GENERALIZATIONS OF THE NOTION OF CLASS GROUP[1,2]

BY

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Introduction

There are currently available two equivalent descriptions for the class group of a Noetherian integrally closed domain. The older, more direct approach, can be summarized as follows: Let A be a Noetherian integrally closed domain and let D denote the free abelian group with the prime ideals of A of height one as generators. Let $x \neq 0$ be an element of A and consider the element $\sum_{\mathrm{bt}\,\mathfrak{p}=1} l_{A_{\mathfrak{p}}}(A_{\mathfrak{p}}/xA_{\mathfrak{p}})\cdot\mathfrak{p}$ of D. Let R denote the subgroup of D generated by all such elements. Then the class group of A, $C(A)$, is the group D/R.

The second approach will now be described. Let A be a Noetherian integrally closed domain. Let $\mathfrak{M}_i$ denote the category of all finitely generated A-modules M such that $M_{\mathfrak{p}} = 0$ for all prime ideals of height less than i. In other words, $\mathfrak{p} \in \operatorname{Supp} M$ if and only if the height of $\mathfrak{p}$ is at least i. From the exact sequence of categories

$$0 \to \mathfrak{M}_1/\mathfrak{M}_2 \to \mathfrak{M}_0/\mathfrak{M}_2 \to \mathfrak{M}_0/\mathfrak{M}_1 \to 0$$

derives an exact sequence of Grothendieck groups

$$K^0(\mathfrak{M}_1/\mathfrak{M}_2) \to K^0(\mathfrak{M}_0/\mathfrak{M}_2) \to K^0(\mathfrak{M}_0/\mathfrak{M}_1) \to 0.$$

Now $K^0(\mathfrak{M}_0/\mathfrak{M}_1)$ is $\mathbf{Z}$; the isomorphism is given by

$$M \to \dim_F(F \otimes_A M)$$

where F is the field of quotients of A. Therefore

$$K^0(\mathfrak{M}_0/\mathfrak{M}_2) \cong \mathbf{Z} \oplus \operatorname{Im}(K^0(\mathfrak{M}_1/\mathfrak{M}_2)).$$

$\operatorname{Im}(K^0(\mathfrak{M}_1/\mathfrak{M}_2))$ can be identified as the class group, $C(A)$, of A [2, Chap. 7, § 4, n° 7, Prop. 17].

In this article we generalize both these definitions to prime ideals of height greater than 1. Generalizing from the first description a sequence of groups, to be called $C_i(A)$ $(0 \leq i \leq \dim A)$, is obtained; from the second description a sequence of groups, to be called $W_i(A)$ $(0 \leq i \leq \dim A)$, is obtained.

The groups $W_i(A)$ are defined for each commutative Noetherian ring A. The groups $C_i(A)$ are defined for those commutative Noetherian rings A which are locally Macaulay.

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[2] This article relates the completion and extension of work of which [4] was the research announcement.

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Following the definition of the groups C_i and W_i in Section 1 we give in Section 2 an alternative treatment of the groups W_i. In Section 3 connections between the two sequences are obtained by using the alternative description of the W_i given in Section 2.

It is convenient to discuss following Section 3 some relations between the groups W_i and the Grothendieck group of the category of finitely generated A-modules, which we do in Section 4. In Sections 5 through 8 we consider some "functorial" properties which these groups enjoy. These properties are strict analogues of those of the ordinary class group. In Section 5 we give a general mapping principle for flat algebras over A. We use this principle to examine the particular algebras A_S, where S is a multiplicatively closed subset of A, (Section 6) and $A[X]$, X an indeterminant (Section 7). Section 8 contains results which connect the groups of A with groups of A/I when I is a particularly well behaved ideal of A.

Section 9 contains several miscellaneous results, among which is the fact that $C_i(A) = 0$ when A is a power series ring over a complete discrete rank one valuation ring or a field. In Section 10 we compute the groups C_i and W_i for various rings. These computations show that some results are best possible.

We close the article with a brief discussion, in Section 11, of relations of these groups with algebraic geometry. We also pose several problems which remain.

Several conventions need mention. A always denotes a commutative Noetherian ring. Whenever the groups C_i are being discussed we assume, as well, that A is locally Macaulay.

Any A-module is unitary and finitely generated. The length of an A-module M is denoted by $l_A(M)$, and occasionally the subscript A is omitted when no confusion can arise. Upper and lower case $\mathfrak{p}$ denotes, almost without exception, a prime ideal in A, and ht $\mathfrak{p}$ denotes its height in A.

1. Definitions

Let A denote a Noetherian, locally Macaulay ring. For each i, $0 \leq i \leq \dim A$, let $D_i = D_i(A)$ denote the free abelian group based on the symbols $\langle \mathfrak{P} \rangle$ where $\mathfrak{P}$ is a prime ideal of height i of A.

By an A-sequence of length i is meant a sequence of elements $x_1, \cdots, x_i$ of A such that

$$\sum_{j=1}^{k} x_j A : x_{k+1} A = \sum_{j=1}^{k} x_j A \qquad \text{for } k = 0, \cdots, i-1.$$

It follows that if $x_1, \cdots, x_i$ is an A-sequence, then $\sum_{j=1}^{i} x_j A$ is an unmixed ideal of A of height i or is A. To each A-sequence of length i, $x_1, \cdots, x_i$, attach the element

$$\sum_{\text{ht } \mathfrak{P}=i} e(x_1, \cdots, x_i \mid A_{\mathfrak{P}}) \langle \mathfrak{P} \rangle$$

in D_i; here $e(x_1, \cdots, x_i \mid A_{\mathfrak{P}})$ denotes the multiplicity of the ideal $\sum x_j A_{\mathfrak{P}}$

on the module $A_{\mathfrak{P}}$ (and we take it to be zero if $A_{\mathfrak{P}} = \sum x_j A_{\mathfrak{P}}$). Since $A_{\mathfrak{P}}$ is a Macaulay ring, $e(x_1, \cdots, x_i \mid A_{\mathfrak{P}})$ is simply $l_{A_{\mathfrak{P}}}(A_{\mathfrak{P}}/\sum x_j A_{\mathfrak{P}})$.

Now let $R_i = R_i(A)$ denote the subgroup of D_i generated by all A-sequences of length i. The i^{th} *class group of* A is $D_i(A)/R_i(A)$ which we denote by $C_i(A)$. For convenience denote $\coprod_{0 \leq i} C_i(A)$ by $C_\bullet(A)$.

If $\mathfrak{P}$ is a prime ideal of A of height i, then the image of $\langle \mathfrak{P} \rangle$ in $C_i(A)$ is denoted by $\text{cl}(\mathfrak{P})$.

