DERIVATIONS OF COMMUTATIVE ALGEBRAS

BY

HOWARD OSBORN[1]

In memoriam Charles Loewner 1893-1968

For any commutative algebra A with unit over a field k of characteristic 0 a *derivation* is a k-linear map Δ of A into some A-module F such that $\Delta ab = a\Delta b + b\Delta a$ for all $a, b \in A$; in particular, the *Kähler derivation* D of A into the *Kähler module* $E(A)$ has the universal mapping property that any derivation of A can be factored in the form

$$A \xrightarrow{D} E(A) \xrightarrow{h} F$$

for a unique A-module homomorphism h (see [1] or [2]). Clearly polynomial relations in A imply A-linear relations in $E(A)$, and the first goal of this paper is to show to what extent the converse is true. The next results show that if A is an algebra of k-valued functions on Spec A (defined later, with the Zariski topology) then $a \in A$ assumes only finitely many values if and only if $Da = 0$. Finally D can always be extended to an exterior derivation of the exterior algebra $\Lambda_A E(A)$ (see [2]), and one would like to obtain information about possible analogs of the de Rham theorem, relating the cohomology ring of the cochain complex $(\Lambda_A E(A), D)$ or one of its near relatives to a reasonable cohomology ring of Spec A itself; we present partial results on this question.

Let I denote any subset of A containing 0, let A^I denote the ideal of those $a \in A$ such that to each $b \in I$ there is an $r \geqq 0$ (depending on a and b) with $b^r a = 0$, and let r_I be the canonical epimorphism of A onto the *restriction* $A_I = A/A^I$.

PROPOSITION 1. *If*

$$I = \{0, b_1, \cdots, b_n\} \quad \text{and} \quad b_1 Da_1 + \cdots + b_n Da_n = 0$$

for some $a_1, \cdots, a_n, b_1, \cdots, b_n \in A$, *then there is a non-trivial polynomial* π *over* k *such that*

$$\pi(r_I a_1, \cdots, r_I a_n) = 0 \in A_I ;$$

in particular, if $A^I = (0)$ *or if one of* $b_1, \cdots, b_n$ *is itself a non-trivial polynomial in* $a_1, \cdots, a_n$ *then there is a non-trivial polynomial* π *over* k *such that*

$$\pi(a_1, \cdots, a_n) = 0.$$

Proof. It suffices to show for each b_i that there exists a non-trivial polynomial π and an $r \geqq 0$ with $b_i^r \pi(a_1, \cdots, a_n) = 0$; the product of the n polynomials π will then satisfy the conclusion of the proposition. If the set S of

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elements $b_i^r\,\pi(a_1, \cdots, a_n) \in A$ for arbitrary $r \geqq 0$ and non-trivial π were multiplicative (not containing 0) then Zorn's lemma would provide an ideal $\mathfrak{p}$ maximal with respect to the property $\mathfrak{p} \cap S = \emptyset$, and such a $\mathfrak{p}$ is automatically prime. Let K be the field of quotients of the integral domain $A/\mathfrak{p}$ and let $\xi_1, \cdots, \xi_n \in K$ be the images of $a_1, \cdots, a_n \in A$ under the canonical homomorphism

$$A \to A/\mathfrak{p} \to K.$$

If $\pi(\xi_1, \cdots, \xi_n) = 0$ for some non-trivial polynomial π then one would have the contradiction $\pi(a_1, \cdots, a_n) \in \mathfrak{p}$; thus $\xi_1, \cdots, \xi_n$ are mutually transcendental over k so that there is a unique derivation

$$k(\xi_1, \cdots, \xi_n) \xrightarrow{\delta_i} K$$

with $\delta_i\,\xi_j$ the Kronecker $\delta_{ij} \in K$. Since k is of characteristic 0 one again uses Zorn's lemma to extend $k(\xi_1, \cdots, \xi_n)$ via successive simple extensions K_α to K itself, for ordinals α such that $K_1 = k(\xi_1, \cdots, \xi_n)$ and $K_{\alpha+1} = K_\alpha(\zeta_\alpha)$ for some $\zeta_\alpha \in K$; if ζ_α is algebraic over K_α then δ_i automatically extends to a derivation

$$K_{\alpha+1} \xrightarrow{\delta_i} K,$$

and if ζ_α is transcendental over K_α one extends

$$K_\alpha \xrightarrow{\delta_i} K$$

by setting $\delta_i\,\zeta_\alpha = 0$. Thus δ_i extends to a derivation δ_i of K into itself, and the composition

