is dominated by a finite complex.

Let now X denote a connected CW complex in $\mathfrak{F}$ and $\pi = \pi_1(X)$. If $\dot{p}: \dot{E} \to X$ is a sphere bundle over X of fiber dimension $k - 1 \geqq 2$ then $\dot{E}$ is connected and $\dot{p}_*: \pi_1(\dot{E}) \to \pi_1(X)$ is an isomorphism. We use this isomorphism to identify $\pi_1(\dot{E})$ with π. By Lemma 3.2, $\dot{E}$ is a space dominated by a finite complex and hence one can talk of the obstruction $\sigma(\dot{E})$ to finiteness of $\dot{E}$ which is an element of $\tilde{K}_0(\mathbf{Z}(\pi))$. The main result proved in this section is the following.

THEOREM 3.3. *Let $X \in \mathfrak{F}$ and $\pi_0(X) = 0$, $\pi_1(X) = \pi$. Let*

$$\sigma(X) \in \tilde{K}_0(\mathbf{Z}(\pi))$$

be the obstruction to finiteness of X. Then for any orientable sphere bundle $\dot{p}:\dot{E} \to X$ of fibre dimension $k - 1 \geqq 2$ we have

$$\begin{aligned} \sigma(\dot{E}) &= 0 && \text{if } (k-1) \text{ is odd} \\ &= 2\sigma(X) && \text{if } (k-1) \text{ is even.} \end{aligned}$$

Remark. This theorem can be thought of as a generalization of the product formula for Wall obstruction to orientable sphere bundles of fibre dimension $\geqq 2$.

For the rest of *this section* we will be considering only connected spaces and hence the word space will mean a connected space. Before actually giving the proof of Theorem 3.3 we state some lemmas all of which are easy consequences of standard auguments but as we need them in the proof of the theorem we prefer to state them separately and indicate their proofs.

LEMMA 3.4. *Let (A, B) be a pair of spaces such that A and B are separately dominated by finite complexes. Then $\exists$ a CW pair (P, Q) with $P \in \mathcal{C}$ (and hence $Q \in \mathcal{C}$ also) and a p.h. equivalence $f:(P, Q) \to (A, B)$.*

Proof. Let $i:B \to A$ denote the inclusion. Since A, B are dominated by finite complexes $\exists$ elements Y, Z in $\mathcal{C}$ and homotopy equivalences $\phi:Y \to B$; $\psi:Z \to A$. Let $\theta:A \to Z$ be a homotopy inverse to ψ. Let $g:Y \to Z$ be any cellular map such that $g \sim \theta\iota\phi:Y \to Z$. Let M_g be the mapping cylinder of g. Then clearly $M_g \in \mathcal{C}$. If $\nu:M_g \to Z$ is the canonical retraction (which is also a homotopy equivalence) we have $\nu \circ j = g$ where $j:Y \to M_g$ is the obvious inclusion. Now $\psi \circ \nu \circ j = \psi \circ g \sim \psi \circ \theta\iota\phi \sim \iota\phi$. Since j is a cofibration $\exists$ a map $h:M_g \to A$ with $h \sim \psi \circ \nu$ and $h \circ j = \phi$. Now $P = M_g$, $Q = Y$ and $f = h$ clearly satisfy the requirements of the lemma.

For any CW complex X we denote the cellular chain complex of X by $C(X)$. If $f:X \to Y$ is a cellular map there is an obvious induced map $f_*:C(X) \to C(Y)$. If $\tilde{X}$ is the universal covering of X and $\pi = \pi_1(X)$ then the operations of π on $\tilde{X}$ make $C(\tilde{X})$ a free $\mathbf{Z}(\pi)$-chain complex. Also if $f:X \to Y$ is a cellular homotopy equivalence one can choose a cellular "lift" $\tilde{f}:\tilde{X} \to \tilde{Y}$ of f and the induced map $\tilde{f}_*:C(\tilde{X}) \to C(\tilde{Y})$ is a chain equivalence of $\mathbf{Z}(\pi)$-chain complexes, when we agree to identify $\pi_1(Y)$ with π by means of the inverse of the isomorphism $f_*:\pi_1(X) \approx \pi_1(Y)$.

