TRANSFORMATION GROUPS OF AUTOMORPHISMS OF $C(X)$

BY

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In this article we discuss certain relationships which exist between a transformation group (X, T, π) and the ring $C(X)$ of all continuous real-valued functions defined on X. If (X, T, π) is a transformation group there is a standard way [2, 1.68] to induce a transformation group on $C(X)$. We first show that under certain conditions this construction can be reversed. In the second half of this paper we indicate a technique by which many dynamical statements about (X, T, π) can be faithfully reflected in $C(X)$.

Throughout the rest of this paper we will use the following notation. When (X, T, π) is a transformation group we will write xt in place of $\pi(x, t)$. For $f \in C(X)$ we let $Z(f) = \{x \in X : f(x) = 0\}$; $[f]$ denotes the principal ideal generated by f.

Let X be a topological space and consider the ring $C(X)$. We give $C(X)$ the compact-open topology. In order to guarantee that $C(X)$ has some non-constant functions we will henceforth assume that X is a completely regular T_1-space. It is well known that, for such X, $C(X)$ provided with the compact-open topology is a topological ring.

Suppose that (X, T, π) is a transformation group where T is a locally compact topological group and X satisfies the conditions of the preceding paragraph. If we define

$$\pi^* : C(X) \times T \to C(X)$$

by

$$\pi^*(f, t)(x) = f(xt^{-1})$$

as in [2, 1.68], then $(C(X), T, \pi^*)$ is a transformation group. $(C(X), T, \pi^*)$ is effective if and only if (X, T, π) is effective. Furthermore, for each $t \in T$, $\pi_t^* : C(X) \to C(X)$ is a ring isomorphism.

1. DEFINITION. Let J be an ideal in $C(X)$. Associated with the ideal J there is a unique ideal $m(J)$ [4] defined by

$$m(J) = \{f \in C(X): \text{ there exists a } g \in J \text{ such that } fg = f\}.$$

These ideals are discussed in [1] and [4].

Recall that an ideal J in $C(X)$ is said to be *fixed* if and only if $\bigcap_{f \in J} Z(f) \neq \emptyset$. The only maximal fixed ideals are of the form

$$J_x = \{f \in C(X) : f(x) = 0\}.$$

An ideal J is called *real-maximal* if and only if $C(X)/J = R$. As in [3], we

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call a topological space X a Q-space if and only if every real-maximal ideal in $C(X)$ is fixed. For the remainder of this paper we will assume that our topological space is, in addition to our previous stipulations, a Q-space. The proof of the following lemma follows immediately from the definitions.

2. LEMMA. *Let* $(C(X), T, \pi^*)$ *be as above, and let*

$$M_p = \{f \in C(X) : f(p) = 0\}.$$

Then

(1) $\pi_t^*(M_p) = M_{pt}$

(2) $\pi_t^*(m(M_p)) = m(\pi_t^*(M_p))$.

There is a large literature concerning the relation between isomorphisms of $C(X)$ and homeomorphisms of X. The interested reader can consult [1] for more information.

3. THEOREM. *If* $(C(X), T, \varphi)$ *is a transformation group of ring isomorphisms on* $C(X)$, *and if* X *is locally compact, then there exists a transformation group* (X, T, π) *such that the induced transformation group* $(C(X), T, \pi^*)$ *is* $(C(X), T, \varphi)$.

Proof. We begin by using the equality of Lemma 2 to define a mapping

$$\pi : X \times T \to X$$

by

$$\pi(p, t) = q \quad \text{if and only if} \quad \varphi_t(M_p) = M_q .$$

For each $t \in T$, φ_t is a ring isomorphism by hypothesis. It follows that for each $p \in X$, $\varphi_t(M_p)$ is a maximal ideal, consequently,

$$\varphi_t(C(X))/\varphi_t(M_p) = C(X)/M_p = R.$$

Hence $\varphi_t(M_p)$ is a real maximal ideal and is therefore fixed. Thus there exists a unique $q \in X$ such that $\varphi_t(M_p) = M_q$. This implies that the mapping π is well defined.