As the only A-sequence of length 0 generates the 0-ideal of A, R_0 consists of the cyclic subgroup generated by $\sum_{\text{ht } \mathfrak{P}=0} l_{A_{\mathfrak{P}}}(A_{\mathfrak{P}}) \langle \mathfrak{P} \rangle$. This yields at once the fact that $C_0(A)$ is torsion if and only if 0 is a primary ideal of A, and $C_0(A) = 0$ if and only if A is a domain. When A is an integrally closed domain, $C_1(A)$ is the ordinary class group of A.

Suppose that A is a commutative Noetherian ring. Following the notation of the introduction, let $\mathfrak{M}_i = \mathfrak{M}_i(A)$ be the category of finitely generated A-modules M such that $\mathfrak{P} \in \text{Supp } M$ only if $\text{ht } \mathfrak{P} \geq i$. If

$$0 \to M' \to M \to M'' \to 0$$

is an exact sequence of A-modules, then $\text{Supp } M = \text{Supp } M' \cup \text{Supp } M''$ [2 Chap. II, §4, n° 4, Prop. 16]. Thus M is in $\mathfrak{M}_i$ if and only if M' and M'' are in $\mathfrak{M}_i$. Hence $\mathfrak{M}_j$ is a Serre subcategory of $\mathfrak{M}_i$ for $j \geq i$ (see [6] for terminology).

For a cateory $\mathcal{C}$, let $K^0(\mathcal{C})$ denote the Grothendieck group of $\mathcal{C}$.

For each triple $(i - 1, i, i + 1)$ of integers there is a functor

$$\mathfrak{M}_i/\mathfrak{M}_{i+1} \to \mathfrak{M}_{i-1}/\mathfrak{M}_{i+1}$$

induced from the inclusion functor $\mathfrak{M}_i \to \mathfrak{M}_{i-1}$ and which in turn induces a homomorphism

$$K^0(\mathfrak{M}_i/\mathfrak{M}_{i+1}) \to K^0(\mathfrak{M}_{i-1}/\mathfrak{M}_{i+1}).$$

Let $W_i(A)$ be the image of this homomorphism. In the next section we show that in fact $W_i(A)$ is a direct summand of $K^0(\mathfrak{M}_{i-1}/\mathfrak{M}_{i+1})$.

By convention set $W_0(A) = (0)$. Let $W_\bullet(A)$ denote $\coprod_{0 \leq i} W_i(A)$.

When A is integrally closed, $W_1(A) = C_1(A)$, as remarked in the introduction.

2. Alternative description of $W_i(A)$

If $\mathcal{C}$ is a category and C is an object in $\mathcal{C}$, then $[C]$ denotes the class of C in $K^0(\mathcal{C})$.

If M is in $\mathfrak{M}_i$ and $\mathfrak{P}$ is a prime ideal of A of height i, then $M_{\mathfrak{P}}$ has finite length as an $A_{\mathfrak{P}}$-module.

LEMMA 2.1. *For each* $M \in \mathfrak{M}_i$, *let* $\chi_i(M) = \sum_{\text{ht } \mathfrak{P}=i} l_{A_{\mathfrak{P}}}(M_{\mathfrak{P}})$.

(a) *If* $0 \to M' \to M \to M'' \to 0$ *is an exact sequence in* $\mathfrak{M}_i$ *then* $\chi_i(M) = \chi_i(M') + \chi_i(M'')$.

(b) M *is in* $\mathfrak{M}_{i+1}$ *if and only if* $\chi_i(M) = 0$.

Proof. (a) follows from the additivity of $l_{A_\mathfrak{P}}$ for each $\mathfrak{P}$. As for (b), if $M \in \mathfrak{M}_{i+1}$, then $M_\mathfrak{P} = 0$ for each prime ideal $\mathfrak{P}$ of A of height i, so $\chi_i(M) = 0$. On the other hand $\chi_i(M) = 0$ implies that $l_{A_\mathfrak{P}}(M_\mathfrak{P}) = 0$ for each prime ideal $\mathfrak{P}$, ht $\mathfrak{P} = i$. Hence $M_\mathfrak{P} = 0$ for these prime ideals, so $M \in \mathfrak{M}_{i+1}$.

COROLLARY 2.1. *Each object of* $\mathfrak{M}_i/\mathfrak{M}_{i+1}$ *is of finite length.*

Proof. That χ_i is a length function follows directly from the lemma.

LEMMA 2.3. *Let M be a simple object in* $\mathfrak{M}_i/\mathfrak{M}_{i+1}$. *Then there is a unique prime ideal $\mathfrak{P}$ of height i such that* $M \cong A/\mathfrak{P}$ *in* $\mathfrak{M}_i/\mathfrak{M}_{i+1}$.

Proof. Since M is simple $\chi_i(M) = 1$, so there is a prime ideal $\mathfrak{P}$ of height i such that $l_{A_\mathfrak{P}}(M_\mathfrak{P}) = 1$ and $l_{A_\mathfrak{Q}}(M_\mathfrak{Q}) = 0$ for all other prime ideals $\mathfrak{Q}$ of height i. Hence $\mathfrak{P} \in \mathrm{Ass}_A M$. Thus there is an exact sequence of A-modules

$$0 \to A/P \to M \to N \to 0.$$

Now $N_\mathfrak{Q} = 0$ for all prime ideals $\mathfrak{Q}$ of height i, so $N \in \mathfrak{M}_{i+1}$, hence $A/\mathfrak{P} \to M$ is an isomorphism in $\mathfrak{M}_i/\mathfrak{M}_{i+1}$.

Let $\mathcal{S}_i$ be the semisimple full subcategory of $\mathfrak{M}_i/\mathfrak{M}_{i+1}$ whose objects are sums of the simple objects. In the terminology of [6], we know that $\mathcal{S}_i$ is both substantial and bisubstantial in $\mathfrak{M}_i/\mathfrak{M}_{i+1}$ (see [2, Chap. IV, §1, n° 4, Thm. 2]). Then by (9.4) and (9.5) of [6] the inclusion functor induces isomorphisms

$$K^0(\mathcal{S}_i) \cong K^0(\mathfrak{M}_i/\mathfrak{M}_{i+1}) \quad \text{and} \quad K^1(\mathcal{S}_i) \cong K^1(\mathfrak{M}_i/\mathfrak{M}_{i+1}).$$

By (7.5) of [6], the sequence of abelian groups

$$K^1(\mathfrak{M}_{i-1}/\mathfrak{M}_i) \xrightarrow{\delta} K^0(\mathfrak{M}_i/\mathfrak{M}_{i+1}) \xrightarrow{\iota} K^0(\mathfrak{M}_{i-1}/\mathfrak{M}_{i+1}) \xrightarrow{\nu} K^0(\mathfrak{M}_{i-1}/\mathfrak{M}_i) \to 0$$

is exact. We now proceed to describe these groups and the homomorphisms in terms of A and its ideals. The group $W_i(A)$ is just Im ι.

