ON MOMENT SEQUENCES OF OPERATORS

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1. Introduction

Let X, Y be Banach spaces over the complex field and denote by $B \equiv B(X, Y)$ the space of continuous linear operators on X into Y. Recently Tucker [6] has introduced a weak extension Y^+ of the Banach space Y and has proved that $B^+ \subseteq B(X, Y^+)$. The weak extension Y^+ is by construction a subspace of Y^{**}, consequently if $\overline{B^+}$ denotes the closure of B^+ in $B^{**}(X, Y)$ topologized in the natural way we obtain $\overline{B^+} \subseteq B(X, Y^{**})$.

DEFINITION 1. Given a sequence $\{\psi_n(t)\} (n \geq 0) \subseteq C[0, 1]$, the sequence $\{A_n\} \subseteq B(X, Y)$ is called a weak moment sequence with respect to $\{\psi_n(t)\}$ if there exists a vector-valued measure μ, defined on the σ-field of Borel sets in [0, 1] into $\overline{B^+}$ such that

(i) $\mu(\cdot)b^*$ is in rca [0, 1] for each $b^* \in B^*(X, Y)$;
(ii) the mapping $b^* \to \mu(\cdot)b^*$ is continuous with the $B(X, Y)$ and $C[0, 1]$ topologies of $B^*(X, Y)$ and rca [0, 1] respectively;
(iii) $b^*A_n = \int_0^1 \psi_n(t)\mu(dt)b^* \quad n = 0, 1, 2, \cdots, b^* \in B^*(X, Y)$;
(iv) $\| \mu \|[0, 1] = \sup \| \sum \alpha_i \mu(E_i) \| < \infty$,

where the supremum is taken over all finite collections of disjoint Borel sets in [0, 1] and all finite sets of scalars α_i with $|\alpha_i| \leq 1$.

DEFINITION 2. Given a sequence $\{\psi_n(t)\} \subseteq C[0, 1]$, the sequence $\{A_n\} \subseteq B(X, Y)$ is called a strong moment sequence with respect to $\{\psi_n(t)\}$ if there exists a vector-valued measure μ, defined on the σ-field of Borel sets in [0, 1] into $B(X, Y)$ such that

(i) $b^*\mu(\cdot)$ is in rca [0, 1], $b^* \in B^*(X, Y)$;
(ii) $A_n = \int_0^1 \psi_n(t)\mu(dt) \quad n = 0, 1, 2, \cdots$;
(iii) $\| \mu \|[0, 1] < \infty$.

(For definitions and details see [2].)

It is our purpose to obtain necessary and sufficient conditions on a sequence $\{A_n\}$ $(n \geq 0)$ of operators in $B(X, Y)$ in order that it will be a weak or a strong moment sequence with respect to $\{\psi_n(t)\}$ $(n \geq 0)$ in various cases of sequences $\{\psi_n(t)\}$. We shall be interested, especially, in the case where $\psi_n(t) = t^{\lambda_n}$, $n \geq 0$, where the sequence $\{\lambda_n\}$ $(n \geq 0)$ satisfies

$$(1.1) \quad 0 \leq \lambda_0 < \lambda_1 < \cdots < \lambda_n < \cdots \uparrow \infty, \qquad \sum_{i=1}^{\infty} 1/\lambda_i = \infty.$$

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2. Preliminaries

Let $\mathfrak{A} = \| a_{nm} \|, n \geq 0, m \geq 1$, be an infinite matrix of real numbers where $a_{n1} = 1$ for $n = 0, 1, 2, \cdots$.
Denote

$$(i_1, \cdots, i_m) = \det \| a_{i_k,r} \|, \qquad 0 \leq i_1 < \cdots < i_m, r = 1, \cdots, m$$