In order to establish that (X, T, π) is a transformation group, we will show that π is continuous. The verification of the other properties is straightforward.

Since the collection $\{Z(f) : f \in C(X)\}$ forms a basis for the closed sets in X, it is sufficient to show that for each $f \in C(X)$, $\pi^{-1}(Z(f))$ is closed. Let $f \in C(X)$. Then $\pi(x, t) \in Z(f)$ if and only if $f \in M_{xt}$, or $\varphi_{t^{-1}} f \in M_x$. Let (x_0, t_0) be a cluster point of $\pi^{-1}(Z(f))$. Then there is a net $\{(x_\alpha, t_\alpha) : \alpha \in A\}$ contained in $\pi^{-1}(Z(f))$ with $\lim_A (x_\alpha, t_\alpha) = (x_0, t_0)$. Since φ is continuous, if K is a compact subset of X then $\lim_A \varphi_{t_\alpha^{-1}} f = \varphi_{t_0^{-1}} f$ uniformly on K. If $(x_0, t_0) \notin \pi^{-1}(Z(f))$ then $\varphi_{t_0^{-1}}(f)(x_0) \neq 0$. Since X is locally compact, there exists a compact neighborhood U of x_0 such that $\inf_{x \in U} |f_{t_0^{-1}}(x)| = \varepsilon > 0$. Since $\lim_A \varphi_{t_\alpha^{-1}} f = \varphi_{t_0^{-1}} f$ uniformly on U there exists a $\beta \in A$ such that $\alpha > \beta$

implies

$$|\varphi_{t_\alpha^{-1}} f(x) - \varphi_{t_0^{-1}} f(x)| < \varepsilon \qquad \text{for all} \quad x \in U. \quad (1)$$

Since $\lim_A x_\alpha = x_0$ there exists an $\alpha > \beta$ such that $x_\alpha \in U$. For this α, (1) implies that

$$|\varphi_{t_\alpha^{-1}} f(x) - \varphi_{t_0^{-1}} f(x)| = |\varphi_{t_0^{-1}} f(x)| < \varepsilon.$$

This contradicts the definition of ε. Thus $(x_0, t_0) \in \pi^{-1}(Z(f))$ and $\pi^{-1}(Z(f))$ is closed.

Let $(C(X), T, \pi^*)$ be defined as before. By Lemma 2 and the definition of π, for $p \in X$ and $t \in T$ we have $\pi^*(M_p) = M_{pt} = \varphi_t M_p$. Thus we need only show that φ_t and π_t^* act in the same way on each $f \in C(X)$ for all $t \in T$. If this is not the case, there exists an $f \in C(X)$, a $t \in T$, and a $p \in X$ such that $\varphi_t f(p) \neq \pi_t^* f(p)$. Since φ_t is a ring isomorphism we may assume that $\varphi_t^* f(p) = 0$ and $\pi_t^* f(p) \neq 0$. Thus $\varphi_t f$ is in M_q whereas $\pi_t^* f$ is not. Now

$$f = \varphi_{t^{-1}} (\varphi_t f) \in \varphi_{t^{-1}} (M_p) = M_{pt^{-1}}$$

and

$$f = \pi_{t^{-1}}^* (\pi_t^* f) \notin \pi_{t^{-1}}^* (M_p) = M_{pt^{-1}}.$$

In other words $f(pt^{-1}) \neq f(pt^{-1})$, which is a contradiction. This proves the theorem.

Observe that the assumption that T is locally compact is not used in this proof. Also, the hypothesis that X is locally compact is used only to construct a particular compact set. Other assumptions would work as well. For example, if we assume that both X and T are first countable, the statement and proof of the theorem follow mutatis mutandis.

We now show how many of the classical dynamical properties of (X, T, π) can be carried over to $(C(X), T, \pi^*)$. These will take the form of purely algebraic statements about maximal ideals in $C(X)$. The results discussed in this section generalize and simplify some recent work of Jenkins and Johnson [3].