PROPOSITION 2.4. $X_i : K^0(\mathfrak{M}_i/\mathfrak{M}_{i+1}) \to D_i$ *defined by*

$$X_i([M]) = \textstyle\sum_{\mathrm{ht}\,\mathfrak{P}=i} l_{A_\mathfrak{P}}(M_\mathfrak{P})\langle\mathfrak{P}\rangle$$

is an isomorphism.

Proof. We use the isomorphism

$$K^0(\mathcal{S}_i) \cong K^0(\mathfrak{M}_i/\mathfrak{M}_{i+1}).$$

Each object in $\mathcal{S}_i$ is isomorphic to an object of the form $\coprod_{\mathrm{ht}\,\mathfrak{P}=i} (A/\mathfrak{P})^{n_\mathfrak{P}}$ ($(A/\mathfrak{P})^n$ is a direct sum of n copies of $A/\mathfrak{P}$) where all but a finite number of the $n_\mathfrak{P}$ are zero. Thus an element of $K^0(\mathcal{S}_i)$ can be written in the form $\sum_{\mathrm{ht}\,\mathfrak{P}=i} m_\mathfrak{P}[A/\mathfrak{P}]$, $m_\mathfrak{P} \in \mathbf{Z}$, almost all $m_\mathfrak{P} = 0$. It is clear that $K^0(\mathcal{S}_i)$ is free on the set $\{[A/\mathfrak{P}] : \mathrm{ht}\,\mathfrak{P} = i\}$. The proposition now follows from the definition of X_i.

PROPOSITION 2.5. $K^1(\mathfrak{M}_i/\mathfrak{M}_{i+1})$ *is isomorphic to* $\coprod_{\mathrm{ht}\,\mathfrak{P}=i} (A_\mathfrak{P}/\mathfrak{P}A_\mathfrak{P})^*$.

Proof. Once again we use the isomorphism established above and consider the group $K^1(\mathcal{S}_i)$. Let $S \in \mathcal{S}_i$ and denote by $S(\mathfrak{P})$ the subobject of S which is the sum of the simple submodules of S isomorphic to $A/\mathfrak{P}$. Then $S = \coprod_{\text{ht } \mathfrak{P}=i} S(\mathfrak{P})$ with $S(\mathfrak{P}) = (0)$ for almost all $\mathfrak{P}$. If α is an automorphism of S then the composite

$$S(\mathfrak{P}) \to S \xrightarrow{\alpha} S \to S(\mathfrak{P}')$$

is zero unless $\mathfrak{P} = \mathfrak{P}'$ (where the end maps are the injection and projection in the *finite* direct sum). If $\mathfrak{P} = \mathfrak{P}'$, then this homomorphism is an automorphism which we denote by $\alpha(\mathfrak{P})$.

Hence the pair $(S, \alpha) = (\coprod S(\mathfrak{P}), \coprod \alpha(\mathfrak{P}))$, so in $K^1(\mathcal{S}_i)$,

$$[S, \alpha] = \prod_{\text{ht } \mathfrak{P}=i} [S(\mathfrak{P}), \alpha(\mathfrak{P})].$$

We now consider the pair $(S(\mathfrak{P}), \alpha(\mathfrak{P})) = (T, \tau)$, where T is a direct sum of n copies of $A/\mathfrak{P}$ and τ is an automorphism of T. Then τ can be considered to be a matrix (τ_{ij}) with τ_{ij} in $\text{Hom}_{\mathcal{S}_i}(A/\mathfrak{P}, A/\mathfrak{P})$ which is a division ring.

LEMMA 2.6. $\text{Hom}_{\mathcal{S}_i}(A/\mathfrak{P}, A/\mathfrak{P}) \cong A_{\mathfrak{P}}/\mathfrak{P}A_{\mathfrak{P}}$.

Remark. The referee has suggested the proof below which is shorter than the original proof.

Proof. Let $\bar{A} = A/\mathfrak{P}$. Then

$$\text{Hom}_{\mathcal{S}_i}(\bar{A}, \bar{A}) = \text{Hom}_{\mathfrak{M}_i/\mathfrak{M}_{i+1}}(\bar{A}, \bar{A}) = \varinjlim \text{Hom}_A(M', \bar{A}/N')$$

where the limit is over those M' (resp. N') such that $\bar{A}/M' \in \mathfrak{M}_{i+1}$ (resp. $N' \in \mathfrak{M}_{i+1}$) (see page 365 of P. Gabriel, *Des Categories Abeliennes*, Bull. Soc. Math. France, vol. 90(1962), pp. 323–448). Hence

$$\text{Hom}_{\mathcal{S}_i}(\bar{A}, \bar{A}) = \varinjlim_{\bar{\mathfrak{a}}} \text{Hom}_A(\bar{\mathfrak{a}}, \bar{A}) = \bigcup \bar{\mathfrak{a}}^{-1} = \bar{K}$$

where $\bar{\mathfrak{a}}$ runs through all the ideals of $\bar{A}$ and $\bar{K}$ is the field of quotients of $\bar{A}$.

To complete the proof of 2.5 we only remark that now one can use elementary row operations to get that $[T, \tau] = [A/\mathfrak{P}, \det \tau]$ in $K^1(\mathcal{S}_i)$. This defines a homomorphism

$$K^1(\mathcal{S}_i) \to \coprod_{\text{ht } \mathfrak{P}=i} (A_{\mathfrak{P}}/\mathfrak{P}A_{\mathfrak{P}})^*$$

which is easily checked to be an isomorphism.

We now describe the homomorphisms δ, ι and ν in terms of the descriptions of the groups just obtained.

Let $(\bar{x}_{\mathfrak{P}}) \in \coprod_{\text{ht } \mathfrak{P}=i-1} (A_{\mathfrak{P}}/\mathfrak{P}A_{\mathfrak{P}})^*$. Then the vector $(\bar{x}_{\mathfrak{P}})$ is the product of its components $\bar{x}_{\mathfrak{P}}$, so we may tell what happens to each component, since $\delta((\bar{x}_{\mathfrak{P}})) = \sum_{\mathfrak{P}} \delta(\bar{x}_{\mathfrak{P}})$. Write $\bar{x}_{\mathfrak{P}} = \bar{a}_{\mathfrak{P}}/\bar{b}_{\mathfrak{P}}$ with $a_{\mathfrak{P}}, b_{\mathfrak{P}} \in A$, both not in $\mathfrak{P}$. Then

$$\delta(\bar{x}_{\mathfrak{P}}) = [A/(\mathfrak{P} + b_{\mathfrak{P}}A)] - [A/(\mathfrak{P} + a_{\mathfrak{P}}A)]$$

in $K^0(\mathfrak{M}_i/\mathfrak{M}_{i+1})$.