(if $m = 1(i_1) = a_{i_1,1} = 1$) and assume that $(i_1, \cdots, i_m) > 0$ for every $0 \leq i_1 < \cdots < i_m$. For a given sequence of operators $\{A_n\}$ $(n \geq 0)$ define

$$(2.1) \quad D^k A_i = \textstyle\sum_{j=0}^k (-1)^j (i, \cdots, i+j-1, i+j+1, \cdots, i+k) A_{i+j}, \qquad k > 0,$$

$$D^0 A_i = A_i$$

and

$$(2.2) \quad \lambda_{nm} = \frac{(0, m+1, \cdots, \cdots n)}{(m+1, \cdots, n)(m, \cdots, n)} D^{n-m} A_m, \quad 0 \leq m < n = 1, 2, \cdots$$

$$\lambda_{nn} = A_n, \qquad n = 0, 1, 2, \cdots.$$

For every fixed n, assuming the λ_{nm}, $0 \leq m \leq n$, are known, (2.2) are $n + 1$ linear equalities with $n + 1$ unknowns $A_0, \cdots, A_n$. It was shown by Schoenberg [4] that solving the equalities (2.2) we get

$$(2.3) \quad A_k = \sum_{m=0}^{n} \frac{(k, m+1, \cdots, n)}{(0, m+1, \cdots, n)} \lambda_{nm}, \qquad 0 \leq k \leq n = 0, 1, 2, \cdots$$

(the coefficient of λ_{nn} is $(k)/(0) = 1$).
Denote

$$C_{kmn} = \frac{(k, m+1, \cdots, n)}{(0, m+1, \cdots, n)}, \qquad 0 \leq k, m \leq n = 0, 1, 2, \cdots$$

and

$$t_{nm} = C_{1mn}, \quad 0 \leq m \leq n = 0, 1, 2, \cdots.$$

We shall use the following results due to Schoenberg [4] (see (8.11), (8.17), (8.23) and the proof of Theorem 8.1).

(a) $0 = t_{n0} < t_{n1} < \cdots t_{nn} = 1$.

(b) Let the points $\{(t_{nm}, C_{kmn})\}$ $(0 \leq m \leq n, n \geq k)$ be the vertices of a polygon $P_k^{(n)}$ and let $P_k^{(n)}(t), 0 \leq t \leq 1$, be the function describing that polygon. Then for each fixed k, $k = 0, 1, 2, \cdots$ the functions $P_k^{(n)}(t)$ tend, as $n \to \infty$, to a continuous function $\phi_k(t)$, uniformly in $0 \leq t \leq 1$.

(c) Define as in (2.1) and (2.2)

$$D^k \phi_i(t) = \textstyle\sum_{j=0}^k (-1)^j (i, \cdots, i+j-1, i+j+1, \cdots, i+k) \phi_{i+j}(t), \quad k > 0$$

$$D^0 \phi_i(t) = \phi_i(t)$$

and

$$\lambda_{nm}(t) = \frac{(0, m+1, \cdots, n)}{(m+1, \cdots, n)(m, \cdots n)} D^{n-m}\phi_m(t), \quad 0 \le m < n = 1, 2, \cdots$$

$$\lambda_{nn}(t) = \phi_n(t), \qquad n = 0, 1, 2, \cdots,$$

then

(2.4) $\lambda_{nm}(t) \ge 0$ for $0 \le t \le 1$ and $0 \le m \le n = 0, 1, 2, \cdots$

and

$$\sum_{m=0}^{n} \lambda_{nm}(t) \equiv 1.$$

3. Weak moment sequences

THEOREM 1. *Suppose that the sequence $\{\phi_n(t)\}$ $(n \ge 0)$ is a fundamental set in $C[0, 1]$, that is, $\{\phi_n(t)\}$ $(n \ge 0)$ spans $C[0, 1]$ in the maximum norm. Then the sequence $\{A_n\}$ $(n \ge 0)$ of operators in $B(X, Y)$ is a weak moment sequence with respect to the sequence $\{\phi_n(t)\}$ $(n \ge 0)$ if and only if*

$$\sup \| \textstyle\sum_{m=0}^{n} \alpha_m \lambda_{nm} \| \equiv M < \infty \tag{3.1}$$

where the supremum is taken over all the finite set of scalars $\alpha_0, \cdots, \alpha_n$ with $|\alpha_m| \le 1$ and all $n \ge 0$. Moreover the semi-variation $\| \mu \|[0, 1] = M$.