$$A \to A/\mathfrak{p} \to K \xrightarrow{\delta_i} K$$

is a derivation Δ_i of A into the A-module K with

$$\Delta_i\,a_j = \delta_{ij} \in K.$$

Since $b_1\,Da_1 + \cdots + b_n\,Da_n = 0$ for the (universal) Kähler derivation it follows for the image η_i of b_i under $A \to A/\mathfrak{p} \to K$ that

$$\eta_i = b_i \cdot 1 = b_1\,\Delta_i\,a_1 + \cdots + b_n\,\Delta_i\,a_n = 0,$$

implying the contradiction $b_i \in \mathfrak{p}$. Thus S is not multiplicative, so that

$$b_i^r\,\pi(a_1, \cdots, a_n) = 0$$

for some $r \geqq 0$ and non-trivial π as claimed.

The following corollary is the best possible version of Proposition 3 in [3]:

COROLLARY. *Let A be the real algebra of all real-valued C^∞ functions on a real n-dimensional vector space, with coordinate functions $x_1, \cdots, x_n \in A$; then*

$$Da = \frac{\partial a}{\partial x_1}\,Dx_1 + \cdots + \frac{\partial a}{\partial x_n}\,Dx_n$$

if and only if $a \in A$ is an algebraic function of $x_1, \cdots, x_n$.

Proof. The coefficient 1 of Da is a non-trivial polynomial in $a, x_1, \cdots, x_n$, and there are no non-trivial polynomial relations in $x_1, \cdots, x_n$ alone.

Since a relation $b_1 Da_1 + \cdots + b_n Da_n = 0$ in $E(A)$ implies

$$\pi(a_1, \cdots, a_n) = 0$$

for at least one non-trivial polynomial π whenever the coefficients $b_1, \cdots, b_n$ don't vanish too badly, one would expect that the n-tuple $(b_1, \cdots, b_n) \in A^n$ is a linear combination of n-tuples

$$(\pi_1(a_1, \cdots, a_n), \cdots, \pi_n(a_1, \cdots, a_n))$$

for non-trivial π such that $\pi(a_1, \cdots, a_n) = 0$, where $\pi_1, \cdots, \pi_n$ are the first partial derivatives of π. We shall prove exactly such a result, but only under the additional assumption that A has no divisors of zero; thus, although Proposition 1 is valid for the algebra A of germs of C^∞ functions at a point the next proposition is not known in that case, although both results are valid for the algebra A of germs of analytic functions at a point, for example. We remark that if A has no divisors of zero then $A^I = (0)$ whenever I contains a non-zero member, so that the final conclusion of Proposition 1 holds except when $b_1 = \cdots = b_n = 0$.

LEMMA. *Suppose A has no divisors of zero and*

$$b_1 Da_1 + \cdots + b_n Da_n = 0$$

for $a_1, \cdots, a_n, b_1, \cdots, b_n \in A$ such that $a_1, \cdots, a_{n-1}$ satisfy no non-trivial polynomial relation. Then if π is a polynomial of minimum degree such that $\pi(a_1, \cdots, a_n) = 0$ it follows that

$$\pi_n(a_1, \cdots, a_n)(b_1, \cdots, b_n) = b_n(\pi_1(a_1, \cdots, a_n), \cdots, \pi_n(a_1, \cdots, a_n)) \in A^n,$$

where $\pi_n(a_1, \cdots, a_n) \neq 0$.

Proof. Minimality implies $\pi_n(a_1, \cdots, a_n) \neq 0$, and the relation

$$\pi_1(a_1, \cdots, a_n)Da_1 + \cdots + \pi_n(a_1, \cdots, a_n)Da_n = 0$$

combined with

$$b_1 Da_1 + \cdots + b_n Da_n = 0$$

gives

$$\sum_{i=1}^{n-1} (\pi_n(a_1, \cdots, a_n)b_i - b_n \pi_i(a_1, \cdots, a_n))Da_i = 0.$$

If any of $\pi_n(a_1, \cdots, a_n)b_i - b_n \pi_i(a_1, \cdots, a_n)$ were non-zero then Proposition 1 would supply a non-trivial polynomial relation among $a_1, \cdots, a_{n-1}$.

