ONE-FLAT SUBMANIFOLDS WITH CODIMENSION TWO

BY

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The following is in the PL-category. Manifolds are orientable and oriented and homeomorphisms are onto and orientation-preserving.

Some of the deep results recently obtained by C. T. C. Wall [8] and the theory of block bundles [7], [3] enable us to generalize a result by the author [5], [6] as follows. (For terminology, see [5], [6].)

Theorem. *Let (M_i, W_i) be one-flat $(n, n+2)$-manifold pairs, let N_i be regular neighborhoods of M_i in W_i, $i = 1, 2$, and let $f : M_1 \to M_2$ be a homeomorphism.*

Then f extends to a homeomorphism $g : N_1 \to N_2$ if and only if $f_* \chi_1 = \chi_2$ and the singularities at x and fx are the same for each point $x \in M_1$, where χ_i is the Euler class of (M_i, W_i).

(In [5] χ_i was called by the Stiefel-Whitney class.)

It is shown to be true by C. T. C. Wall [8] that each locally flat $(p, p+2)$-elementary (i.e., sphere or ball) pair (M, W) is *collared*, that is to say, a regular neighborhood N of M in W is $M \times B^2$, where B^2 is a 2-ball.

Let T be the frontier of N which is an admissible regular neighborhood (i.e., $N \cap W^{\cdot}$ is a regular neighborhood of $M^{\cdot}$ in $W^{\cdot}$ and N admissibly collapses to M) of a locally flat elementary pair (M, W). Then $T = M \times S^1$ where S^1 is a 1-sphere $B^{\cdot 2}$. A p-cycle mod $T^{\cdot}$ denoted by $M \times 0^1$, and a 1-cycle $0^p \times S^1$ are called *longitude* l and *meridian* m of T respectively where 0^1, 0^p are points of S^1 and of the interior M°. (The cycles should be consistent with the orientation of M, W, see [6].)

The theorem has been proved for $n \geq 3$ in [5], [6] (where it is assumed that the manifolds M_i are closed). The stumbling block was the following lemma for dimension $p = 3$. In the lemma, ρ_* means the homomorphism between homology groups with integer coefficients induced by ρ and $\sim$ means homologous.

Lemma 1. *Let (M, W) be a locally flat $(p, p+2)$-elementary pair and let N be an admissible regular neighborhood of (M, W). Let $\rho : N \to N$ be a homeomorphism such that $\rho \mid M$ = identity and $\rho_* m \sim m$ on T and $\rho_* l \sim l$ on T* mod $T^{\cdot}$. *Then ρ is pseudo-isotopic to the identity in N.*

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Proof. The similar lemma is true for $p = 1$ by the classical Baer theorem, see [6]. Let us assume that $p \geq 2$.

At first we note that $\pi_p(\tilde{P}L_2) = 0$ if $p \neq 1$, for notation, see [7]. By Theorem 3 of [8] and Theorems 5.6 and 5.7 of [7] we have $\pi_3(\tilde{P}L_2, PL_2(I)) = 0$. Since $PL_2(I)$ has the homotopy type of the orthogonal group O_2 by Lemma 3 of [8], $\pi_2(PL_2(I)) = 0$, and hence $\pi_3(\tilde{P}L_2) = 0$ by the exactness of the homotopy sequence. Then by the corollary B1 of [3, Part II], $\pi_p(\tilde{P}L_2) = 0$ if $p \neq 1$.

Let K, H, J be subdivisions of M, N, W such that K, H are subcomplexes of J. Then by Theorem 4.3 of [7] there is a 2-block bundle ξ over K with N as the total space. Let ξ_P be the principal $\tilde{P}L_2^p$-bundle over K associated with ξ. Since N is a collar of (M, W), ξ is trivial and there is a cross section $s : K \to E(\xi_P)$, the total space. By Theorem 4.4 of [7] it may be assumed that $\rho : E(\xi) \to E(\xi)$ is an automorphism. For each k-simplex Δ_i^k of K,

$$s(\Delta_i^k)^{-1} \cdot \rho \mid E(\xi \mid \Delta_i^k) \cdot s(\Delta_i^k) : \sigma^k \times I^2 \to \sigma^k \times I^2$$

is an automorphism of the trivial block bundle, where σ^k is the standard k-simplex. Then we may define a map $f : K \to \tilde{P}L_2$ by taking

$$f(\Delta_i) = s(\Delta_i)^{-1} \cdot \rho \mid E(\xi \mid \Delta_i) \cdot s(\Delta_i)$$

for each simplex Δ_i of K; for a map see [7]. Since the process can be reversed, we say that ρ and f are related to each other (with respect to the cross section s).