Proof. Suppose, first, that (3.1) holds and define the operator

$$T : C[0, 1] \to B(X, Y)$$

as follows. For the finite linear combination $P(t) = \sum_{i=0}^{k} a_i\phi_i(t)$ define

$$T(P) = \textstyle\sum_{i=0}^{k} a_i A_i \tag{3.2}$$

We shall prove that $\| T(P) \| \le M \| P \|$ (where $\| P \| = \sup_{0\le t\le 1} | P(t) |$) and as the finite linear combinations of the $\phi_n(t)$, $n \ge 0$, are dense in $C[0, 1]$, we extend T by continuity to the whole $C[0, 1]$. Now, formula (2.3) can be written in the following way

$$A_k = \sum_{m=0}^{n} P_k^{(n)}(t_{nm})\lambda_{nm}, \qquad 0 \le k \le n = 0, 1, 2, \cdots.$$

Hence for $n \ge k$

$$\begin{aligned} T(P) = \sum_{i=0}^{k} a_i A_i &= \sum_{i=0}^{k} a_i \sum_{m=0}^{n} P_i^{(n)}(t_{nm})\lambda_{nm} \\ &= \sum_{m=0}^{n} [\sum_{i=0}^{k} a_i P_i^{(n)}(t_{nm})]\lambda_{nm} \end{aligned}$$

and by (3.1) we have for $n \ge k$

$$\| T(P) \| \le M \sup_{0\le t\le 1} | \textstyle\sum_{i=0}^{k} a_i P_i^{(n)}(t) |. \tag{3.3}$$

Since $P_i^{(n)}(t) \to \phi_i(t)$ uniformly in $0 \le t \le 1$ we obtain by (3.3)

$$\| T(P) \| \le M \sup_{0\le t\le 1} | \textstyle\sum_{i=0}^{k} a_i\phi_i(t) | = M \| P \|.$$

Hence $\| T \| \leq M$ and since it is readily seen by (3.1) that $\| T \| \geq M$ we have $\| T \| = M$. By the representation theorem of operators from $C[0, 1]$ to $B(X, Y)$ (see [2, Theorem VI. 7.2] or [1, Theorem 3.1]) there exists a vector valued measure μ, from the σ-field of Borel sets in [0, 1] to $B^{**}(X, Y)$ satisfying conditions (i) and (ii) of Definition 1 such that

$$(3.4) \qquad b^*T(f) = \int_0^1 f(t)\mu(dt)b^*, \qquad f \in C[0, 1],\ b^* \in B^*(X, Y)$$

and

$$(3.5) \qquad \| T \| = \| \mu \|[0, 1].$$

By (3.4)

$$b^*A_n = b^*T(\phi_n) = \int_0^1 \phi_n(t)\mu(dt)b^*, \quad n = 0, 1, 2, \cdots, \ b^* \in B^*(X, Y)$$

and by (3.5)

$$\| \mu \|[0, 1] = \sup \| \sum \alpha_i \mu(E_i) \| = M < \infty .$$

By the construction of μ in the proof of Theorem VI. 7.2 [2] and using the arguments similar to [5] and [6] one can easily prove that for each closed set $F \subseteq [0, 1]$, $\mu(F) \in B^+$ and thus it is readily seen that

$$\mu(E) \in \overline{B^+} \quad \text{for every Borel set } E \subseteq [0, 1].$$

Thus we proved that $\{A_n\}$ $(n \geq 0)$ is a weak moment sequence with respect to $\{\phi_n(t)\}$ $(n \geq 0)$.

Conversely, suppose that $\{A_n\}$ $(n \geq 0)$ is a weak moment sequence with respect to $\{\phi_n(t)\}$ $(n \geq 0)$. The vector-valued measure existing by Definition 1 defines an operator

$$T : C[0, 1] \rightarrow B(X, Y)$$

by the equation (3.4) (see [2, Theorem VI. 7.2]). The operator T is bounded and satisfies (3.5).