ι is the canonical homomorphism induced from the inclusion of the categories.

$$\nu(M) = \sum_{\text{ht } \mathfrak{P}=i-1} l_{A_\mathfrak{P}}(M_\mathfrak{P})\langle \mathfrak{P} \rangle.$$

Because $K^0(\mathfrak{M}_{i-1}/\mathfrak{M}_i)$ is a free group, the epimorphism ν splits to give

$$K^0(\mathfrak{M}_{i-1}/\mathfrak{M}_{i+1}) \cong \text{Ker } \nu \oplus D_{i-1}(A) = \text{Im } \iota \oplus D_{i-1}(A) = W_i(A) \oplus D_{i-1}(A)$$

by the definition of $W_i(A)$.

Now $\text{Im } \iota = K^0(\mathfrak{M}_\iota/\mathfrak{M}_{i+1})/\text{Ker } \iota \cong D_i(A)/\text{Im } \delta$. The description of δ given above shows that $\text{Ker } \iota = \text{Im } \delta$ is generated by the elements $[A/(\mathfrak{P} + xA)]$ where $\mathfrak{P}$ is a prime ideal of height $i - 1$ of A and $x \notin \mathfrak{P}$. This element is just $\sum_{\text{ht } \mathfrak{Q}=i} l_{A_\mathfrak{Q}}(A_\mathfrak{Q}/\mathfrak{P}_\mathfrak{Q} + xA_\mathfrak{Q})\langle \mathfrak{Q} \rangle$ in $D_i(A)$.

3. Relations between $C_i(A)$ and $W_i(A)$

The relation which is easiest to obtain is that $C_i(A)$ is a stronger invariant than is $W_i(A)$, for each i.

PROPOSITION 3.1. *For each i, $0 \leq i \leq \dim A$, there is an epimorphism $C_i(A) \to W_i(A)$.*

Proof. It suffices to show that each relation $r \in R_i(A)$ maps to zero under the homomorphism

$$K^0(\mathfrak{M}_i/\mathfrak{M}_{i+1}) \to K^0(\mathfrak{M}_{i-1}/\mathfrak{M}_{i+1}).$$

Let $r = \sum_{\text{ht } \mathfrak{P}=i} l_{A_\mathfrak{P}}(A_\mathfrak{P}/ \sum x_j A_\mathfrak{P})\langle \mathfrak{P} \rangle$ where $x_1, \cdots, x_i$ is an A-sequence of length i. Then the sequence of A-modules

$$0 \to A/ \sum_{j=1}^{i-1} x_j A \xrightarrow{x_i} A/ \sum_{j=1}^{i-1} x_j A \to A/ \sum_{j=1}^{i} x_j A \to 0$$

is exact. So

$$[A/ \sum_{j=1}^{i} x_j A] = [A/ \sum_{j=1}^{i-1} x_j A] - [A/ \sum_{j=1}^{i-1} x_j A] = 0$$

in $K^0(\mathfrak{M}_{i-1}/\mathfrak{M}_{i+1})$.

PROPOSITION 3.2. *Suppose that $C_i(A) = 0$ for some i. Then the epimorphism $C_{i+1}(A) \to W_{i+1}(A)$ of Proposition 3.1 is an isomorphism.*

Proof. Using the description of $\text{Im } \delta$ in Section 2 one sees that to prove the proposition it is sufficient to show that each element $[A/(\mathfrak{P} + xA)]$ ($\text{ht } \mathfrak{P} = i$, $x \notin \mathfrak{P}$) in $K^0(\mathfrak{M}_{i+1}/\mathfrak{M}_{i+2})$ is in the subgroup $R_{i+1}(A)$ of $D_{i+1}(A)$.

Since $C_i(A) = 0$ there are A-sequences $x_{1k}, \cdots, x_{ik}$; $k = 1, \cdots, m$ and integers $n_1, \cdots, n_m$ such that

$$\langle \mathfrak{P} \rangle = \sum_{k=1}^{m} n_k \sum_{\text{ht } \mathfrak{p}=i} l_{A_\mathfrak{p}}(A_\mathfrak{p}/ \sum_{j=1}^{i} x_{jk} A_\mathfrak{p})\langle \mathfrak{p} \rangle.$$

Among the prime ideals $\mathfrak{p}$ of height i which are associated with at least one of the A-sequences above, let $\mathfrak{p}_1, \cdots, \mathfrak{p}_r$ contain x, while $\mathfrak{p}_{r+1}, \cdots, \mathfrak{p}_s$ do not contain x. Since $\bigcap_{j=r+1}^{s} \mathfrak{p}_j \not\subset \bigcup_{j=1}^{r} \mathfrak{p}_j$, we can choose an element $w \in A$ such that $w \in \bigcap_{j=r+1}^{s} \mathfrak{p}_j$ while $w \notin \mathfrak{p}_j$ for $j = 1, \cdots, r$. Then $t = x + w$ is not in any $\mathfrak{p}_j$ and the sequences $x_{1k}, \cdots, x_{ik}, t$; $k = 1, \cdots, m$ are A-sequences. We now compute the element

$$\text{(a)} \quad \sum_{k=1}^{m} n_k \sum_{\text{ht}\,\mathfrak{Q}=i+1} l_{A_{\mathfrak{Q}}}(A_{\mathfrak{Q}}/ \textstyle\sum_{j=1}^{i} x_{jk} A_{\mathfrak{Q}} + tA_{\mathfrak{Q}})\langle \mathfrak{Q} \rangle \qquad \text{in } R_{i+1}(A).$$

Apply the associativity law for multiplicities [8] to

$$l_{A_{\mathfrak{Q}}}(A_{\mathfrak{Q}}/ \textstyle\sum_j x_{jk} A_{\mathfrak{Q}} + tA_{\mathfrak{Q}}) = e(x_{1k}, \cdots, x_{ik}, t \mid A_{\mathfrak{Q}})$$

to obtain

$$e(x_{1k}, \cdots, x_{ik}, t \mid A_{\mathfrak{Q}}) = \sum_{\text{ht}\,\mathfrak{p}=i} e(x_{1k}, \cdots, x_{ik} \mid A_{\mathfrak{p}}) e(t \mid A_{\mathfrak{Q}}/\mathfrak{p}A_{\mathfrak{Q}}).$$

Substituting in (a) we get

$$\text{(b)} \quad \sum_{k=1}^{m} \sum_{\text{ht}\,\mathfrak{Q}=i+1} n_k \sum_{\text{ht}\,\mathfrak{p}=i} l_{A_{\mathfrak{p}}}(A_{\mathfrak{p}}/ \textstyle\sum_j x_{jk} A_{\mathfrak{p}}) l((A/\mathfrak{p})_{\mathfrak{Q}}/t(A/\mathfrak{p})_{\mathfrak{Q}})\langle \mathfrak{Q} \rangle.$$

