TWO THEOREMS ON SPECTRAL SEQUENCES[1]

BY

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In this note, we prove the following theorems.

THEOREM 1. *Let X and Y be two* CW*-complexes. Then the exact couple associated with the Postnikov decomposition of Y and the modified homotopy functor $\pi_*(X,\quad)$ is isomorphic to the first derived couple associated with the skeleton decomposition of X and the modified homotopy functor $\pi_*(\quad, Y)$.*

THEOREM 2. *The spectral sequence associated with the skeleton decomposition of X and the modified homotopy functor $\pi_*(\quad, Y)$ is isomorphic to that associated with a homology decomposition of X for all* CW*-complexes Y, if and only if the space X has no torsion.*

We recall some basic facts in Section 1 and present the proofs of the theorems in Section 2 and 3.

1. We recall that the Postnikov decomposition of a CW-complex Y is a sequence of maps[2]

$$(1) \qquad \begin{array}{c} Y \longrightarrow Y_{p+1} \xrightarrow{\;l_p\;} Y_p \longrightarrow \cdots \longrightarrow * \\ \text{with } h_p : Y \to Y_p \end{array}$$

where

(i) every map is a fibration, and

$$\text{(ii)} \qquad \begin{aligned} \pi_r(Y_p) &= \pi_r(Y) \quad \text{for} \quad r \leq p \\ \pi_r(Y_p) &= 0 \qquad\;\; \text{for} \quad r > p. \end{aligned}$$

Thus we see that the fiber of l_p is the Eilenberg-MacLane space $K(\pi_{p+1}(Y), p + 1)$. If we apply the modified homotopy functor $\pi_*(X,\quad)$ to the sequences of maps

$$K(\pi_{p+1}(Y), p + 1) \to Y_{p+1} \xrightarrow{\;l_p\;} Y_p$$

the resulting exact couple is called the exact couple associated with the Post-

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[2] We assume that all spaces have a based point * and all maps are based maps.

nikov decomposition of Y and the modified homotopy functor $\pi_*(X, \quad)$ with

$$(2)\qquad \begin{aligned} D^{p,q} &= \pi_{q+1}(X, h_p) = \pi_q(X, F_p), \\ E^{p,q} &= \pi_{q+1}(X, l_p) = \pi_q(X, K(\pi_{p+1}(Y), p+1)). \end{aligned}$$

A homology decomposition [4] of the CW-complex X is a sequence of maps

$$(3)\qquad * \longrightarrow X_p \xrightarrow{i^{p+1}} X_{p+1} \longrightarrow \cdots \longrightarrow X, \qquad k^{p+1}: X_p \to X$$

such that

(i) every map i^{p+1} is a cofibration, and

$$(\text{ii})\qquad \begin{aligned} H_r(X_p) &= H_r(X) \quad \text{for} \quad r \leq p \\ H_r(X_p) &= 0 \qquad\quad \text{for} \quad r > p. \end{aligned}$$

Since the cofiber of i^{p+1} is the Moore space $K'(H_{p+1}(X), p+1)$, we see [4, Theorem 7.1′] that there is a map

$$u_p : K'(H_{p+1}(X), p) \to X_p$$

such that i^{p+1} is equivalent to the canonical cofibration

$$Y_p \to X_p \cup_{u_p} CK'(H_{p+1}(X), p).$$

We will henceforth consider X_{p+1} as so obtained. Again if we apply the modified homotopy functor $\pi_*(\quad , Y)$ to the sequence (3), we have an exact couple with

$$(4)\qquad \begin{aligned} D^{p,q} &= \pi_{q+1}(k^p, Y) = \pi_q(X/X_p, Y), \\ E^{p,q} &= \pi_{q+1}(i^p, Y) = \pi_q(X_p/X_{p-1}, Y), \end{aligned}$$

which we called the exact couple associated with the homology decomposition of X and the functor $\pi_*(\quad , Y)$.

We need the following facts.

1. Let $f : Y_1 \to Y_2$ be a fibration. Then the exact sequence

$$\to \pi_q(X, Y_1) \to \pi_q(X, Y_2) \to \pi_q(X, f) \to \pi_{q-1}(X, Y_1) \to$$

is induced by the Eckmann-Hilton sequence

$$\to \Omega Y_1 \to \Omega Y_2 \to E_f \to Y_1 \xrightarrow{f} Y_2$$

where E_f is the pull-back of the diagram

$$\begin{array}{ccc} E_f & \longrightarrow & PY_2 \\ \downarrow & & \downarrow \pi \\ Y_1 & \xrightarrow{f} & Y_2 \end{array}$$

in which $PY_2 = \{l : I \to Y_2 \mid l(1) = *\}$ and $\pi(l) = l(0)$.

2. Let $g : X_2 \to X_1$ be a cofibration. Then the exact sequence

$$\to \pi_q(X_1, Y) \to \pi_q(X_2, Y) \to \pi_q(g, Y) \to \pi_{q-1}(X_1, Y) \to$$

is induced by the Puppe sequence

$$X_2 \xrightarrow{g} X_1 \to X_1 \cup_g CX_2 \to \Sigma X_2 \xrightarrow{\Sigma g} \Sigma X_1 \to$$

3. Let h_1, h_2 be two cohomology theories satisfying the Eilenberg-Steenrod axioms and the "wedge axiom"

$$h_j^q(\vee_{i \in I} S_i^p) = \prod_{i \in I} h_j^q(S_i^p), \qquad j = 1, 2.$$

Let $\tau_1, \tau_2 : h_1 \to h_2$ be two functor-morphisms. If

$$\tau_1(*) = \tau_2(*) : h_1(*) \to h_2(*),$$

then

$$\tau_1(X) = \tau_2(X) : h_1(X) \to h_2(X),$$

provided that X is finite dimensional.