PROPOSITION 2. *Suppose that A has no divisors of zero and that*

$$b_1 Da_1 + \cdots + b_n Da_n = 0$$

for some $a_1, \cdots, a_n, b_1, \cdots, b_n \in A$. Then for some $m \leqq n$ there are elements

$c_1, \cdots, c_m \in A$ and non-trivial polynomials $\pi^0, \pi^1, \cdots, \pi^m$ with

$$\pi^0(a_1, \cdots, a_n) \neq 0$$

and

$$\pi^1(a_1, \cdots, a_n) = \cdots = \pi^m(a_1, \cdots, a_n) = 0$$

such that

$$\pi^0(a_1, \cdots, a_n)(b_1, \cdots, b_n) = \sum_{i=1}^m c_i(\pi_1^i(a_1, \cdots, a_n), \cdots, \pi_n^i(a_1, \cdots, a_n)),$$

where π_j^i is the j^{th} first partial derivative of π^i.

Proof. If $b_1 Da_1 = 0$ for $b_1 \neq 0$ then by Proposition 1 there is a non-trivial π^1 of minimum degree such that $\pi^1(a_1) = 0$, hence an identity $\pi^0(a_1)b_1 = \pi_1^1(a_1)b_1$ for $\pi^0(a_1) = \pi_1^1(a_1) \neq 0$. By induction on n, if $b_n = 0$ the inductive hypothesis applies directly, and if $b_n \neq 0$ there are positive integers $n_1 < \cdots < n_q = n$ and a non-trivial polynomial relation $\pi^m(a_1, \cdots, a_n) = 0$ of minimum degree involving only the variables $a_{n_1}, \cdots, a_{n_q}$ such that no proper subset of $a_{n_1}, \cdots, a_{n_q}$ satisfies a non-trivial polynomial relation. As in the Lemma one obtains

$$\pi_n^m(a_1, \cdots, a_n) \neq 0$$

and

$$\sum_{i=1}^{n-1} (\pi_n^m(a_1, \cdots, a_n)b_i - b_n \pi_i^m(a_1, \cdots, a_n))Da_i = 0,$$

so that the inductive hypothesis provides non-trivial $\pi, \pi^1, \cdots, \pi^{m-1}$ with

$$\pi(a_1, \cdots, a_n) \neq 0$$

and

$$\pi^1(a_1, \cdots, a_n) = \cdots = \pi^{m-1}(a_1, \cdots, a_n) = 0$$

for some $m - 1 \leqq n - 1$ such that

$$\pi(\pi_n^m b_1 - b_n \pi_1^m, \cdots, \pi_n^m b_n - b_n \pi_n^m) = \sum_{i=1}^{m-1} c_i(\pi_1^i, \cdots, \pi_n^i),$$

where all polynomials are evaluated on $a_1, \cdots, a_n$; that is,

$$\pi^0(b_1, \cdots, b_n) = \sum_{i=1}^m c_i(\pi_1^i, \cdots, \pi_n^i)$$

as required, where $\pi^0 = \pi\pi_n^m \neq 0$ and $c_m = \pi b_n$.

We interpret Proposition 2 briefly as follows. For any commutative algebra A with unit let $\sum_A A$ be the direct sum of copies of A indexed by the elements of A itself, with generator D^*a corresponding to $a \in A$, and let $F(A)$ be the submodule generated by all elements of the form

$$\pi_1(a_1, \cdots, a_n)D^*a_1 + \cdots + \pi_n(a_1, \cdots, a_n)D^*a_n$$

for all $n > 0$ and $(a_1, \cdots, a_n) \in A^n$ satisfying polynomial relations

$$\pi(a_1, \cdots, a_n) = 0.$$

Then $\sum_A A/F(A)$ is the Kähler module and $Da = D^*a + F(A)$. In general

$$b_1 D^*a_1 + \cdots + b_n D^*a_n \in F(A)$$

if and only if for some $m \geqq 0$ and $N \geqq 0$ there exist $c_1, \cdots, c_m, a_{n+1}, \cdots, a_{n+N} \in A$ and polynomials $\pi^1, \cdots, \pi^m$ such that

$$\pi^1(a_1, \cdots, a_{n+N}) = \cdots = \pi^m(a_1, \cdots, a_{n+N}) = 0$$

and

$$(b_1, \cdots, b_n, 0, \cdots, 0) = \textstyle\sum_{i=1}^m c_i(\pi_1^i(a_1, \cdots, a_{n+N}), \cdots, \pi_{n+N}^i(a_1, \cdots, a_{n+N})).$$

According to Proposition 2, however, if A is an extension field of k then one does not need to introduce any additional elements $a_{n+1}, \cdots, a_{n+N}$.