We rearrange (b) to obtain

$$\begin{aligned}
&\sum_{\text{ht}\,\mathfrak{Q}=i+1} \sum_{\text{ht}\,\mathfrak{p}=i} \sum_{k=1}^{m} n_k l_{A_{\mathfrak{p}}}(A_{\mathfrak{p}}/ \textstyle\sum x_{jk} A_{\mathfrak{p}}) l((A/\mathfrak{p})_{\mathfrak{Q}}/t(A/\mathfrak{p})_{\mathfrak{Q}})\langle \mathfrak{Q} \rangle \\
&\quad = \sum_{\text{ht}\,\mathfrak{Q}=i+1} \sum_{\text{ht}\,\mathfrak{p}=i} \delta_{\mathfrak{p},\mathfrak{P}} l((A/\mathfrak{p})_{\mathfrak{Q}}/t(A/\mathfrak{p})_{\mathfrak{Q}}) \langle \mathfrak{Q} \rangle \text{ (where } \delta \text{ is Kronecker } \delta) \\
&\quad = \sum_{\text{ht}\,\mathfrak{Q}=i+1} l_{A_{\mathfrak{Q}}}(A_{\mathfrak{Q}}/(\mathfrak{P} + tA)_{\mathfrak{Q}}) \langle \mathfrak{Q} \rangle \\
&\quad = \sum_{\text{ht}\,\mathfrak{Q}=i+1} l_{A_{\mathfrak{Q}}}(A_{\mathfrak{Q}}/(\mathfrak{P} + xA)_{\mathfrak{Q}}) \langle \mathfrak{Q} \rangle \\
&\quad = [A/\mathfrak{P} + xA].
\end{aligned}$$

The penultimate equality follows since $x \notin \mathfrak{P}$ implies $w \in \mathfrak{P}$, so

$$\mathfrak{P} + tA = \mathfrak{P} + (x + w)A = \mathfrak{P} + xA.$$

COROLLARY 3.3. *If A is a domain, then $W_1(A)$ is isomorphic to $C_1(A)$.*

COROLLARY 3.4. *$C_\bullet(A) = 0$ if and only if A is a domain and $W_\bullet(A) = 0$.*

In Section 10 we give an example which shows that $C_2(A) \neq W_2(A)$ for a domain A.

4. Connections with $K^0(A)$

The inclusion functor $\mathfrak{M}_j \to \mathfrak{M}_i$ for $j \geq i$ induces a group homomorphism

$$\varphi_{ij} : K^0(\mathfrak{M}_j) \to K^0(\mathfrak{M}_i)$$

whose cokernel is $K^0(\mathfrak{M}_i/\mathfrak{M}_j)$. For each pair (i, j), $i \leq j$, let $G_{ij}(A) = G_{ij}$ denote the image of φ_{ij}. Let $i \leq j, j \leq k$; then $\varphi_{ik} = \varphi_{ij}\varphi_{jk}$, so for fixed i, the G_{ij} give a filtration on the group $K^0(\mathfrak{M}_i)$. Since $K^0(\mathfrak{M}_i/\mathfrak{M}_{i+1}) = D_i(A)$ is free, the group $G_{i,i+1}$ is a direct summand of $K^0(\mathfrak{M}_i)$.

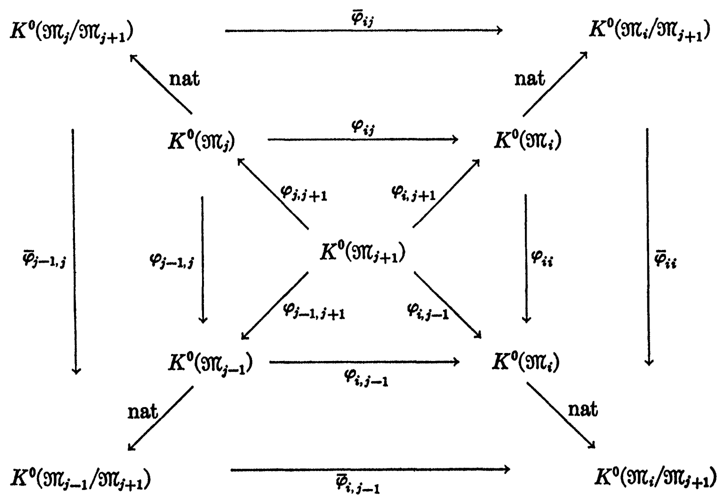
PROPOSITION 4.1. *Let i be an integer, $0 \leq i \leq \dim A$.*

(a) $G_{ii}/G_{i,i+1} = D_i(A)$.

(b) *For each j, $i < j$, $G_{ij}/G_{i,j+1}$ is a homomorphic image of W_j.*

Proof. $G_{ii}/G_{i,i+1} = \operatorname{Coker} \varphi_{i,i+1} = K^0(M_i/M_{i+1})$ so (a) follows from Proposition 2.4.

To prove (b) consider the commutative diagram



Now

$$G_{ij}/G_{i,j+1} \cong \operatorname{Im} \bar{\varphi}_{ij} = \operatorname{Im} \bar{\varphi}_{i,j-1} \bar{\varphi}_{j-1,j} .$$

But $\operatorname{Im} \bar{\varphi}_{j-1,j} = W_j(A)$, so $\bar{\varphi}_{i,j-1}$ is the desired epimorphism.

COROLLARY 4.2. *The groups G_{0j} give a filtration on the Grothendieck group of the category of finitely generated A-modules $K^0(\mathfrak{M}_0)$ whose associated graded group is a homomorphic image of $D_0(A) \oplus W_\bullet(A)$ and hence of $D_0(A) \oplus C_\bullet(A)$.*

COROLLARY 4.3. *If A is such that $W_\bullet(A) = 0$, and* (*Krull*) $\dim A < \infty$, *then $K^0(\mathfrak{M}_0) = D_0(A)$.*

Proof. $D_0(A)$ is a direct summand of $K^0(\mathfrak{M}_0)$. The statement now follows from Cor. 4.2.

In Section 10 we show that when A is the coordinate ring of the real three-sphere ($A = \mathbf{R}[X_0, X_1, X_2, X_3]/(X_0^2 + X_1^2 + X_2^2 + X_3^2 - 1)$) then $K^0(\mathfrak{M}_0(A)) = \mathbf{Z}$, but $W_3(A) = \mathbf{Z}/2\mathbf{Z}$. So the converse of 4.3 does not hold. This example also shows that the next prroposition is best possible.