4. Cf. [4]. Let $Y \to X$ be a cofibration with cofiber F. Assume that all spaces are 1-connected. If Y is $(k - 1)$-connected and F is $(l - 1)$-connected, then the homomorphism $\varepsilon' : \pi_m(i) \to \pi_m(F)$ is an isomorphism for $m < k + l - 1$ and an epic if $m = k + l - 1$.

2. *Proof of Theorem* **1.** We break the proof of theorem 1 into the following lemmas.

$$(5) \qquad \begin{array}{ccccccc} * & \longrightarrow & X^p & \xrightarrow{j^{p+1}} & X^{p+1} & \longrightarrow \cdots \longrightarrow & X. \\ & & \big\lfloor & \xrightarrow{\qquad\qquad g^{p+1} \qquad\qquad} & & & \uparrow \end{array}$$

Applying the modified homotopy functor $\pi_*(\ \ , Y)$ to (5), we have an exact couple with

$$D'^{p,q} = \pi_{q+1}(g^p, Y) = \pi_p(X/X^{p-1}, Y),$$

$$E'^{p,q} = \pi_{q+1}(j^p, Y) = \pi_q(X^p/X^{p-1}, Y).$$

LEMMA 1. *There is a natural isomorphism*

$$(\kappa^{p-q-1})^* : \pi_q(X/X^{p-q-1}, F_p) \to D^{p,q}$$

where F_p is the fiber of $h_p : Y \to Y_p$.

LEMMA 2. $$D^{p,q} \overset{\text{nat}}{\cong} D_1'^{p-q,q}$$

Proof. We consider the diagram[3]

$$\begin{array}{ccccc}
\pi_{q+1}(X/X^{p-q},Y_p) & \longrightarrow & \pi_{q+1}(X/X^{p-q-1},Y_p) & & \\
\downarrow & & \downarrow & & \\
\pi_q(X/X^{p-q},F_p) & \xrightarrow{\alpha} & \pi_q(X/X^{p-q-1},F_p) & \longrightarrow & \pi_q(X^{p-q}/X^{p-q-1},F_p) \\
\downarrow & & \downarrow i_{p*} & & \\
\pi_q(X/X^{p-q},Y) & \xrightarrow{\alpha'} & \pi_q(X/X^{p-q-1},Y) & & \\
\downarrow & & & & \\
\pi_q(X/X^{p-q},Y_p) & & & &
\end{array}$$

where $i_p : F_p \to Y$ and all the rows and columns are exact. Since all homotopy groups of Y_p higher than p are zero and F_p is p-connected, we have

$$\pi_{q+1}(X/X^{p-q},\ Y_p) = \pi_q(X/X^{p-q},\ Y_p) = \pi_q(X^{p-q}/X^{p-q-1},\ F_p) = 0$$

Therefore we have

$$\pi_q(X/X^{p-q-1},\ F_p) \overset{i_{p*}}{\cong} \operatorname{Im}\alpha' = D_1'^{\,p-q,q}$$

To show i_{p*} is natural, we consider the diagram

$$\begin{array}{ccccc}
\pi_q(X/X^{p-q},\ F_{p+1}) & & \xrightarrow{(g'^{p-q})^*(e_p)_*} & & \pi_q(X/X^{p-q-1},\ F_p) \\
\Big\downarrow & \searrow^{e_{p*}} & 2 & \nearrow^{(g'^{p-q})_*} & \Big\downarrow \\
(i_{p+1})_* & 1 & \pi_q(X/X^{p-q},\ F_p) & & \\
\Big\downarrow & \swarrow^{(i_p)_*} & & 3 & \Big\downarrow \\
\pi_q(X/X^{p-q},\ Y) & & \longrightarrow & & \pi_q(X/X^{p-q-1},\ Y)
\end{array}$$

where $e_p : F_{p+1} \to F_p$ and $g'^{p-q} : X/X^{p-q-1} \to X/X^{p-q}$. We see that triangle 1 is commutative because $i_{p+1} = i_p e_p$; triangle 2 commutes evidently; and square 3 commutes because $(i_p)_*$ is natural. Hence the outside square is commutative. Next we note that the following diagram

$$\begin{array}{ccc}
\pi_q(X,F_{p+1}) & \xrightarrow{\alpha = (e_p)_*} & \pi_q(X,F_p) \\
\uparrow (\kappa^{p-q})^* & & \uparrow (\kappa^{p-q-1})^* \\
\pi_q(X/X^{p-q},F_{p+1}) & \xrightarrow{(g^{p-q})^*(e_p)_*} & \pi_q(X/X^{p-q-1},F_p)
\end{array}$$

is commutative. Hence we have the natural isomorphism

$$i_{p*}(\kappa^{p-q-1})^{-1} : D^{p,q} \to D_1'^{\,p-q,q}$$

[3] To avoid too many symbols, we use the same notation for two homomorphisms if they are induced by the same map.

LEMMA 3. *There is a natural isomorphism*

$$j : E_1'^{p-q+1,q} \to E^{p,q}.$$

Proof. We produce the isomorphism as follows. Let Z be the kernel of $\beta'\gamma'$

$$Z \subseteq \pi_q(X^{p-q+1}/X^{p-q},\ Y) \xrightarrow{\gamma'} \pi_{q-1}(X/X^{p-q+1},\ Y) \to \pi_{q-1}(X^{p-q+1}/X^{p-q+1},\ Y)$$