We now turn to results which require A to be a function algebra. For any commutative algebra A with unit over a field k of characteristic 0 let Spec A denote the set of epimorphisms

$$A \xrightarrow{P} k$$

in the Zariski topology, closed sets being of the form $\{P \mid PI = 0\}$ where I is any subset of A containing 0. If A is semi-simple in the sense that

$$\textstyle\bigcap_{P \in \operatorname{Spec} A} \ker P = (0)$$

then A is isomorphic to an algebra of k-valued functions on Spec A which will be identified with A itself, the value of $a \in A$ at $P \in \operatorname{Spec} A$ being its image Pa under P.

PROPOSITION 3. *If A is semi-simple then $a \in A$ assumes at most finitely many values in k if and only if $bDa = 0$ for some $b \in A$ which is no-where zero.*

Proof. If a assumes only the distinct values $\alpha_1, \cdots, \alpha_n \in k$ then $\pi(a) = 0$ for $\pi(X) = (X - \alpha_1) \cdots (X - \alpha_n)$, hence $\pi'(a)Da = D\pi(a) = 0$ where $\pi'(a)$ is nowhere zero. Conversely if $bDa = 0$ for b nowhere zero then Proposition 1 provides a non-trivial polynomial π with $\pi(a) = 0$, so that a can assume at most the roots of π lying in k. (Alternatively, one can use ultrafilters as in Lemma 7.4 and Proposition 7.5 of [2].)

Any $a \in A$ assuming only finitely many values is necessarily constant on each component of Spec A, i.e., *locally constant*; for if a assumes only the values $\alpha_1, \cdots, \alpha_n$ then the sets

$$\{P \mid Pa = \alpha_1\}, \cdots, \{P \mid Pa = \alpha_n\}$$

are both open and closed. However, there are locally constant functions which assume infinitely many values whenever Spec A has infinitely many components, so that in order to test for local constancy by means of derivations one must modify the Kähler derivation. We do this as in Propositions 8.3 and 8.4 of [2]; a brief summary is presented here, beginning with an alternate characterization of Kähler derivation.

Let $\operatorname{Spec}_m A$ denote the *maximal* spectrum of A, consisting of all epimorphisms of A onto arbitrary field extensions of k, including k itself, and for any

$P \in \mathrm{Spec}_m A$ let A_P denote the image of the localization homomorphism of A at P, in the Zariski topology as usual. Then each of the induced epimorphisms

$$A \xrightarrow{r_P} A_P$$

induces an epimorphism

$$E(A) \xrightarrow{(r_P, r_P)^*} E(A_P)$$

of Kähler modules, where $E(A_P)$ is regarded as an A-module via r_P, and according to Proposition 7.3 of [2] the direct product $\prod_{P \in \mathrm{Spec}_m A} (r_P, r_P)^*$ is a monomorphism. If D_P is the Kähler derivation of A_P then Lemma 3.4 of [2] gives $D_P r_P a = (r_P, r_P)^* Da$ so that D can be represented as a composition of the direct products $\prod r_P$ and $\prod D_P$ computed over all $P \in \mathrm{Spec}_m A$; the module generated by the image of this composition is isomorphic to $E(A)$. If one restricts these direct products to the index set $\mathrm{Spec}\, A \subset \mathrm{Spec}_m A$ one obtains a derivation

$$A \xrightarrow{d} \iota\pi E(A),$$

where $\iota\pi E(A)$ denotes the submodule of $\prod E(A_P)$ generated by the image of d.

PROPOSITION 4. *If A is semi-simple and*

$$A \xrightarrow{d} \iota\pi E(A)$$

the derivation defined above, then $a \in A$ is locally constant on Spec A *if and only if* $da = 0$.