LEMMA 3.5. *Let X be a countable CW complex and $\tilde{X}$ the universal covering* *of X and let $\pi = \pi_1(X)$. There exists a* $\mathbf{Z}(\pi)$*-chain equivalence*

$$\mu: CS(\tilde{X}) \to C(\tilde{X}).$$

Proof. Since X is countable it is of the homotopy type of a locally finite simplicial complex K [7]. We can choose cellular homotopy equivalences $\psi: K \to X$; $\phi: X \to K$ which are inverses of one another in homotopy. The universal covering $\tilde{X}$ is also a locally finite simplicial complex and $\pi_1(K)$ operates without fixed points on K. By a result of S. Eilenberg [1] $\exists$ a $\mathbf{Z}(\pi)$-chain map $\gamma: CS(\tilde{K}) \to C(\tilde{K})$ inducing isomorphisms in homology. Now

$$\mu = \tilde{\psi}_* \circ \nu \circ CS(\tilde{\varphi})$$

where $\tilde{\psi}_* : C(\tilde{K}) \to C(\tilde{X})$ is gotten from $\tilde{\psi}: \tilde{K} \to \tilde{X}$ clearly satisfies the requirements of Lemma 3.5.

Let (X, Y) be a CW pair with $\iota_*: \pi_1(Y) \to \pi_1(X)$ an isomorphism, where $i: Y \to X$ is the inclusion map. Let $\alpha: \tilde{X} \to X$ be the projection from the universal covering $\tilde{X}$ of X. Then $\tilde{Y} = \alpha^{-1}(Y)$ is the univeral covering of Y. The operations of $\pi = \pi_1(X)$ on $\tilde{X}$ carry $\tilde{Y}$ into itself and $(C(\tilde{X}), C(\tilde{Y}))$ and $(CS(\tilde{X}), CS(\tilde{Y}))$ become $\mathbf{Z}(\pi)$ chain complex pairs.

LEMMA 3.6. *There exists a* $\mathbf{Z}(\pi)$ *chain equivalence*

$$\lambda: (C(\tilde{X}), C(\tilde{Y})) \to (CS(\tilde{X}), CS(\tilde{Y}).$$

Proof. Using J. H. C. Whitehead's geometric realization of the singular complex with the triangulation described on page 103 of [8] it is not difficult to show that $\exists$ a simplicial pair (K, L) and a p.h. equivalence $g: (K, L) \to (X, Y)$. Using the fact that $i: Y \to X$ is a cofibration it is easy to get a map $f: (X, Y) \to (K, L)$ such that $g \circ f \sim \mathrm{Id}_X$ and $g \circ f \mid Y \sim \mathrm{Id}_Y : Y \to Y$. Also f can be chosen to be cellular. Let $\beta: \tilde{K} \to K$ be the projection from the universal covering $\tilde{K}$ of K. Then since the inclusion of L in K induces an isomorphism of fundamental groups it follows that $\beta^{-1}(L) = \tilde{L}$ is the universal covering of L. If $\tilde{f}: \tilde{X} \to \tilde{K}$ is a cellular "lift" of f in the sense that $\beta \circ \tilde{f} = f \circ \alpha$ then $\tilde{f}(\tilde{Y}) \subset \tilde{L}$ and $\tilde{f}: (\tilde{X}, \tilde{Y}) \to (\tilde{K}, \tilde{L})$ is a p.h. equivalence. Similarly if $\tilde{g}$ is a "lift" of g (i.e. to say $\alpha \circ \tilde{g} = g \circ \beta$) then $\tilde{g}(\tilde{L}) \subset \tilde{Y}$ and $\tilde{g}: (\tilde{K}, \tilde{L}) \to (X, Y)$ is a p.h. equivalence. Let

$$j: (C(\tilde{K}), C(\tilde{L})) \to (CS(\tilde{K}), CS(\tilde{L}))$$

be the natural inclusion of the simplicial chain complex pair into the singular chain complex pair then $\lambda = CS(\tilde{g}) \circ j \circ \tilde{f}_*$ satisfies the requirements of the Lemma.

LEMMA 3.7. *Consider the same situation as explained in the paragraph preceding Lemma 3.6. In addition assume that both X and Y are elements in* $\mathfrak{C} \cap \mathfrak{F}$. *Then the* $\mathbf{Z}(\pi)$*-chain complex* $C(\tilde{X})/C(\tilde{Y})$ *satisfies condition* G_N *of Gersten* [2] *for some integer* N.