4. Definition [3]. Let Q be an ideal in $C(X)$ and S any subset of T. We define an ideal associated with Q as follows

$$J(Q; S) = \bigcap_{t \in S} \pi_t^*(Q).$$

If $S = T$ we write $J(Q)$; i.e. $J(Q) = J(Q; T)$.

5. Lemma. *Let f be in $C(X)$, and let A and B be any subsets of T. Then $J([f]; A) \subset J(M_p; B)$ if and only if $pB \subset \text{cl}\,(Z(f) \cdot A)$.*

Proof. We first establish the sufficiency. Suppose that $f \in C(X)$, and that A and B are subsets of T. Assume $pB \subset \text{cl}\,(A(f) \cdot A)$, and let $g \in J([f]; A)$. Then $Z(g) \supset Z(\pi_t^* f) = Z(F) \cdot t$ for all $t \in A$. Since $Z(g)$ is closed

$$Z(g) \supset \text{cl}\,(\bigcup_{t \in A} Z(f) \cdot T).$$

However, by hypothesis $pB \subset \text{cl } (Z(f)\cdot A)$. Hence $pB \subset Z(g)$. This implies

$$g \in \bigcap_{t \in B} \pi_t^*(M_p) = J(M_p\,;B)$$

which shows that $J([f]; A) \subset J(M_p\,; B)$.

To prove the necessity, we suppose that $J([f]; A) \subset J(M_p\,; B)$ and that $g \notin \text{cl } (Z(f)\cdot A)$. Since X is completely regular there exists a function $g \in C(X)$ such that $g(q) = 1$, $\text{cl } (Z(f)\cdot A) \subset Z(g)$, and $0 \leq g \leq 1$. Let ε denote a constant function such that $0 < \varepsilon < 1$. We define functions h and k as follows: $h = -\varepsilon + \max (g, \varepsilon)$, $k = -\varepsilon + \min (g, \varepsilon)$. We first observe that if $x \in X - Z(k)$ then $k(x) \neq 0$. Hence $\min (g(x), \varepsilon) = g(x)$. This implies that $h(x) = 0$. Thus $Z(h) \supset X - Z(k)$. Furthermore, if

$$x \in \text{cl } (Z(f)\cdot A)$$

then $g(x) = 0$. Thus $x \in X - Z(f)$. Hence we have

$$Z(h) \supset X - Z(k) \supset \text{cl } (Z(f)\cdot A) \supset Z(\pi_t^*(f)).$$

Since π_t^* is an isomorphism, $[\pi_t^*(f)] = \pi_t^*([f])$. Let m denote the function $\pi_t^*(f)$. We show that $h \in [\pi_t^*(f)] = [m]$. Let $n = h/(m + k) \in C(X)$. We observe that for $x \in X - Z(h)$, $k(x) = 0$ while $m(x) \neq 0$. Thus

$$n\cdot m(x) = \frac{h(x)}{m(x)}\, m(x) = h(x).$$

On the other hand, if $x \in Z(h)$ $n\cdot m(x) = 0$. Thus $n\cdot m = h$ which implies $h \in [m] = \pi_t^*([f])$. This is valid for all $t \in A$. Hence $h \in J([f]; A)$. By hypothesis this implies $h \in J(M_p\,; B)$. Therefore $Z(h) \supset pB$. However $h(q) = 1 - \varepsilon$ which implies $q \notin pB$. Thus we have $pB \subset \text{cl } (Z(f)\cdot A)$.

6. Definition. We say that the real maximal ideal M_p is *periodic* under T if and only if there exists a compact set $K \subset T$ such that $J(M_p\,; K) \subset J(M_p)$. (Note that the opposite inclusion is trivial, so that the above statement is in fact an equality.)