Now suppose that (M, W) is a sphere pair. Since $\pi_p(\tilde{P}L_2) = 0$, there is a homotopy

$$F : K \times I \to \tilde{P}L_2$$

between f and the identity. Let $r : K \times I \to \tilde{P}L_2$ be an extension of s (for example, r is the composition of the projection $p : K \times I \to K$ and s). Then the automorphism

$$\eta : E(\xi \times I) \to E(\xi \times I)$$

of the product bundle $\xi \times I$ related to F (with respect to r) is a pseudo-isotopy between ρ and the identity.

Next suppose that (M, W) is a ball pair. Then

$$\rho^{\cdot} = \rho \mid E(\xi \mid K^{\cdot}) : E(\xi \mid K^{\cdot}) \to E(\xi \mid K^{\cdot})$$

is a homeomorphism which satisfies the condition of Lemma 1 where M, N, W are replaced by $M^{\cdot}$, $N \cap W^{\cdot}$, $W^{\cdot}$ respectively. Since $(M^{\cdot}, W^{\cdot})$ is a sphere pair, there is a pseudo-isotopy between $\rho^{\cdot}$ and the identity by Lemma 1 for sphere pairs. Let

$$F : K^{\cdot} \times I \to \tilde{P}L_2$$

be a map related to the pseudo-isotopy. Then $F \mid K^{\cdot} \times \{0\} = f^b$ (the map related to $\rho^\cdot$) and $F \mid K^\cdot \times \{1\}$ = identity. Let us define a map

$$g : (K \times I)^\cdot \to \tilde{P}L_2$$

by $g \mid K \times \{0\} = f$ (the map related to ρ), $g \mid K \times \{1\}$ = identity and $g \mid K \times I = F$. Since $(K \times I)^\cdot$ is a p-sphere and $\pi_p(\tilde{P}L_2) = 0$, g extends to $G : K \times I \to \tilde{P}L_2$. Then the pseudo-isotopy related to G is the required one.

Now following Gluck [2], we have

COROLLARY 1. *The group of pseudo-isotopy classes of automorphisms of* $S^p \times S^1$ *for* $p > 1$ *is isomorphic to* $Z_2 + Z_2 + Z_2$.

See [1], [4] for comparison.

COROLLARY 2. *There are at most two knots* (S^p, S^{p+2}) $p > 1$, *which have equivalent complements.*

For higher dimensional knots, see [6].

Let us review some notions used in [5], [6]. Let (M, W) be an $(n, n + 2)$-manifold pair and (K, J) a full subdivision of the pair. Let Δ be a q-simplex of K. Let ∇, $\square$ denote $n - q$, $n - q + 2$-balls which are dual to Δ in K, J respectively. Then $(\nabla, \square)$ is an $(n - q, n - q + 2)$-ball pair such that $(\nabla, \square) = x * (\gamma, \Gamma)$, the join of the barycenter x of Δ and (γ, Γ) where γ, Γ are isomorphic to the first barycentric subdivision of links Lk $(\Delta, K)'$, Lk $(\Delta, J)'$ respectively, see [6], so that (γ, Γ) is an $(n - q - 1, n - q + 1)$-elementary pair, i.e., (γ, Γ) is a sphere pair if the interior Δ° is in the interior M° and (γ, Γ) is a ball pair otherwise. By $\mathfrak{R}^q$, $\mathfrak{K}^{q+2}$ we denote polyhedra consisting of dual balls ∇, $\square$ respectively where $\Delta \in K - K^{n-q-1}$. They may be regarded as subcomplexes of K', J' respectively such that $\mathfrak{R}^n = K'$ and $\mathfrak{K}^{n+2} = N(K, J')$, the star neighborhood. We say that (M, W) is *flat* at a point x of M if the star pair (St (x, K), St (x, J)) is flat and that (M, W) is *q-flat* if (M, W) is flat at each point $x \in K - K^{q-1}$. We say that (M, W) is *locally flat* if it is 0-flat.

The following lemma will be proved by induction on p assuming the lemma is true for $p - 1$, because it has been proved for $p = 1, 2$, see [5], [6].