Proof. That $C(\tilde{X})/C(\tilde{Y})$ is a free $\mathbf{Z}(\pi)$-chain complex of finite type is clear. (Finite type means each $C_p(\tilde{X})/C_p(\tilde{Y})$ is a finitely generated $\mathbf{Z}(\pi)$-module.) Also since X and Y are separately dominated by finite complexes we can find an integer $L > 2$ having the following properties:

(1) $H^j(X, Y; \mathcal{B}) = 0$ for all local coefficient systems $\mathcal{B}$ over X and $j > L$

(2) $H_j(\tilde{X}, \tilde{Y}; \mathbf{Z}) = 0$ for all $j > L$.

Denoting the complex $C(\tilde{X})/C(\tilde{Y})$ by C we have $B_L = \partial(C_{L+1})$ a $\mathbf{Z}(\pi)$-module and it determines a local coefficient system $\mathcal{B}$ over X. Let $\partial: C_{L+1} \to C_L$ be factored into

$$C_{L+1} \xrightarrow{c} B_L \xrightarrow{j} C_L$$

with j the inclusion and c the canonical map. $H^*(X, Y; \mathcal{B})$ is the homology of the complex of π-homomorphisms of C into B_L and thus c determines an $(L + 1)$ cochain. Since $c \circ \partial = 0$ it is a cocyle. Since $H^{L+1}(X, Y; \mathcal{B}) = 0$, c has to be a coboundary and hence $c = s \circ \partial$ for some $s: C_L \to B_L$. Thus $c = s \circ j \circ c$ and since c is an epimorphism we have $sj = \mathrm{Id}_{B_L}$. This shows that B_L is a direct summand of C_L and is hence projective. Now condition (2) yields $Z_{L+1} = B_{L+1} = \partial(C_{L+2})$ where $Z_{L+1} = \operatorname{Ker} \partial: C_{L+1} \to C_L$. Thus $C_{L+1}/B_{L+1} \approx B_L$ is $\mathbf{Z}(\pi)$-projective. Also it is finitely generated being a direct summand of C_L.

This shows that C satisfies condition G_{L+1} of Gersten. The above proof is actually a reproduction of Wall's proof in [6].

Proof of Theorem 3.3. Without loss of generality X can be assumed to be an element in $\mathcal{C} \cap \mathcal{F}$. In fact $\exists$ a $Y \in \mathcal{C}$ and a homotopy equivalence $g: Y \simeq Y$. It is now clear that $Y \in \mathcal{C} \cap \mathcal{F}$ and that the total space $g^*(\dot{E})$ of the pull back bundle has the same homotopy type as $\dot{E}$ and it suffices to prove the theorem for Y instead of X. We therefore assume $X \in \mathcal{C} \cap \mathcal{F}$. If E is the mapping cylinder of $\dot{p}: \dot{E} \to X$ and $p: E \to X$ the canonical retraction then $(E, \dot{E})$ is a bundle pair with (D^k, S^{k-1}) as the fibre pair and further it is orientable. By Lemma 3.2 both E and $\dot{E}$ are dominated by finite complexes. Since

$$p_*: \pi_n(E) \to \pi_n(X)$$

is an isomorphism for all n it follows that $p: E \to X$ is a homotopy equivalence and hence $\sigma(E) = \sigma(X)$, (i.e. when we identify $\pi_1(E)$ with π by means of the isomorphism p_*). By Lemma 3.4 $\exists$ a CW pair (P, Q) with $P, Q \in \mathcal{C}$ and a p.h. equivalence $f: (P, Q) \to (E, \dot{E})$. Since $k - 1 \geqq 2$ by assumption, we have $\pi_1(\dot{E}) \to_{\approx} \pi_1(E)$ under the map induced by the inclusion. Hence the inclusion $Q \subset P$ induces an isomorphism $\pi_1(P) \approx \pi_1(Q)$. Applying Lemma 3.6 we get a chain equivalence

$$\lambda: (C(\tilde{P}), C(\tilde{Q})) \to (CS(\tilde{P}), CS(\tilde{Q}))$$

of $\mathbf{Z}(\pi)$-chain complexes. It follows that

$$CS(\tilde{f}) \circ \lambda : (C(\tilde{P}), C(\tilde{Q})) \to (CS(\tilde{E}), CS(\tilde{E}^{\cdot}))$$

is a chain equivalence of $\mathbf{Z}(\pi)$-chain complexes, where $\tilde{E}$ and $\tilde{E}^{\cdot}$ have the same meanings as in Section 2. Denoting the induced map

$$C(\tilde{P})/C(\tilde{Q}) \to CS(\tilde{E})/CS(\tilde{E}^{\cdot}) = CS(\tilde{E}, \tilde{E}^{\cdot})$$

by ε we see that

$$\varepsilon : C(\tilde{P})/C(\tilde{Q}) \to CS(\tilde{E}, \tilde{E}^{\cdot})$$

is a $\mathbf{Z}(\pi)$ chain equivalence.