7. Theorem. *M_p is periodic under T if and only if p is periodic under T.*

Proof. Suppose first that p is periodic. Then there exists a compact set $K \subset T$ such that $pK = pT$. This implies that

$$J(M_p\,; K) = \bigcap_{t \in K} \pi_t^*(M_p) = \bigcap_{t \in K} M_{pt} = \bigcap_{t \in T} M_{pt} = J(M_p).$$

On the other hand, let K be a compact subset of T such that $J(M_p\,; K) \subset J(M_p)$. Then for each $f \in M_p$, $J([f]; K) \subset J(M_p\,; K) \subset J(M_p)$. By Lemma 5, $pT \subset Z(f)\cdot K$. Now assume that there exists a $t \in T$ such that $pt \notin pK$. Since pK is closed there exists an open set U such that $pK \subset U$ and $pt \notin U$. Since K is compact there exists an open set V containing p such that $VK \subset U$. Let f be a function in M whose zero set is contained in V.

Then $pT \subset Z(f) \cdot K \subset VK \subset U$ which is a contradiction. Thus $pT = pK$, and p is periodic.

8. DEFINITION. Let $\mathcal{A}$ be a class of subsets of T. The elements of $\mathcal{A}$ are called *admissible sets*. We say that the maximal ideal M_p is $\mathcal{A}$-recursive under T provided that for each $f \in m(M_p)$ there is an admissible set $A \in \mathcal{A}$ such that $[f] \subset J(M_p ; A)$.

9. THEOREM. *M_p is recursive under T if and only if p is recursive under T.*

Proof. Assume that M_p is recursive and let U be an open set containing p. Since X is completely regular there exists a function $f \in m(M_p)$ such that $Z(f) \subset U$. By the definition of recursive there exists an admissible set $A \subset T$ such that $[f] \subset J(M_p ; A)$. Lemma 5 implies that

$$pA \subset \text{cl } (Z(f) \cdot e) = Z(f) \subset U.$$

Thus p is recursive under T.

Now suppose that p is recursive, and let $f \in m(M_p)$. Then there exists an open set U containing p such that $U \subset Z(f)$. Hence there is an admissible set $A \subset T$ such that $pA \subset U \subset Z(f) = \text{cl } (Z(f) \cdot e)$. This implies that $[f] \subset J(M_p ; A)$.

It is now clear how to proceed in general. In order to define all of the classical recursive properties for maximal ideals M_p in $C(X)$, we simply replace the term admissible set by an appropriate phrase, as in [2, 3.38].

Finally, we give a necessary and sufficient condition in order that (X, T) be minimal.

10. THEOREM. *A necessary and sufficient condition that the transformation group (X, T, π) be minimal is that the only ideals in $C(X)$ which are invariant under $\mathcal{G} = \{\pi_t^* : t \in T\}$ are the ideals, $\{0\}$, and $C(X)$.*

Proof. We first prove the necessity. Let (X, T, π) be minimal and let Q be an ideal in $C(X)$ which is invariant under $\mathcal{G}$. For all $t \in T$, $\pi_t^*(Q) = Q$. Thus $J(Q) = \bigcap_{t \in T} \pi_t^*(Q) = Q$. If there exists an $f \in Q$ such that $Z(f) = \emptyset$ then $Q = C(X)$, and the theorem is proved. Otherwise let $f \in Q$ and $p \in Z(f)$. Then $Q \subset M_p$ which implies $J(Q) \subset J(M_p)$. Since (X, T, π) is minimal, cl $(pT) = X$. Thus if $g \in J(M_p)$ then $g \in \pi_t^*(M_p) = M_{pt}$. This implies that $g(x) = 0$ for all $x \in pT$. Since g is continuous and pT is dense in X we have $g(x) = 0$ for all $x \in X$. Thus $J(M_p) = \{0\}$. But $J(Q) \subset J(M_p)$, and therefore $J(Q) = 0$.

To prove the sufficiency, let $p \in X$. By definition $J(M_p)$ is invariant under $\mathcal{G}$. Clearly $1 \notin J(M_p)$. This implies $J(M_p) \neq C(X)$. Thus $J(M_p) = \{0\}$. If $f \in C(X)$ such that $Z(f) \supset$ cl (pT) then $f \in J(M_p)$. If cl $(pT) \neq X$ then there is a $f \in C(X)$ such that $f \neq 0$ and $f \in J(M_p)$ which is a contradiction. Thus cl $(pT) = X$, and (X, T, π) is minimal.

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