We look at the diagram

$$\begin{array}{ccccc}
0 \to \pi_q(X/X^{p-q}, K(\pi_{p+1}(Y), p+1)) & \xrightarrow{\beta'''} & \pi_q(X^{p-q+1}/X^{p-q}, K(\pi_{p+1}(y), p+1)) & \xrightarrow{\gamma'''} & \pi_{q-1}(X/X^{p-q+1}, K(\pi_{p+1}(Y), p+1)) \\
\downarrow (k_{p+1})_* & & \downarrow & & \downarrow \\
\pi_q(X/X^{p-q}, Y_{p+1}) & \xrightarrow{\beta''} & \pi_q(X^{p-q+1}/X^{p-q}, Y_{p+1}) & \xrightarrow{\gamma''} & \pi_{q-1}(X/X^{p-q+1}, Y_{p+1}) \\
 & & \uparrow (h_{p+1})_* & & \uparrow (h_{p+1})_* \\
 & & \pi_q(X^{p-q+1}/X^{p-q}, Y) & \xrightarrow{\gamma'} & \pi_{q-1}(X/X^{p-q+1}, Y) \\
 & & & & \uparrow (i_{p+1})_* \\
 & & & & \pi_{q-1}(X/X^{p-q}, F_{p+1}).
\end{array}$$

From lemma 2, we know that $\ker \beta' = D_1'^{p-q+1,q-1}$ is isomorphic to $\operatorname{Im} i_{p*}$. Hence

$$\ker \beta'\gamma' = \ker (h_{p+1})_*\gamma' = \ker \gamma'' = \operatorname{Im} \beta''$$

which is isomorphic to $\pi_q(X/X^{p-q}, K(\pi_{p+1}(Y), p+1))$, i.e.,

$$s = (k_{p+1})^{-1} \beta''^{-1}(h_{p+1})_* : Z \rightarrowtail\!\!\!\!\twoheadrightarrow \pi_p(X/X^{p-q}, K(\pi_{p+1}(Y), P+1))$$

Now we let j' be the homomorphism

$$\begin{aligned}
Z \xrightarrow{s} \pi_q(X/X^{p-q}, K(\pi_{p+1}(Y), p+1)) & \\
\xrightarrow{(g'^{p-q})^*} \ \pi_q(X/X^{p-q-1}, K(\pi_{p+1}(Y)p+1)) & \\
\xrightarrow{(\kappa^{p-q-1})^*} \pi_q(X, K(\pi_{p+1}(Y), p+1)) &= E^{p,q}.
\end{aligned}$$

The homomorphism $(g'^{p-q})^*$ is epic because of the exact sequence

$$\begin{aligned}
\pi_q(X/X^{p-q}, K(\pi_{p+1}(Y), p+1)) & \\
\to \pi_q(X/X^{p-q-1}, K(\pi_{p+1}(Y), p+1)) & \\
\to \pi_q(X^{p-q}/X^{p-q-1}, K(\pi_{p+1}(Y), p+1)) &= 0
\end{aligned}$$

Consequently, j' is an epic. Since each homomorphism constituting j' is natural, j' is a natural epic.

Let B be the image of $\beta'\gamma'$. Then $E_1^{p-q+1,q} = Z/B$.

If we write out the appropriate diagram, we see that j' induces an isomorphism

$$j : E_1'^{p-q+1,q} \to E^{p,q}.$$

COROLLARY 4. $E_1^{p-q+1,q} \simeq H^{p-q+1}(X, \pi_{p+1}(Y))$.

Proof. The assertion follows from the fact that

$$\pi_q(X, K(\pi_{p+1}(Y), p+1)) \overset{\text{ob}}{\cong} H^{p-q+1}(X, \pi_{p+1}(y))$$

by obstruction theory.

We can, of course, prove Corollary 4 by direct computation and then deduce the fact that

$$\pi_q(X, K(\pi_{p+1}(Y), p+1)) \overset{\eta}{\cong} H^{p-q+1}(X, \pi_{p+1}(Y)).$$

Then it is not difficult to verify that this isomorphism η is the same as the obstruction isomorphism ob by using Proposition 3 of §1 to show that $\text{ob}\cdot\eta^{-1}$ is the identity transformation of $H^*(\ \ , \pi_*(Y))$. We need this fact later and will give a proof later (lemma 7).

LEMMA 5. *The diagram*

$$\begin{array}{ccc} D_1'^{p-q,q} & \xrightarrow{\beta_1'} & E_1'^{p-q+1,q} \\ {\scriptstyle i}\uparrow & & \uparrow{\scriptstyle j^{-1}} \\ D^{p,q} & \xrightarrow{\beta} & E^{p,q} \end{array}$$

is commutative.

Proof. The assertion amounts to showing that the square 1 of the following diagram is commutative.

$$\begin{array}{ccccc} \pi_q(X/X^{p-q}, Y) & \xrightarrow{(g'^{p-q})^*} & D_1'^{p-q,q} & \xrightarrow{\beta_1'} & Z/B \\ \uparrow & 2 & \uparrow \quad 1 & & \nwarrow \\ \pi_q(X/X^{p,q}, F_p) & \xrightarrow{(g'^{p-q})^*} & \pi_q(X/X^{p-q-1}, F_p) & \xrightarrow{\beta} & \pi_q(X/x^{p-q-1}, k(\pi_{p+1}(Y), p+1)) \end{array}$$