PROPOSITION 4.4. *Let A be an integrally closed domain with $K^0(\mathfrak{M}_0) \cong \mathbf{Z}$. Then $W_1(A) = 0$ and $W_2(A) = 0$.*

Proof. $K^0(\mathfrak{M}_0) \cong \mathbf{Z}$ implies that $K^0(\mathfrak{M}_0/\mathfrak{M}_i) \cong \mathbf{Z}$ for all i, in particular for $i = 2$. Hence $W_1(A) = 0$. So A is a unique factorization domain [2, Chap. 7, §4, n° 4, Prop. 17]. We show that $K^0(\mathfrak{M}_1) = D_1$. Since $K^0(\mathfrak{M}_0) = \mathbf{Z}$, the homomorphism

$$K^1(\mathfrak{M}_0/\mathfrak{M}_1) \to K^0(\mathfrak{M}_1)$$

is an epimorphism, so each element of $K^0(\mathfrak{M}_1)$ is of the form $[A/xA] - [A/yA]$ $x, y \neq 0$ in A. If $x = uv$, then the sequence

$$0 \to A/vA \xrightarrow{u} A/xA \to A/uA \to 0$$

is exact. Hence $[A/xA] = [A/uA] + [A/vA]$. Since A is a UFD we may factor x and y into irreducible elements, say $x = p_1 \cdots p_r$, $y = q_1 \cdots q_s$. Hence

$$[A/xA] - [A/yA] = \sum [A/p_i A] - \sum [A/q_i A].$$

But this element is in $D_1(A)$. Hence

$$K^0(\mathfrak{M}_1) = D_1(A), \quad \text{so} \quad K^0(\mathfrak{M}_1/\mathfrak{M}_3) = D_1(A);$$

therefore $W_2(A) = 0$.

5. The mapping principle

Let B be an A-algebra which is flat as an A-module, and which is Noetherian. We show that under these hypotheses there are natural homomorphisms $W_i(A) \to W_i(B)$ and when A and B are locally Macaulay $C_i(A) \to C_i(B)$ for all i, $0 \leq i \leq \dim A$.

The groups C_i can be treated as follows. Let $\mathfrak{p}$ be a prime ideal of A of height i. Since A is locally Macaulay, there is an A-sequence $x_1, \cdots, x_i$ such that $\mathfrak{p}$ is a minimal prime ideal associated with $I = \sum_{j=1}^{i} x_j A$. Then each prime ideal $\mathfrak{P}$ associated with $\mathfrak{p}B$ will be an associated prime ideal of IB [2, Chap. IV, §2, n° 6, Thm. 2], and so ht $\mathfrak{P} = i$ ($x_1, \cdots, x_i$ is also a B-sequence since B is flat as an A-module). To each element $\sum_{\text{ht } \mathfrak{p}=i} n_\mathfrak{p} \langle \mathfrak{p} \rangle$ of $D_i(A)$ assign the element $\sum_{\text{ht } \mathfrak{P}=i} \sum_{\text{ht } \mathfrak{p}=i} n_\mathfrak{p}\, l_{B_\mathfrak{P}}(B_\mathfrak{P}/\mathfrak{p}B_\mathfrak{P}) \langle \mathfrak{P} \rangle$ of $D_i(B)$. If a relation in $R_i(A)$ goes to a relation in $R_i(B)$, this homomorphism $D_i(A) \to D_i(B)$ induces the desired homomorphism $C_i(A) \to C_i(B)$. This we now check.

THEOREM 5.1. *The homomorphism*

$$D_i(A) \to D_i(B) : \langle \mathfrak{p} \rangle \to \textstyle\sum_{\text{ht } \mathfrak{P}=i} l_{B_\mathfrak{P}}(B_\mathfrak{P}/\mathfrak{p}B_\mathfrak{P}) \langle \mathfrak{P} \rangle$$

induces a homomorphism $C_i(A) \to C_i(B)$.

Proof. The discussion above shows that it is sufficient to prove that $R_i(A)$ is mapped into $R_i(B)$. Let $x_1, \cdots, x_i$ be an A-sequence of length i and consider the relation $\sum_{\text{ht } \mathfrak{p}=i} l_{A_\mathfrak{p}}(A_\mathfrak{p}/I_\mathfrak{p}) \langle \mathfrak{p} \rangle$ where $I = \sum_{j=1}^{i} x_j A$. Applying the homomorphism we obtain the element

$$\textstyle\sum_{\text{ht } \mathfrak{P}=i} \sum_{\text{ht } \mathfrak{p}=i} l_{A_\mathfrak{p}}(A_\mathfrak{p}/I_\mathfrak{p})\, l_{B_\mathfrak{P}}(B_\mathfrak{P}/\mathfrak{p}B_\mathfrak{P}) \langle \mathfrak{P} \rangle$$

of $D_i(B)$. Using the Theorem of Transition [8, 19.1] applied to $A_\mathfrak{p}$ and $B_\mathfrak{P}$ [8, 19.2] we obtain

$$l_{B_\mathfrak{P}}(B_\mathfrak{P}/IB_\mathfrak{P}) = l_{A_\mathfrak{P}}(A_\mathfrak{p}/I_\mathfrak{p}) l_{B_\mathfrak{P}}(B_\mathfrak{P}/\mathfrak{p}B_\mathfrak{P})$$

and so our element is $\sum_{\text{ht}\,\mathfrak{P}=i} l_{B_{\mathfrak{P}}}(B_{\mathfrak{P}}/IB_{\mathfrak{P}})\ \langle\mathfrak{P}\rangle$ which is in $R_i(B)$ since $x_1, \cdots, x_i$ is a B-sequence.

To treat the groups W_i, let $\mathfrak{M}_i = \mathfrak{M}_i(A)$ and $\mathfrak{N}_i = \mathfrak{M}_i(B)$ for each i. If $M \in \mathfrak{M}_i$ then $B \otimes_A M \in \mathfrak{N}_i$ [2, Chap. II, §4, n° 4, Prop. 18]. We therefore have a commutative diagram of categories

$$\begin{array}{ccc} \mathfrak{M}_i/\mathfrak{M}_{i+1} & \longrightarrow & \mathfrak{M}_{i-1}/\mathfrak{M}_{i+1} \\ \downarrow & & \downarrow \\ \mathfrak{N}_i/\mathfrak{N}_{i+1} & \longrightarrow & \mathfrak{N}_{i-1}/\mathfrak{N}_{i+1} \end{array}$$

induced by the functor $B \otimes_A -$. Since B is a flat A-module, there is induced a commutative diagram of Grothendieck groups

$$\begin{array}{ccc} K^0(\mathfrak{M}_i/\mathfrak{M}_{i+1}) & \xrightarrow{f} & K^0(\mathfrak{M}_{i-1}/\mathfrak{M}_{i+1}) \\ \downarrow & & \downarrow e \\ K^0(\mathfrak{N}_i/\mathfrak{N}_{i+1}) & \xrightarrow{g} & K^0(\mathfrak{N}_{i-1}/\mathfrak{N}_{i+1}) \end{array}$$

Since $W_i(A) = \operatorname{Im} f$, the desired homomorphism is

$$e : \operatorname{Im} f \rightarrow \operatorname{Im} g = W_i(B).$$

Summarizing, we obtain the following

THEOREM 5.2. *If B is a noetherian A-algebra which is flat as an A-module then there is a homomorphism $W_i(A) \rightarrow W_i(B)$ obtained by sending $[M]$ to $[B \otimes_A M]$.*

In the next two sections we apply these homomorphisms to the cases $B = A_S$, S a multiplicatively closed subset of A, and $B = A[X]$, and obtain more precise information.