By Lemma 2.3, $\tau : CS(\tilde{E}, \tilde{E}^{\cdot}) \to CS(\tilde{X}) \otimes_{\mathbf{Z}} H$ is a $\mathbf{Z}(\pi)$ chain equivalence. Applying Lemma 3.5 and noting that π operators on H are trivial we see that

$$\mu \otimes \mathrm{Id}_H : CS(\tilde{X}) \otimes_{\mathbf{Z}} H \to C(\tilde{X}) \otimes_{\mathbf{Z}} H$$

is a $\mathbf{Z}(\pi)$-chain equivalence. If we choose the usual triangulation for (D^k, S^{k-1}) (i.e. as $(\Delta^k, \Delta^{\cdot k})$ with Δ^k the k-dim simplex) the complex H is the same as $C(D^k, S^{k-1})$. Another description for $C(\tilde{X}) \otimes_{\mathbf{Z}} H$ is that it is the k fold suspension $\sum^k C(\tilde{X})$ of $C(\tilde{X})$ as defined in [2].

We now recall the definition of the Wall obstruction for a free chain complex C of finite type over $\mathbf{Z}(\pi)$ satisfying condition G_N for some integer N. It is defined to be the element $(-1)^N[C_N/B_N]$ in $\tilde{K}_0(\mathbf{Z}(\pi))$. It is shown in [2] that if C satisfies G_N it also satisfies G_{N+i} for any $i \geqq 0$ and that

$$(-1)^{N+i}[C_{N+i} \mid B_{N+i}] = (-1)^N[C_N \mid B_N]$$

so that the Wall obstruction is a well defined element say $k(C) \in \tilde{K}_0(\mathbf{Z}(\pi))$. With this definition the Wall obstruction of $\sum^k C$ is the same as $(-1)^k k(C)$.

Since P and Q are in $\mathcal{C}$ and are also dominated by finite complexes the chain complexes $C(\tilde{P})$ and $C(\tilde{Q})$ are free $\mathbf{Z}(\pi)$ chain complexes of finite type satisfying condition G_N for some N. Also by Lemma 3.7, $C(\tilde{P})/C(\tilde{Q}) = C$ (say) has the same property. Now

$$(\mu \otimes \mathrm{Id}_H) \circ \tau \circ \varepsilon : C \to \textstyle\sum^k C(\tilde{X})$$

is a chain equivalence and hence by a result of Gersten [2] we have

$$k(C) = k(\textstyle\sum^k C(\tilde{X})) = (-1)^k k(C(\tilde{X})) = (-1)^k \sigma(X).$$

Since $f : P \simeq E$ and $p : E \simeq X$ we have $k(C(\tilde{P})) = \sigma(P) = \sigma(E) = \sigma(X)$. Now, the exact sequence

$$0 \to C(\tilde{Q}) \to C(\tilde{P}) \to C \to 0$$

yields

(i) $$k(C(\tilde{P})) = k(C(\tilde{Q})) + k(C).$$

Also since $f \mid Q : Q \simeq \dot{E}$ we have $k(C(\tilde{Q})) = \sigma(Q) = \sigma(\dot{E})$. Thus the equality (i) yields $\sigma(X) = \sigma(\dot{E}) + (-1)^k \sigma(X)$ or what is the same as

$$\begin{aligned} \sigma(\dot{E}) &= 0 && \text{if } k \text{ is even} \\ &= 2\sigma(X) && \text{if } k \text{ is odd} \end{aligned}$$

This completes the proof of Theorem 3.3.

4. The main theorem

The main theorem as already stated in the introduction is the following.

THEOREM 4.1. *If M^n is a closed, connected, orientable topological manifold of dimension n then* $\sigma(M) = (-1)^n \overline{\sigma(M)}$.