Since square 2 is commutative and $(g'^{p-q})^*$ are epic, it suffices to show that diagram 1 plus 2 is commutative. Recall that we have maps

$$h_p : Y \to Y_p, \qquad \kappa^{p-q-1} : X \to X/X^{p-q-1},$$

and

$$k_{p+1} : K(\pi_{p+1}(Y), p+1)) \to Y_{p+1}.$$

Let us introduce the following notations:

$$f_p : F_p \to K(\pi_{p+1}(Y), p+1))$$

and

$$q^{p-q} : X^{p-q-1}/X^{p-q} \to X/X^{p-q}.$$

From the definition of β_1', we have

$$\beta_1'(g'^{p-q})^*(i_p)_* = (q^{p-q})^*(i_p)_*.$$

Hence

$$j_1\beta_1'(g'^{p-q})^*(i_p)_* = (g'^{p-q})^*(k_{p+1})^{-1}\beta''^{-1}(h_{p+1})_*(q^{p-q})^*(i_p)_*$$
$$= (g'^{p-q})^*(k_{p+1})_*^{-1}(h_{p+1})_*(i_p)_*$$

where $(\kappa^{p-q-1})^*j_1 = j$. In view of the homotopy commutative square

$$\begin{array}{ccc} F_p & \xrightarrow{i_p} & Y \\ \downarrow{\scriptstyle f_p} & & \downarrow{\scriptstyle h_{p+1}} \\ K(\pi_{p+1}(Y), p+1) & \xrightarrow{k_{p+1}} & Y_{p+1} \end{array}$$

we may write[4]

$$j_1\beta_1'(g'^{p-q})^*(i_p)_* = (g'^{p-q})^*(f_p)_*$$
$$= (f_p)_*(g'^{p-q})_*$$
$$= \beta(g'^{p-q})^*.$$

This completes the proof.

LEMMA 6. *The diagram*

$$\begin{array}{ccc} E_1'^{p-q+1} & \xrightarrow{\gamma_1'} & D'^{p-q+1,q-1} \\ {\scriptstyle j^{-1}}\updownarrow & & \updownarrow{\scriptstyle i} \\ E^{p,q} & \xrightarrow{\gamma} & D^{p+1,q-1} \end{array}$$

commutes.

Proof. Let $f : X \to \Omega^q K(\pi_{p+1}(Y), p + 1))$ represent an element in

$$\pi_q(X, K(\pi_{p+1}(Y), p + 1)) = E^{p,q}.$$

Since $K(\pi_{p+1}(Y), p + 1)$ is p-connected we may assume that $f \mid X^{p-q} = 0$. In view of Proposition 1 of Section 1, γ is identified with the homomorphism induced by the map

$$\Omega^{q-1}r_{p+1} : \Omega^q K(\pi_{p+1}(Y). p + 1)) \to \Omega^{q-1}F_{p+1}$$

as in the cartesisn diagram

$$\begin{array}{ccc} K(\pi_{p+1}(Y), p+1) & = & K(_{p+1}(Y), p+1) \\ \downarrow{\scriptstyle r_{p+1}} & & \downarrow \\ F_{p+1} & \longrightarrow & PK(\pi_{p+1}(Y), p+1) \\ \downarrow & & \downarrow{\scriptstyle \pi} \\ F_p & \xrightarrow[f_p]{} & K(\pi_{p+1}(Y), p+1). \end{array}$$

[4] To avoid too many symbols, we use these rather ambiguous notations. There should be no confusion if one writes out the corresponding commutative diagram.

Then $i\gamma(f)$ is given by the element represented by f' in the commutative diagram

$$\begin{array}{ccccccc} X/X^{p-q} & \xrightarrow{f} & \Omega^q K(\pi_{p+1}(Y), p+1) & \xrightarrow{\Omega^{q-1} r_{p+1}} & \Omega^{q-1} F_{p+1} & \xrightarrow{\Omega^{q-1} i_{p+1}} & \Omega^{q-1} Y \\ \downarrow & & & & \uparrow f' & & \\ X/X^{p-q} \cup CX^{p-q+1}/X^{p-q} & \longrightarrow & \longrightarrow & \longrightarrow & \lrcorner & & \end{array}$$

The map f' can be described as follows:

Since F_{p+1} is $(p+1)$-connected, we have an isomorphism

$$\pi_q(X^{p-q+1}/X^{p-q}, F_p) \xrightarrow{f_{p*}} \pi_q(X^{p-q+1}/X^{p-q}, K(\pi_{p+1}(Y), p+1))$$

Let f'' represent the counterimage of $[f \mid X^{p-q+1}/X^{p-q}]$. Let

$$f' : X/X^{p-1} \cup CX^{p-q+1}/X^{p-q} \to \Omega^{q-1} F_{p+1}$$

be defined by

$$f'(x) = \Omega^{q-1} r_{p+1} f(x) = (*, f(x)), \qquad x \in X/X^{p-q}$$

$$f'(x, t) = (f''(x)(t), f(x)_t), \qquad (x, t) \in CX^{p-q+1}/X^{p-q}$$

where $f(x)_t$ is a path on $\Omega^{q-1} K(\pi_{p+1}(Y), p+1)$ given by $f(x)_t(\tau) = f(x)(\tau t)$. We see that f' is actually a map for $f'(x, 1) = (f''(x)(1), f(x)_1) = (\bar{x}, f(x))$ since $f''(x)$ is a loop on $\Omega^{q-1} F_p$ and

$$f_p f''(x)(t) = f_{p*}(f'')(x, t) = (f \mid X^{p-q+1}/X^{p-q})(x, t) = f(x)_t$$

for $(x, t) \in CX^{p-q+1}/X^{p-q}$.

If we assume that the natural isomorphism j^{-1} coincides with the obstruction isomorphism ob, we see that[5]

$$j^{-1}(f) = d^{p,q}(f, *)$$

which can be represented by the map

$$f''' : \Sigma X^{p-q+1}/X^{p-q} \to \Omega^{q-1} Y$$

given by

$$f'''(x, t) = (h_{p+1})_*^{-1}(k_{p+1})_* f(x)(t)$$

Thus

$$\gamma_1' j^{-1}(f)(x, t) = (s^{p-q+1})^* (h_{p+1})_*^{-1}(k_{p+1})_* f(x)(t),$$

where s^{p-q+1} is the map,

$$X/X^{p-q} \cup CX^{p-q+1}/X^{p-q} \to \Sigma X^{p-q+1}/X^{p-q},$$

in the Puppe sequence for the map $X^{p-q+1}/X^{p-q} \to X/X^{p-q}$. Now

$$i\gamma(f)(x) = i_{p+1}(\bar{x}, f(x)), \qquad x \in X/X^{p-q}$$

$$i\gamma(f)(x, t) = i_{p+1}(f''(x)(t), f(x)_t), \qquad (x, t) \in CX^{p-q+1}/X^{p-q}.$$

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[5] $d^{p,q}(f, *)$ is the difference cochain of f and the constant map $*$.