6. From A to A_S

Throughout this section, S denotes a multiplicatively closed subset of A. Let $B = A_S$. The homomorphism $D_i(A) \rightarrow D_i(A_S)$ given in Section 5 can be described as follows. Let $\mathfrak{p}$ be a prime ideal of A of height i. Then

$$\langle\mathfrak{p}\rangle \rightarrow \sum_{\text{ht}\,\mathfrak{P}=i} l_{B_{\mathfrak{P}}}(B_{\mathfrak{P}}/\mathfrak{p}B_{\mathfrak{P}})\ \langle\mathfrak{P}\rangle.$$

Since $\mathfrak{p}B_{\mathfrak{P}} = B_{\mathfrak{P}}$ if $\mathfrak{p} \cap S \neq \emptyset$ or if $\mathfrak{p}B \neq \mathfrak{P}$, this element becomes $\langle\mathfrak{p}B\rangle$ if $\mathfrak{p} \cap S = \emptyset$ and 0 otherwise.

To obtain further information, the next lemma is required.

LEMMA 6.1. *Let S be a multiplicatively closed subset of A. If $y_1, \cdots, y_i$ is an A_S-sequence, then there is an A-sequence $x_1, \cdots, x_i$ such that*

$$\sum_{j=1}^{i} x_j A_S = \sum_{j=1}^{i} y_j A_S.$$

Proof. It is sufficient, by induction, to treat the case $i = 1$. Let $(0) = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_s$ be an irredundant representation of (0) as the intersection of primary ideals of A. Let $\mathfrak{p}_i$ be the radical of $\mathfrak{q}_i$, $i = 1, \cdots, s$. Since A is locally Macaulay, ht $\mathfrak{p}_i = 0$ for each i. Assume that $\mathfrak{p}_1, \cdots, \mathfrak{p}_k$ meet S, while $\mathfrak{p}_{k+1}, \cdots, \mathfrak{p}_s$ do not meet S. It may be assumed (multiplying by an element of S if necessary) that

$$y = y_1 \in \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_k .$$

Choose $w \in \mathfrak{q}_{k+1} \cap \cdots \cap \mathfrak{q}_s - \bigcup_{i=1}^k \mathfrak{p}_i$. Set $x = y + w$. Then $x \notin \mathfrak{p}_i$ for $1 \leq i \leq s$, and since $wA_S = 0$ we get $xA_S = yA_S$.

Remark. We are indebted to the referee for the above proof which represents a substantial simplification of the original argument.

The content of this lemma is that every element of $R_i(A_S)$ comes from an element of $R_i(A)$. This yields the following as corollaries.

THEOREM 6.2. (cf. [2, Chap. VII, §1, n° 10, Prop. 17]). *Let S be a multiplicatively closed subset of A. Then for each $i \geq 0$, there is an epimorphism*

$$C_i(A) \to C_i(A_S)$$

deduced from $\langle \mathfrak{p} \rangle \to 0$ if $\mathfrak{p} \cap S \neq \emptyset$ and $\langle \mathfrak{p} \rangle \to \langle \mathfrak{p}A_S \rangle$ if $\mathfrak{p} \cap S = \emptyset$. The kernel is generated by the set $\{\mathrm{cl}\,(\mathfrak{p})\}$ where $\mathfrak{p} \cap S \neq \emptyset$.

COROLLARY 6.3. (cf. [10, Lemma 1.7]). *If $\mathfrak{p} \cap S \neq \emptyset$ implies that* $\mathrm{cl}(\mathfrak{p}) = 0$ *for all prime ideals $\mathfrak{p}$ of A of height i, then the epimorphism*

$$C_i(A) \to C_i(A_S)$$

is an isomorphism.

COROLLARY 6.4. *If $C_i(A_S) = 0$, then $C_i(A)$ is generated by the set $\{\mathrm{cl}(\mathfrak{p})\}$ where* ht $\mathfrak{p} = i$ *and* $\mathfrak{p} \cap S \neq \emptyset$.

PROPOSITION 6.5. *There is an epimorphism*

$$C_i(A) \to \coprod_{\mathrm{ht}\,\mathfrak{p}=i} C_i(A_\mathfrak{p})$$

deduced from $\langle \mathfrak{p} \rangle \to \langle \mathfrak{p}A_\mathfrak{p} \rangle$.

Proof. Clearly $D_i(A)$ is isomorphic to $\coprod_{\mathrm{ht}\,\mathfrak{p}=i} D_i(A_\mathfrak{p})$ under the assignment $\langle \mathfrak{p} \rangle \to \langle \mathfrak{p}A_\mathfrak{p} \rangle$. All that needs to be remarked is that if $x_1, \cdots, x_i$ is an A-sequence of length i, then $x_1, \cdots, x_i$ is an $A_\mathfrak{p}$-sequence of length i.

We now treat the groups W_i.

THEOREM 6.5. *Let S be a multiplicatively closed subset of A. The homomorphism $W_i(A) \to W_i(A_S)$ of Section 5 is an epimorphism. The kernel is generated by the $[A/\mathfrak{P}]$ in $K^0(\mathfrak{M}_{i-1}/\mathfrak{M}_{i+1})$ where $\mathfrak{P}$ ranges over the prime ideals of A of height i with $\mathfrak{P} \cap S \neq \emptyset$.*

Proof. As in Section 5, let $\mathfrak{M}_i = \mathfrak{M}_i(A)$ and $\mathfrak{N}_i = \mathfrak{M}_i(B)$. The functor

$$\mathfrak{M}_i \to \mathfrak{N}_i : M \to M_S$$

is onto the objects, for $\mathfrak{M}_i(A_S)$ is equivalent to $\mathfrak{M}_i(A)/\mathcal{K}_i$, where $\mathcal{K}_i$ denotes the Serre subcategory of $\mathfrak{M}_i(A)$ consisting of those $N \in \mathfrak{M}_i$ with $N_S = 0$.