LEMMA 2. *Let* (M_i, W_i) *be* $(n - p + 1)$*-flat* $(n, n + 2)$*-pairs and let* (K_i, J_i) *be full subdivisions,* $i = 1, 2$. *Let* $f : M_1 \to M_2$ *be a homeomorphism which is simplicial with respect to* K_1 *and* K_2 *such that* $f_* \chi_1 = \chi_2$ *and the pair* $(\gamma_{1j}, \Gamma_{1j})$ *is homeomorphic to* $(\gamma_{2j}, \Gamma_{2j})$ *for each pair of corresponding* $(n - p)$*-simplexes* Δ_{1j} *of* K_1 *and* $\Delta_{2j} = f\Delta_{1j}$ *of* K_2. *Then there is a homeomorphism* $g^p : \mathfrak{K}_1^{p+2} \to \mathfrak{K}_2^{p+2}$ *such that* $g^p \mid \mathfrak{R}_1^p = f$ *and* $g^p \square_{1j} = \square_{2j}$ *for each pair of r-simplexes* Δ_{ij} *of* K_i $(r \geq n - p, i = 1, 2)$.

Proof. Since (M_i, W_i) is $(n - p + 1)$-flat, $(\gamma_{ijk}, \Gamma_{ijk})$ is flat for each $(n - p + 1)$-simplex Δ_{ijk} of K_i [5], [6]. By the inductive hypothesis there is a homeomorphism $g^{p-1} : \mathcal{H}_1^{p+1} \to \mathcal{H}_2^{p+1}$ satisfying the conditions. By [6] $\bigcup_k \square_{ijk}$, say N_{ij}, is an admissible regular neighborhood of γ_{ij} in Γ_{ij} where Δ_{ijk} are $(n - p + 1)$-simplexes incident with an $(n - p)$-simplex Δ_{ij} of K_i. Since (M_i, W_i) are $(n - p + 1)$-flat, the $(p - 1, p + 1)$-pairs $(\gamma_{ij}, \Gamma_{ij})$ are locally flat [6]. By the corollary to Theorem 3 of [8] $(\gamma_{ij}, \Gamma_{ij})$ are collared. By the construction of g^{p-1} it is verified that

$$g^{p-1} \mid N_{1j} : N_{1j} \to N_{2j}$$

is a homeomorphism such that $g^{p-1}\gamma_{1j} = \gamma_{2j}$, $g_*^{p-1} m_{1j} \sim m_{2j}$ on T_{2j}, $g_*^{p-1} l_{1j} \sim l_{2j}$ on T_{2j} mod $T_{2j}^{\cdot}$ where m_{ij} and l_{ij} are meridians and longitudes of T_{ij} which are the frontiers of N_{ij}.

Since $(\gamma_{1j}, \Gamma_{1j})$ is homeomorphic to $(\gamma_{2j}, \Gamma_{2j})$, there is a homeomorphism $\theta : \Gamma_{1j} \to \Gamma_{2j}$ such that $\theta\gamma_{1j} = \gamma_{2j}$. By Theorem 4.4 of [7] it is assumed that $\theta N_{1j} = N_{2j}$ and

$$\rho = g^{p-1}\theta^{-1} \mid N_{2j} : N_{2j} \to N_{2j}$$

is a homeomorphism which satisfies the conditions in Lemma 1. Hence ρ is pseudo-isotopic to the identity in N_{2j}. Using a collar of N_{2j} in Γ_{2j} we have a homeomorphism $g_j' : \Gamma_{1j} \to \Gamma_{2j}$ such that $g_j' \mid N_{1j} = \rho\theta \mid N_{1j} = g^{p-1} \mid N_{1j}$. Since $(\nabla_{ij}, \square_{ij}) = x_{ij} * (\gamma_{ij}, \Gamma_{ij})$, we have the conical extension $g_j : \square_{1j} \to \square_{2j}$ of g_j'. Then $g^p : \mathcal{H}_1^{p+2} \to \mathcal{H}_2^{p+2}$ is obtained by taking $g^p \mid \square_{1j} = g_j$ for each Δ_{1j} of K_1, completing the inductive step for p.

Proof of theorem. The necessity follows from [5], [6]. Let (K_i, J_i) be full subdivisions of (M_i, W_i) satisfying the conditions of Lemma 2. If $p = n$, then $n - p + 1 = 1$, $\mathcal{R}_i^p = \mathcal{R}_i^n = M_i$ and $\mathcal{H}_i^{p+2} = \mathcal{H}_i^{n+2} = N(K_i, J_i')$, proving the theorem.

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