Since $i_{p+1} = i_p\, e_p$, we may write

$$i\gamma(f)(x) = * \quad \text{for } x \in X/X^{p-q}$$

$$i\gamma(f)(x,\, t) = i_p f'(x)(t) \quad \text{for } (x,\, t) \in \mathbf{C}X^{p-q+1}/X^{p-q}$$

If we regard Y_{p+1} as obtained from Y by killing homotopy groups higher than $p+1$, we see immediately that

$$\gamma_1' j^{-1}(f)(x) = *, \quad x \in X/X^{p-q}$$

for s^{p-q+1} is the collapsing map, and

$$\begin{aligned} \gamma_1' j^{-1}(f)(x,\, t) &= (k_{p+1})_* f(x)(t), \quad (x,\, t) \in \mathbf{C}X^{p-q+1}/X^{p-q} \\ &= k_{p+1} f(x)(t) \end{aligned}$$

which is clearly equal to $i_p f''(x)(t)$ in view of the homotopy commutative diagram

$$\begin{array}{ccc} & K(\pi_{p+1}(Y),\, p+1) & \\ {}^{f_p}\nearrow & & \searrow{}^{k_{p+1}} \\ F_p & \longrightarrow & Y_{p+1} \end{array}$$

So now it remains to prove

LEMMA 7. *The isomorphism j^{-1} is just the obstruction isomorphism.*

Proof. By the remark we made after Corollary 4, we need only show that j^{-1} and the obstruction isomorphism ob coincide when $X = S^0$. In this case $j^{-1} : \pi_q(S^0, K(\pi_{p+1}(Y), p+1)) \to H^{p-q+1}(S^0, \pi_{p+1}(Y))$ for $q \neq p+1$, is the zero homomorphism and

$$j^{-1} : \pi_{p+1}(S^0,\, K(\pi_{p+1}(Y),\, p+1)) \to H^0(S^0,\, \pi_{p+1}(Y))$$

is given by

$$\begin{aligned} j^{-1}[f] &= (h_{p+1})_*^{-1}(q^{p-q})(k_{p+1})_*(g'^{p-q})^{*-1}[f] \\ &= (h_{p+1})_*^{-1}(k_{p+1})_*[f]. \end{aligned}$$

But

$$\text{ob}\,[f] = d^0(f,\, *) = (h_{p+1})_*^{-1}(k_{p+1})_*[f].$$

Hence the proof is complete for Theorem 1.

COROLLARY 8. *When X is finite dimensional, there is a spectral sequence with $E^{p,q} = \pi_q(X^p/X^{p-1},\, Y)$ converging to the graded group associated with $\pi_*(X,\, Y)$ filtered by the kernels of $\pi_*(X,\, Y) \to \pi_*(X^p,\, Y)$. In particular, we show that if X is a finite dimensional polyhedron and $Y = G$ a topological group, the Shih spectral sequence coincides with that of Atiyah-Hirzebruch for K-theory* [2].

Remarks 1. The proof of Theorem 1 is quite conceptual until it breaks down in Lemma 6. In proving Lemma 6, we actually assume Corollary 4 in order to have Lemma 7. However, Corollary 4 holds without Theorem 1. In the Shih spectral sequence, the Kan definition of homotopy groups is used. As X is a finite-dimensional polyhedron, the two definitions coincide.

2. We actually proved Theorem 1 in the relative form, i.e., we use the functors $\pi_*(f,\ \ )$ and $\pi_*(\ \ , g)$ instead of $\pi_*(X,\ \ )$ and $\pi_*(\ \ , Y)$ since f is a cofibration and g a fibration, we can always replace them by its cofiber and fiber.

3. *Proof of Theorem* 2. It is well known that homology decompositions of a space are not homotopy invariants [1]. The following example[6] shows that their associated spectral sequences are not homotopy invariants.

Let $n \geq 7$ be an integer. Let h represent the generator of the group $\pi_{n+6}(S^n) = \mathbf{Z}_2$. Let h' be an extension of h to $S^{n+6} \cup {}_2e^{n+7}$ where integer 2 indicates the degree of the attaching map. Let $k : S^{n+5} \cup {}_2e^{n+6} \to S^{n+6}$ be the collapsing map.

$$
\begin{array}{ccc}
S^{n+6} \cup {}_2e^{n+7} & \xrightarrow{\ h'\ } & S^n \\
\downarrow j & \nearrow h & \\
S^{n+6} & & \\
\uparrow k & & \\
S^{n+5} \cup {}_2e^{n+6} & &
\end{array}
\tag{6}
$$

Let

$$
\begin{aligned}
X &= S^n \cup_h C(S^{n+6} \cup {}_2e^{n+7}) \vee \Sigma(S^{n+5} \cup {}_2e^{n+6}),\\
X_{n+6} &= S^n \vee \Sigma(S^{n+5} \cup {}_2e^{n+6}),\\
X'_{n+6} &= S^n \cup_{hk} C(S^{n+5} \cup {}_2e^{n+6}).
\end{aligned}
$$

Then the sequences

$$
\begin{aligned}
&* \to S^n \to X_{n+6} \xrightarrow{\ k^{n+7}\ } X\\
&* \to S^n \to X'_{n+6} \xrightarrow{\ k'^{n+7}\ } X
\end{aligned}
$$

are two (non-homotopic) homology decompositions of X, where k^{n+7} is the canonical embedding and k'^{n+7} is defined as follows

$$
\begin{aligned}
k'^{n+7}(x) &= x \quad \text{for} \quad x \in S^n\\
k'^{n+1}(x, t) &= (jkx, t) \quad \text{for} \quad (x, t) \in C(X^{n+5} \cup {}_2e^{n+6}).
\end{aligned}
$$

[6] This example is a special case of the Brown-Copeland example [1].