Now

$$b^*\left(\sum_{m=0}^n \alpha_m \lambda_{nm}\right) = b^*T\left(\sum_{m=0}^n \alpha_m \lambda_{nm}(t)\right) \quad \text{for every } b^* \in B^*(X, Y)$$

hence

$$\sum_{m=0}^n \alpha_m \lambda_{nm} = T\left(\sum_{m=0}^n \alpha_m \lambda_{nm}(t)\right)$$

whence

$$(3.6) \qquad \| \sum_{m=0}^n \alpha_m \lambda_{nm} \| \leq \| T \| \sup_{0 \leq t \leq 1} | \sum_{m=0}^n \alpha_m \lambda_{nm}(t) |.$$

For every finite set of scalars $\alpha_0, \cdots, \alpha_n$ with $|\alpha_m| \leq 1$ we have by (2.4)

$$| \sum_{m=0}^n \alpha_m \lambda_{nm}(t) | \leq \sum_{m=0}^n \lambda_{nm}(t) = 1, \qquad 0 \leq t \leq 1.$$

Hence by (3.6)

$$\sup \| \sum_{m=0}^n \alpha_m \lambda_{nm} \| \leq \| T \| = \| \mu \|[0, 1] < \infty,$$

where the supremum is taken over all the finite sets of scalars $\alpha_0, \cdots, \alpha_n$ with

$|\alpha_m| \leq 1$. This completes the proof of (3.1). In fact it is easily seen that

$$\sup \| \textstyle\sum_{m=0}^{n} \alpha_m \lambda_{nm} \| = \| T \|, \qquad \text{Q.E.D.}$$

For a sequence $\{A_n\}$ $(n \geq 0)$ of operators in $B(X, Y)$ define

(3.7) $$[A_m, \cdots, A_n] = \textstyle\sum_{i=m}^{n} (1/W'_{nm}(\lambda_i))A_i, \quad 0 \leq m < n = 1, 2, \cdots$$
$$[A_n] = A_n, \qquad n = 0, 1, 2, \cdots,$$

where $W_{nm}(x) = (x - \lambda_m) \cdot \cdots \cdot (x - \lambda_n)$.

THEOREM 2. *Let $\{\lambda_n\}$ $(n \geq 0)$ satisfy (1.1). Then the sequence $\{A_n\}$ $(n \geq 0)$ of operators in $B(X, Y)$ is a weak moment sequence with respect to the sequence $\{t^{\lambda_n}\}$ $(n \geq 0)$ if and only if*

(3.8) $$\sup \| \textstyle\sum_{m=0}^{n} \alpha_m \cdot \lambda_{m+1} \cdot \cdots \cdot \lambda_n [A_m, \cdots, A_n] \| \equiv M < \infty,$$

where the supremum is taken over all the finite sets of scalars $\alpha_0, \cdots, \alpha_n$ with $|\alpha_m| \leq 1$ and all $n \geq 0$. Moreover if $\lambda_0 = 0$, then $\| \mu \|[0, 1] = M$.

Proof. We deal, first with the case $\lambda_0 = 0$. Let $\mathfrak{A}$ be an infinite Vandermonde defined by the sequence $\{\lambda_n\}$ $(n \geq 0)$, that is, $\mathfrak{A} = \| \lambda_n^{m-1} \| n \geq 0, m \geq 1$. Then it is readily seen by (2.2) and (3.7) that

(3.9) $$\lambda_{nm} = (-1)^{n-m} \lambda_{m+1} \cdot \cdots \cdot \lambda_n [A_m, \cdots, A_n], \quad 0 \leq m \leq n = 0, 1, 2, \cdots.$$

By Schoenberg [4] Theorem 9.1, we have $\phi_n(t) = t^{\lambda_n/\lambda_1}$, $n \geq 0$, and by the well-known Müntz theorem the sequence $\{t^{\lambda_n/\lambda_1}\}$ $(n \geq 0)$ is fundamental in $C[0, 1]$. Hence by Theorem 1, (3.8) is necessary and sufficient in order that there will be a vector valued measure μ, from the σ-field of Borel sets in $[0, 1]$ to $\overline{B^+}$ satisfying conditions (i), (ii) and (iv) of Definition 1 and such that

(3.10) $$b^* A_n = \textstyle\int_0^1 t^{\lambda_n/\lambda_1} \mu(dt) b^*, \qquad n = 0, 1, 2, \cdots, b^* \in B^*(X, Y)$$

Define a vector-valued measure ν, on the σ-field of Borel sets in $[0, 1]$ by $\nu(E) = \mu(E^{\lambda_1})$ for every Borel set $E (E^{\lambda_1} = \{t^{\lambda_1} \mid t \in E\})$, then by (3.10)

$$b^* A_n = \textstyle\int_0^1 t^{\lambda_n} \nu(dt) b^*, \qquad n = 0, 1, 2, \cdots, b^* \in B^*(X, Y)$$

Conditions (i), (ii) and (iv) or Definition 1 are straightforward. This completes the proof in the case $\lambda_0 = 0$.