Proof. M^n can be imbedded in $\mathbf{R}^{n+k}$ for some large k so as to have a normal microbundle. Since M is orientable the normal microbundle has to be orientable. By taking the imbedding $M^n \subset \mathbf{R}^{n+k} \subset \mathbf{R}^{n+k} \times \mathbf{R}$ into $\mathbf{R}^{n+k+1}$ one gets a normal orientable disk bundle in $\mathbf{R}^{n+k+1}$. If E is the total space of this disc bundle then E is a compact topological manifold with boundary, and if the boundary is denoted by $\dot{E}$ then the pair $(E, \dot{E})$ is an orientable bundle over M with fibre pair (D^{k+1}, S^k). Now Int E being an open subset of $\mathbf{R}^{n+k+1}$ is an open differentiable manifold and whenever $k + 1 \geqq 2$ this manifold Int E has one end. Moreover it is clear that this end say ε is tame and that the fundamental group $\pi_1(\varepsilon)$ at the end ε is isomorphic to $\pi = \pi_1(M)$ whenever $k \geqq 2$. Let $\sigma(\varepsilon)$ be Siebenmann's obstruction [4] to closing the manifold Int E at the end ε. Now $\dot{E}$ admits of a topological collar in E and hence $\dot{E} \times \mathbf{R}$ carries a differentiable structure. Since $\dot{E}$ is compact the manifold $\dot{E} \times \mathbf{R}$ has two ends say ε_+ and ε_- and there is a duality formula $\sigma(\varepsilon_+) = (-1)^{n+k}\overline{\sigma(\varepsilon_-)}$ relating the obstructions for closing $\dot{E}$ at the ends ε_+ and ε_- respectively [4]. However it is clear that one of the ends say ε_+ is the same as the end ε of Int E and that the obstructions $\sigma(\varepsilon)$ and $\sigma(\varepsilon_+)$ are equal. Also it is not difficult to see that $\sigma(E) = \sigma(\varepsilon)$. In addition one has

$$\begin{aligned} \sigma(\dot{E}) = \sigma(\varepsilon_+) + \sigma(\varepsilon_-) &= \sigma(\varepsilon_+) + (-1)^{n+k+1-1}\overline{\sigma(\varepsilon_+)} \\ &= (\sigma\varepsilon_+) + (-1)^{n-k}\overline{\sigma(\varepsilon_+)} \end{aligned}$$

(Corollary 11.3 of [4]). Also $\exists$ a CW complex $X \in \mathfrak{F} \cap \mathfrak{C}$ and a homotopy equivalence $f : X \simeq M$. It is easily seen that the total space $f^*(\dot{E})$ of the inverse image of the bundle $p \mid \dot{E} : \dot{E} \to M$ (where $p : E \to M$ is the projection of the bundle E) is of the same homotopy type as $\dot{E}$. By Theorem 3.3 the obstruction for $f^*(\dot{E})$ vanishes if k is odd and hence $\sigma(\dot{E}) = 0$ when k is odd. We can without loss of generality choose k odd. Then we have

$$\sigma(\varepsilon_+) = (-1)^{n+(k-1)}\overline{\sigma(\varepsilon_+)}$$

i.e. to say $\sigma(E) = (-1)^n \overline{\sigma(E)}$. But $\sigma(E) = \sigma(M)$ and hence we have $\sigma(M) = (-1)^n \overline{\sigma(M)}$.

REFERENCES

1. S. EILENBERG, *Homology of spaces with operators* I, Trans. Amer. Math. Soc., vol. 61 (1947), pp. 378–417.
2. S. M. GERSTEN, *Product formula for Walls' obstruction*, Amer. J. Math., vol. 88 (1966), pp. 337–346.
3. O. HANNER, *Some theorems on absolute neighborhood retracta*, Ark. Mat., (1949–1952), pp. 389–408.
4. L. SIEBENMANN, *The obstruction to finding boundary for an open manifold of dimension greater than five*, to appear.
5. E. H. SPANIER, *Algebraic topology*, McGraw-Hill, New York, 1966.
6. C. T. C. WALL, *Finiteness conditions for CW-complexes*, Ann. of Math., vol. 81 (1965), pp. 56–69.
7. J. H. C. WHITEHEAD, *Combinatorial homotopy* I, Bull. Amer. Math. Soc., vol. 55 (1949), pp. 213–245.
8. ———, *A certain exact sequence*, Ann. of Math., vol. 52 (1950), pp. 51–110.

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