We now show that the spectral sequences associated with these two decompositions are different. Let $Y = S^{n+q+1}$. If we apply the modified homotopy functor $\pi_*(\ \ , Y)$ to the first decomposition, we have a spectral sequence which we may represent by diagram as follows:

$$\begin{array}{ccccccc} & & & & & & \pi_q(X, Y) \\ & & & & & & \downarrow (k^{n+7})^* \\ \pi_{q+1}(X_{n+6}, Y) & \overset{\alpha_1'}{\longleftrightarrow} & \pi_{q+1}(X^n, Y) & \xrightarrow{\beta_1' = 0} & \pi_q(\Sigma S^{n+5} \cup {}_2e^{n+6}, Y) & \rightarrowtail & \pi_q(X_{n+6}, Y). \end{array}$$

The horizontal sequence is induced by the Puppe sequence

$$S^n \to X_{n+6} \to \Sigma(S^{n+5} \cup {}_2e^{n+6}) \to \cdots$$

Now it is evident that from the way γ_1' is induced, γ_1' maps $\pi_q(\Sigma S^{n+5} \cup {}_2e^{n+6}, Y)$ isomorphically onto the direct factor $\pi_q(\Sigma S^{n+5} \cup {}_2e^{n+6})$ of $\pi_q(X_{n+6}, Y)$ and that $\pi_q(\Sigma S^{n+5} \cup {}_2e^{n+6})$ in $\pi_q(x_{n+6}, Y)$ is in the image of $(k^{n+7})^*$. Since β_1' is trivial, we see immediately that

$$\begin{aligned} {}_IE_\infty^{n+6,q} &= \gamma_1'^{-1}(\text{Im}\ (k^{n+7})_*)/\beta_1'(\ker\ (\pi_{q+1}(S^n, Y) \to \pi_{q+1}(*, Y)) \\ &= \pi_q(\Sigma S^{n+5} \cup {}_2e^{n+6}, Y) \end{aligned}$$

which we will soon show is isomorphic to $\mathbf{Z}_2$.

Similarly, we apply the functor $\pi_*(\ \ , Y)$ to the second decomposition to obtain the spectral sequence represented by the diagram

$$\begin{array}{ccccccc} \pi_{q+1}(X'_{n+6}, Y) & \xrightarrow{\alpha_2'} & \pi_{q+1}(S^n, Y) & \xrightarrow{\beta_2'} & \pi_{q+1}(\Sigma S^{n+5} \cup {}_2e^{n+6}, Y) & \xrightarrow{\gamma_2'} & \pi_q(X_{n+6}, Y). \\ & & & {\scriptstyle h^*}\searrow \quad \nearrow{\scriptstyle k^*} & & & \\ & & & \pi_{q+1}(S^{n+6}, Y) & & & \end{array}$$

We want to show that k^*h^* is nontrivial and then to calculate the ${}_{II}E_\infty$ term of the spectral sequence. Since $n + q + 1 < 2(n + q + 1) - 1$, we have, by the suspension theorem,

$$\pi_{q+1}(S^{n+6}, Y) = \pi_{q+1}(S^{n+6}, S^{n+q+1}) \simeq \mathbf{Z}_2.$$

From the sequence

$$\begin{aligned} \pi_{q+1}(S^{n+6}, Y) \xrightarrow{\times 2} \pi_{q+1}(S^{n+6}, Y) \xrightarrow{k^*} \pi_q(\Sigma S^{n+5} \cup {}_2e^{n+6}, Y) \\ \to \pi_q(S^{n+6}, Y) \to \pi_q(S^{n+6}, Y) \end{aligned}$$

we see that

$$\pi_q(\Sigma S^{n+5} \cup {}_2e^{n+6}, Y) \overset{k^*}{\cong} \pi_{q+1}(S^{n+6}, Y) = \mathbf{Z}_2$$

since $\pi_{m+5}(S^m) = 0$ for $m \geq 7$. By our choice, h^* sends the fundamental class in $\pi_{q+1}(S^n, Y)$ to the generator $[h]$ of $\pi_{q+1}(S^{n+6}, Y)$. Hence h^* is epic. In view of all these data, we have

$$_{II}E_\infty^{n+6,q} = (\gamma_2'^{-1}(\text{Im } k'^{n+7})/\text{Im } \beta_2') \subseteq \pi_{q+1}(\Sigma S^{n+5} \cup {}_2e^{n+6}, Y)/\text{Im } k^*h^*$$
$$\simeq \mathbf{Z}_2/\mathbf{Z}_2 = 0.$$

Therefore we conclude that the two sequences are not the same. As a consequence, we see that the two homology decompositions are not homotopic. This example also kills the hope that the spectral sequence associated with a homology decomposition is isomorphic to that associated with the skeleton decomposition since we saw that the latter is homotopy invariant. However, in [1], some sufficient conditions on the space X are given to insure that all homology decompositions of X are homotopic. Then one may ask again whether, in this case, the spectral sequence associated with the homology decomposition is isomorphic to that associated with the skeleton decomposition. The answer to this question is again negative as we see from the following example.