Proof. Suppose that $a \in A$ is not constant in any Zariski neighborhood of some $P \in \mathrm{Spec}\, A$; without loss of generality one can further assume that $Pa = 0 \in k$, i.e., $a \in \ker P$. The ideal I_P of all $b \in A$ which vanish on some neighborhood of P is a radical ideal (or "semi-prime", meaning that $b^n \in I_P$ implies $b \in I_P$) and is therefore an intersection of minimal prime ideals $\mathfrak{p} \supset I_P$, where each $\mathfrak{p}$ is contained in the maximal ideal $\ker P$. Since $a \in \ker P$ and $a \notin I_P \subset \ker P$ there is some prime ideal $\mathfrak{p}$ properly contained in $\ker P$ such that $a \notin \mathfrak{p}$. Let K be the field of quotients of the integral domain $A/\mathfrak{p}$, observing that K contains k as a subfield, and let $\xi \in K$ be the image of a under

$$A \to A/\mathfrak{p} \to K.$$

If ξ were algebraic over k then it would be a root of some irreducible polynomial π over k, for which $\pi(a) \in \mathfrak{p}$; but $a \in \ker P$ and $\mathfrak{p} \subset \ker P$, so that $\pi(0) = 0$, and the irreducibility of π would imply that it is of degree one with no constant term, giving the contradiction $a \in \mathfrak{p}$. Thus ξ is transcendental over k so that there is a unique derivation

$$k(\xi) \xrightarrow{\delta} K$$

which vanishes on k and carries ξ into $1 \in K$, where K is regarded as a $k(\xi)$-module via the inclusion homomorphism. Since k is of characteristic 0 one can extend $k(\xi)$ and δ as in the proof of Proposition 1 to a derivation δ of K into itself with $\delta\xi = 1$, so that the composition

$$A \rightarrow A/\mathfrak{p} \rightarrow K \xrightarrow{\delta} K$$

is a derivation Δ of A such that $\Delta a \neq 0$. But since $P \in \operatorname{Spec} A$ (rather than $\operatorname{Spec}_m A \sim \operatorname{Spec} A$), Δ can be factored in the form

$$A \xrightarrow{d} \iota\pi E(A) \xrightarrow{h} K$$

for some A-module homomorphism h, so that $da \neq 0$. Thus if $da = 0$ it follows that a is constant in some Zariski neighborhood of each $P \in \operatorname{Spec} A$, which completes the proof.

Proposition 4 can be interpreted as a statement about the cohomology of A. According to Remark 9.10 of [2] the derivation d extends to a unique exterior derivation of the exterior algebra $\Lambda\, \iota\pi E(A)$, which thereby becomes a cochain complex. If A is a C^∞ structure this complex is distinct from the usual de Rham complex; for it is shown in [3] that $E(A_P)$ is not the usual module of differentials at any $P \in \operatorname{Spec} A$, hence that $\iota\pi E(A)$ is also not the usual module of differentials on $\operatorname{Spec} A$. One nevertheless expects for any semi-simple A that the cohomology ring $H^0(A) \oplus H^1(A) \oplus \cdots$ of the complex $(\Lambda\, \iota\pi E(A), d)$ is related to a cohomology ring of $\operatorname{Spec} A$ itself, with coefficients in k. In this language Proposition 4 may be re-written as follows:

PROPOSITION 5. *If A is semi-simple then $H^0(A)$ is the first conjugate of the k-linear space whose dimension is the number of connected components of* $\operatorname{Spec} A$.

Thus $H^0(A)$ is always the usual 0-dimensional Cech cohomology (say) of $\operatorname{Spec} A$. For $i > 0$ $H^i(A)$ generally contains more information than any corresponding cohomology module of $\operatorname{Spec} A$, however. In the following example $\operatorname{Spec} A$ is the real line in its usual topology, for which the usual cohomology modules vanish in positive dimensions.

PROPOSITION 6. *Let A be the algebra of all real-valued C^∞ functions on the real line; then $H^1(A) \neq 0$.*

Proof. Let $x \in A$ be the usual coordinate function and observe that

$$d((1 + x^2)^{-1}\, dx) = 0 \in \Lambda\, \iota\pi E(A).$$

If $(1 + x^2)^{-1}\, dx = da$ for some $a \in A$ then the same equality would hold when d is the classical derivation of A into itself, which can be factored through the derivation d of this paper. Hence $a = \tan^{-1} x$ plus a constant so that $d \tan^{-1} x = (1 + x^2)^{-1}\, dx$ for our derivation d. It follows from the definition of d that

$$Dr_P \tan^{-1} x = r_P(1 + x^2)^{-1}\, Dr_P\, x$$

for any $P \in \operatorname{Spec} A$, where D is the Kähler derivation of A_P; but this contradicts Proposition 1 (applied to A_P) since no localization of $\tan^{-1} x$ is algebraic in the corresponding localization of x.

References

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University of Illinois
Urbana, Illinois