Assume, now, that $\lambda_0 > 0$. Suppose that the sequence $\{A_n\}$ $(n \geq 0)$ is a weak moment sequence with respect to the sequence $\{t^{\lambda_n}\}$ $(n \geq 0)$. The vector-valued measure μ, existing by Definition 1, defines (see [2] Theorem VI.7.2) an operator $T : C[0, 1] \to B(X, Y)$ by the equation (3.4). Define sequences $\{\tilde{A}_n\}$, $\{\tilde{\lambda}_n\}$ $(n \geq 0)$ by

(3.11) $$\tilde{A}_0 = T(1), \quad \tilde{A}_n = A_{n-1} \ (n \geq 1), \quad \tilde{\lambda}_0 = 0, \quad \tilde{\lambda}_n = \lambda_{n-1} \ (n \geq 1),$$

then the sequence $\{\tilde{A}_n\}$ $(n \geq 0)$ is a weak moment sequence with respect to the sequence $\{t^{\tilde{\lambda}_n}\}$ $(n \geq 0)$. The sequence $\{\tilde{\lambda}_n\}$ $(n \geq 0)$ satisfies (1.1) with $\tilde{\lambda}_0 = 0$

and for this case we have already proved Theorem 2. It is readily seen that for $n \geq m \geq 1$

$$[\tilde{A}_m, \cdots, \tilde{A}_n] = [A_{m-1}, \cdots, A_{n-1}], \tag{3.12}$$

hence by (3.8) for the sequence $\{\tilde{A}_n\}$, $\{\tilde{\lambda}_n\}$ $(n \geq 0)$,

$$\sup \| \textstyle\sum_{n=0}^{n} \alpha_m \lambda_{m+1} \cdot \cdots \cdot \lambda_n [A_m, \cdots, A_n] \|$$
$$\leq \sup \| \textstyle\sum_{m=0}^{n+1} \alpha_m \tilde{\lambda}_{m+1} \cdot \cdots \cdot \tilde{\lambda}_{n+1} [\tilde{A}_m, \cdots, \tilde{A}_{n+1}] \| < \infty.$$

Conversely, if (3.8) holds, define the sequences $\{\tilde{A}_n\}$, $\{\tilde{\lambda}_n\}$ $(n \geq 0)$ by (3.11) with one exception, $\tilde{A}_0$ is an arbitrary bounded operator. By (2.3) for $k = 0$ and (3.9) for the sequences $\{\tilde{A}_n\}$, $\{\tilde{\lambda}_n\}$ $(n \geq 0)$ we get

$$(-1)^n \tilde{\lambda}_1 \cdot \cdots \cdot \tilde{\lambda}_n [\tilde{A}_0, \cdots, \tilde{A}_n]$$
$$= \tilde{A}_0 - \textstyle\sum_{m=1}^{n} (-1)^{n-m} \tilde{\lambda}_{m+1} \cdot \cdots \cdot \tilde{\lambda}_n [\tilde{A}_m, \cdots, \tilde{A}_n],$$

hence by (3.12)

$$\tilde{\lambda}_1 \cdot \cdots \cdot \tilde{\lambda}_n \| [\tilde{A}_0, \cdots, \tilde{A}_n] \| \tag{3.13}$$
$$\leq \| \tilde{A}_0 \| + \| \textstyle\sum_{m=1}^{n} (-1)^{n-m} \lambda_m \cdot \cdots \cdot \lambda_{n-1} [A_{m-1}, \cdots, A_{n-1}] \|.$$