Let $X = S^p \cup {}_ke^{p+1}$, $p > 1$. Clearly, X admits only the trivial homology decomposition. The infinite term $E_\infty^{p+1,q}$ of the corresponding spectral sequence is clearly trivial since $E^{p+1,q} = \pi_q(X_{p+1}/X_p, Y) = \pi_q(X/X, Y) = 0$. On the other hand, the spectral sequence associated with the skeleton decomposition

$$* \longrightarrow S^p \longrightarrow X$$

can be represented by the diagram

$$\pi_{q+1}(S^p, Y) \xrightarrow{\beta'} \pi_q(S^{p+1}, Y) = \pi_q(X^{p+1}/X^p, Y) \xrightarrow{\gamma'} \pi_q(X, Y).$$

If we let $Y = S^{p+q+1}$, β' is simply multiplication by k. Hence the term

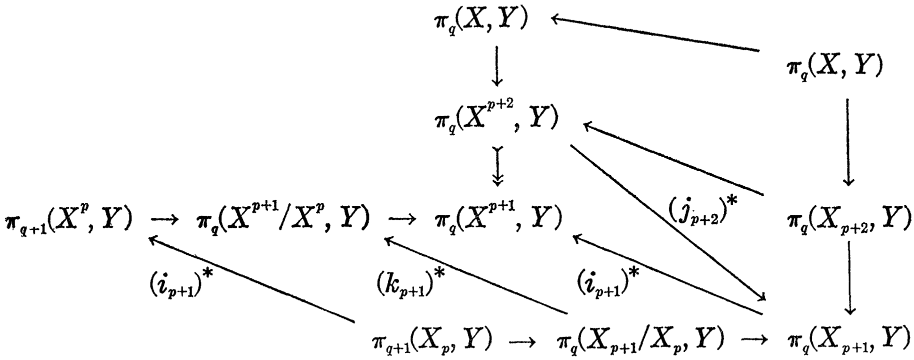
$$E_\infty^{p+1,q} = \gamma^{-1}(\pi_q(X, Y)/\text{Im } \beta' = \mathbf{Z}_k \neq 0.$$

We saw in the above example that the non-triviality of the attaching map is really the reason why the two spectral sequences do not coincide. Thus we are led to think that when there is torsion in the space X, the two spectral sequences are different. This turns out to be true.

We now begin our proof of Theorem 2. Let $j_{p+1} : X_p \longrightarrow X^{p+1}$ be the inclusion map. To prove the theorem, we need only show that if X has torsion, the spectral sequences will be different. So let us suppose there is a least integer p such that $H_p(X)$ has torsion, i.e.,

$$K'(H_p(X), p - 1) = \bigvee S_i^{p-1} \cup_f \bigvee e_j^p$$

with $f \mid S_{j_0} \to S_{i_0}$ essential for some i_0, j_0. The relation between the spectral sequences in consideration can be expressed by the commutative diagram



where $k_{p+1} : X^{p+1}/X^p \to X_{p+1}/X_p$ is the induced map between the cofibers of $X^p \subseteq X^{p+1}$ and $X_p \subseteq X_{p+1}$. We try to show that there is an element in ${}_sE^p_{\infty,q}$ of the spectral sequence associated with the skeleton decomposition which is not in the image of the induced map of spectral sequences. For this purpose, we make explicit the following spaces:

$$X_{p+1} = X_{p-1} \cup \mathrm{C}(\vee S_i^{p-1} \cup_f \vee e_j^p) \cup_g \mathrm{C}K'(\mathrm{H}_{p+1}(X),\ p),$$
$$X^{p+1} = X_{p-1} \cup \mathrm{C}(\vee S_i^{p-1} \cup_f \vee e_j^p) \cup_{g'} \vee e^{p+1},$$

$g' = g \mid (p+1)$ cells of $K'(H_{p+1}(X), p)$.

$$X^{p+1}/X^p = \vee S_j^{p+1} \vee S^{p+1},$$

where S_j^{p+1} comes from the cells e_j^p, etc.

$$X^{p+1}/X_p = \vee S^{p+1}$$
$$X_{p-1} = X^{p-1}$$

Now let Y be obtained from S^{p+q+1} by killing all homotopy groups of S^{p+q+1} higher than $p + q + 1$. Our idea of proof is to pick an element in $\pi_q(X^{p+1}/X^p, Y)$ in such a way that it cannot be pulled back by $(i_{p+1})^*$. To do this, we first show that the homomorphism

$$\pi_{q+1}(X^p, Y) \to \pi_q(X^{p+1}/X^p, Y)$$

does not cover the factor $\pi_q(\vee S_j^{p+1}, Y)$ in $\pi_q(X^{p+1}/X^p, Y)$. We consider the following Puppe sequence

$$\vee S_j^p \xrightarrow{f} X_{p-1} \cup \vee e_i^p = X^p \to X_p \to \vee S_j^{p+1} \to \cdots$$

which induces the exact sequence

$$\pi_{q+1}(X^p, Y) \to \pi_{q+1}(\vee S_j^p) = \pi_q(\vee S_j^{p+1}) \to \pi_q(X_p, Y) \to \cdots.$$

Let

$$X^p \xrightarrow{\pi} X^p/X_{p-1}$$

be the canonical projection. We have the commutive diagram

$$\begin{array}{ccc}
\pi_{q+1}(X_{p-1}, Y) = \pi_{q+1}(X^{p-1}, Y) = 0 & & \\
\uparrow & & \\
\pi_{q+1}(X^p, Y) & \longrightarrow & \pi_{q+1}(\bigvee S_j^p, Y) \\
\uparrow \pi^* & & \uparrow \\
\pi_q(\bigvee S_i^{p+1}, Y) = \pi_{q+1}(X^p/X_{p-1}, Y) & \xrightarrow{f^*} & \pi_q(\bigvee S_j^{p+1}, Y).
\end{array}$$

Since π^* is onto and f^* is not, it is clear that the homomorphism

$$\pi_{q+1}(X^p, Y) \to \pi_{q+1}(\bigvee S_j^{p+1}, Y)$$

is not epic. Next, we observe that the image of $\pi_q(\bigvee S_j^{p+1}, Y)$ in $\pi_{q-1}(X^{p+2}/X^{p+1}, Y)$ is trivial. To prove this, it suffices to show that the attaching map g'' in

$$X_{p+2}/X_{p-1} = (\bigvee S_j^p \cup_f \bigvee e_j^{p+1}) \cup_g CK'(H_{p+1}(X), p)) \cup_{g''} CK'(H_{p+2}(X), p+1$$

induces trivial attaching map from the $(p+1)$-cells of $K'(H_{p+2}(X), p+1)$ to e_j^{p+1}, i.e., the induced map g_1'' in

$$(\bigvee S_j^{p+1} \vee K'(H_{p+1}(X), p+1)) \cup_{g''} \bigvee e^{p+2}$$

(we will justify this later) is trivial when composed with the embedding when restricted to $\bigvee S_j^{p+1}$. For if this were not trivial, we would have a nontrivial homomorphism:

$$H_{p+1}(\bigvee S^{p+1}) \to H_{p+1}(\bigvee S_j^{p+1}).$$

This is not the case because this homomorphism factors thru a trivial homomorphism as in the diagram

$$\begin{array}{ccc}
H_{p+1}(\bigvee S^{p+1}) & & H_{p+1}(S_i^p \cup_f \bigvee e^{p+1} \cup_g CK'(H_{p+1}(X),p) \\
\downarrow & & \| \\
H_{p+2}(\Sigma K'(H_{p+2}(X),p+1) & \xrightarrow{g''_*=0} & H_{p+1}(K'(H_{p+1}(X),p+1)) \\
& & \downarrow \text{projection} \\
& & H_{p+1}(\bigvee S_i^{p+1} \vee K'(H_{p+1}(X),p+1)) \\
& & \downarrow \\
& & H_{p+1}(\bigvee S_j^{p+1}) \oplus H_{p+1}(X) \\
& & \downarrow \\
& & H_{p+1}(\bigvee S_j^{p+1}).
\end{array}$$

To complete the proof, let $h : \bigvee S_j^{p+1} \to Y$ represent an element that is not in Im α'. By the remark above, $[h]$ comes from $\pi_q(X^{p+2}, Y)$; hence from $\pi_q(X, Y)$. Thus h represents a nontrivial element in ${}_sE_\infty^{p,q}$. Let d represent the class $[d]$ such that $(i_{p+1})^*[d] = \gamma'[h]$. We argue that $[d]$ is not in Im $(\pi_q(X_{p+1}/X_p, Y))$. Suppose it were. Then $[d]$ goes to zero in $\pi_q(X_p, Y)$ and hence $\gamma[h]$ comes from $\pi_q(X^{p+1}/X_p, Y) = \pi_q(\bigvee S^{p+1}, Y)$ (see diagram below). This is impossible.

$$\begin{array}{ccc}
\pi_q(X^{p+1}/X_p, Y) = \pi_q(S^{p+1}, Y) & & \\
\downarrow & & \\
\pi_q(X^{p+1}, Y) & \longrightarrow & \pi_q(X^p, Y) \\
(i_{p+1})^* \uparrow & \searrow (j_{p+1})^* & \uparrow (i_p)^* \\
\pi_q(X_{p+1}, Y) & \longrightarrow & \pi_q(X_p, Y)
\end{array}$$

We now justify the fact that after identifying $\bigvee S_i^p$ to a point, the space X_{p+2}/X_{p-1} will become

$$\bigvee S_i^{p+1} \vee K'(H_{p+1}(X) \cup_{g_1''} \vee e^{p+2}).$$

LEMMA. Let

$$X = K'(G, p) \cup_g CK'(G', p) = (\bigvee S_i^p \cup \bigvee e_j^{p+1}) \cup_g CK'(G', p),$$

where g is homologically trivial. Then

$$X/\bigvee S_i^p = \bigvee S_j^{p+1} \vee K'(G', p + 1) \quad \textit{for } p \geq 2$$

Proof. Consider the sequence

$$\pi_{p+2}(K'(G, p), \bigvee S^p) \to \pi_{p+1}(\bigvee S^p) \to \pi_{p+1}(K'(G, p))$$

$$\xrightarrow{\phi} \pi_{p+1}(K'(G, p), \bigvee S^p) \to \pi_p(\bigvee S^p) \to G.$$

Since $\bigvee S^p$ is $(p - 1)$-connected and $\bigvee S_j^{p+1}$ is p-connected, by Proposition 4,

$$\pi_{p+1}(K'(G, p), \bigvee S^p) \cong \pi_{p+1}(\bigvee S_j^{p+1})$$

since $p + 1 < 2p$. Thus the homomorphism ϕ is trivial since $(\pi_{p+1} \bigvee S_j^{p+1})$ is just the kernel of $\pi_p(\bigvee S^p) \to G$. Now the assertion follows from the following diagram

$$\begin{array}{ccccccccc}
0 \to & \text{Ext}\,(G', \pi_{p+1}(K'(G, p))) & \to & \pi_p(G', K'(G, p)) & \xrightarrow{\eta} & \text{Hom}\,(G', G) & \to 0 \\
& \downarrow \phi_* = 0 & & \downarrow & & \downarrow & \\
0 \to & \text{Ext}\,(G', \pi_{p+1}(\bigvee S_j^{p+1})) & \to & \pi_p(G', \bigvee S_j^{p+1}) & \xrightarrow{\eta} & \text{Hom}\,(G', \oplus Z_j) & \to 0
\end{array}$$

because the element $[g]$ lies in the kernel of η.

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