Now, if $|\alpha_m| \leq 1$ for $0 \leq m \leq n$,

$$\| \textstyle\sum_{m=0}^{n} \alpha_m \cdot \tilde{\lambda}_{m+1} \cdot \cdots \cdot \tilde{\lambda}_n [\tilde{A}_m, \cdots, \tilde{A}_n] \|$$
$$\leq \tilde{\lambda}_1 \cdot \cdots \cdot \tilde{\lambda}_n \| [\tilde{A}_0, \cdots, \tilde{A}_n] \|$$
$$+ \| \textstyle\sum_{m=0}^{n-1} \alpha_{m+1} \cdot \lambda_{m+1} \cdot \cdots \cdot \lambda_{n-1} [A_m, \cdots, A_{n-1} \|,$$

hence by (3.8) and (3.13)

$$\leq \| A_0 \| + 2M.$$

Thus we have proved that

$$\sup \| \textstyle\sum_{m=0}^{n} \alpha_m \cdot \tilde{\lambda}_{m+1} \cdot \cdots \cdot \tilde{\lambda}_n [\tilde{A}_m, \cdots, \tilde{A}_n] \| \equiv H < \infty \tag{3.14}$$

where the supremum is taken over all the finite sets of scalars $\alpha_0, \cdots, \alpha_n$ with $|\alpha_m| \leq 1$ and all $n \geq 0$. As the sequence $\{\tilde{\lambda}_n\}$ $(n \geq 0)$ satisfies (1.1) with $\tilde{\lambda}_0 = 0$ we obtain by (3.14) and Theorem 2, which we have proved for this case, that the sequence $\{\tilde{\lambda}_n\}$ $(n \geq 0)$ is a weak moment sequence with respect to $\{t^{\tilde{\lambda}_n}\}$ $(n \geq 0)$. This implies the desired conclusion. Q.E.D.

For the sequence $\{\lambda_n = n\}$ $(n \geq 0)$ we have

$$\lambda_{nm} = \binom{n}{m} \Delta^{n-m} A_m, \qquad 0 \leq m \leq n$$

where $\Delta^0 A_n = A_n$ and $\Delta^k A_n = \Delta^{k-1} A_n - \Delta^{k-1} A_{n+1}$. This leads us to the following consequence of Theorem 2.

Corollary 1. *The sequence $\{A_n\}$ $(n \geq 0)$ of operators in $B(X, Y)$ is a weak* *moment sequence with respect to the sequence* $\{t^n\}$ $(n \geq 0)$ *if and only if*

$$\sup \| \textstyle\sum_{m=0}^{n} \alpha_m \binom{n}{m} \Delta^{n-m} A_m \| \equiv M < \infty$$

where the supremum is taken over all finite sets of scalars $\alpha_0, \cdots, \alpha_n$ with $|\alpha_m| \leq 1$ and all $n \geq 0$. Moreover $\| \mu \|[0, 1] = M$.

Corollary 1 may be looked upon as a generalization of the well-known Hausdorff solution of the moment problem.

4. Strong moment sequences

THEOREM 3. *Suppose that the sequence* $\{\phi_n(t)\}$ $(n \geq 0)$ *is fundamental in* $[0, 1]$ *and that* Y *is reflexive. Then the sequence* $\{A_n\}$ $(n \geq 0)$ *of operators in* $B(X, Y)$ *is a strong moment sequence with respect to* $\{\phi_n(t)\}$ $(n \geq 0)$ *if and only if* (3.1) *holds. Moreover* $\| \mu \|[0, 1] = M$.

Proof. If Y is reflexive, then the measure μ obtained by Theorem 1 takes values in $B(X, Y)$ and the proof of our theorem is similar to that of [2] Theorem VI.7.3, Q.E.D.

Similarly we obtain

THEOREM 4. *Let* $\{\lambda_n\}$ $(n \geq 0)$ *satisfy* (1.1) *and suppose that* Y *is reflexive. Then the sequence* $\{A_n\}$ $(n \geq 0)$ *of operators in* $B(X, Y)$ *is a strong moment sequence with respect to* $\{t^{\lambda_n}\}$ $(n \geq 0)$ *if and only if* (3.8) *holds. Moreover, if* $\lambda_0 = 0$, *then* $\| \mu \|[0, 1] = M$.

Theorem 3 is a generalization of [3, Theorem 1], and Theorem 4 is a generalization of [3, Consequence 2 and Theorem 2].

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