Thus we get induced functors $\mathfrak{M}_i/\mathfrak{M}_j \to \mathfrak{N}_i/\mathfrak{N}_j$ which are onto the objects. Hence the commutative diagram

$$\begin{array}{ccc} K^0(\mathfrak{M}_i/\mathfrak{M}_{i+1}) & \to & K^0(\mathfrak{M}_{i-1}/\mathfrak{M}_{i+1}) \\ \downarrow & & \downarrow \\ K^0(\mathfrak{N}_i/\mathfrak{N}_{i+1}) & \to & K^0(\mathfrak{N}_{i-1}/\mathfrak{N}_{i+1}) \\ \downarrow & & \downarrow \\ 0 & & 0 \end{array}$$

has exact columns. If $z \in W_i(A_S)$ then there is a $d' \in K^0(\mathfrak{N}_i/\mathfrak{N}_{i+1})$ whose image is z. Let $d \in K^0(\mathfrak{M}_i/\mathfrak{M}_{i+1})$ be a preimage of d' and x the image of d' in $K^0(\mathfrak{M}_{i-1}/\mathfrak{M}_{i+1})$. Then z is the image of x.

To compute the kernel note that if $\mathfrak{P} \cap S \neq \emptyset$, then $A_S \otimes_A A/\mathfrak{P} = 0$ so $[A/\mathfrak{P}]$ is in the kernel. On the other hand, if $x \in W_i(A)$ is in the kernel, then there is a $y \in K^0(\mathcal{K}_i/\mathcal{K}_{i+1})$ whose image is x. But y is the sum of the requisite classes, so also is x.

Corollary 6.6. *Let A and S be as in Theorem 6.5. If $\mathfrak{P} \cap S \neq \emptyset$ implies $[A/\mathfrak{P}] = 0$ in $W_i(A)$ for each prime ideal $\mathfrak{P}$ of A of height i, then the epimorphism $W_i(A) \to W_i(A_S)$ is an isomorphism.*

Corollary 6.7. *If $W_i(A_S) = 0$, then $W_i(A)$ is generated by $[A/\mathfrak{P}]$ as $\mathfrak{P}$ runs through the set of prime ideals of A of height i which meet S.*

Proposition 6.8. *The morphisms $W_i(A) \to W_i(A_{\mathfrak{P}})$, ht $\mathfrak{P} = i$, induce an epimorphism $W_i(A) \to \coprod_{\text{ht } \mathfrak{P}=i} W_i(A_{\mathfrak{P}})$.*

Proof. Let $M \in \mathfrak{M}_i$. Then $M_{\mathfrak{P}} = 0$ for almost all prime ideals $\mathfrak{P}$ of A with ht $\mathfrak{P} = i$. Hence $W_i(A) \to \prod_{\text{ht } \mathfrak{P}=i} W_i(A_{\mathfrak{P}})$ has its image in $\coprod_{\text{ht } \mathfrak{P}=i} W_i(A_{\mathfrak{P}})$. The fact that the homomorphism is onto follows easily.

7. From A to $A[X]$

Since $A[X]$ is a flat A-module we apply the considerations of Section 5 to obtain homomorphisms $C_i(A) \to C_i(A[X])$ which sends cl $(\mathfrak{p})$ to cl $(\mathfrak{p}A[X])$ if $\mathfrak{p}$ is a prime ideal of A of height i.

Our first result shows that, under a mild assumption satisfied for instance by all regular rings, these homomorphisms are onto.

Proposition 7.1. *Assume for each prime ideal $\mathfrak{p}$ of height $i - 1$ of A, that $C_{i-1}(A_{\mathfrak{p}}) = 0$. Then $C_i(A) \to C_i(A[X])$ is an epimorphism.*

Proof. It must be shown that $C_i(A[X])$ is generated by the set $\{$cl $(\mathfrak{p}A[X])\}$ where $\mathfrak{p}$ ranges over the prime ideals of A of height i.

Let $\mathfrak{P}$ be a prime ideal of $A[X]$ with ht $\mathfrak{P} = i$. If ht $(\mathfrak{P} \cap A) = i$, then $\mathfrak{P} = (\mathfrak{P} \cap A)A[X]$, so this case is trivial.

Otherwise ht $(\mathfrak{P} \cap A) = i - 1$; set $\mathfrak{p} = \mathfrak{P} \cap A$. Let $y_1, \cdots, y_{i-1}$ be an

$A_\mathfrak{p}$-sequence and choose an A-sequence $x_1, \cdots, x_{i-1}$ such that

$$\sum x_j A_\mathfrak{p} = \sum y_j A_\mathfrak{p} .$$

Let

$$I = \sum x_j A_\mathfrak{p} = \mathfrak{q} \cap \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_r$$

be an irredundant decomposition of I into primary ideals where $\mathfrak{p}$ is the radical of $\mathfrak{q}$ and $\mathfrak{p}_j$ is the radical of $\mathfrak{q}_j$; $1 \leq j \leq r$. Let

$$\mathfrak{r} = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_r .$$

Let S be the complement in A of the set $\mathfrak{p} \cup \mathfrak{p}_1 \cup \cdots \cup \mathfrak{p}_r$. Choose e_1 and e_2 in A_S such that e_1 and e_2 map onto (0, 1) and (1, 0) respectively in the ring $A_S/IA_S = A_S/\mathfrak{q}A_S \oplus A_S/\mathfrak{r}A_S$. Then $e_i = f_i/s$, $i = 1, 2$, for some $f_i \in A$, $s \in S$.

Note that $\mathfrak{P}A_\mathfrak{p}[X]/\mathfrak{p}A_\mathfrak{p}[X]$ is generated by a monic polynomial h in $A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p}[X]$. Since $A_S/\mathfrak{p}A_S = A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p}$, g' monic may be chosen in $A_S[X]$ such that the image of g' in $A_S/\mathfrak{p}A_S[X]$ is h. Write $g' = g/t$ for some $g \in A[X]$, $t \in S$.

A straightforward check shows that

$$x_1, \cdots, x_{i-1}, \quad f_1 + f_2 g$$

is an $A[X]$-sequence. Let I' be the ideal in $A[X]$ generated by this sequence. If $\mathfrak{W}$ is a prime ideal of $A[X]$, ht $\mathfrak{W} = i$, such that $\mathfrak{W} \supseteq I'$, then $f_1^2 \in \mathfrak{W}$, since $f_1 f_2 \in \mathfrak{q} \cap \mathfrak{r} = I$. But f_1^2 is in no $\mathfrak{p}_i$, $i = 1, \cdots, r$, so we have $\mathfrak{W} \cap A \neq \mathfrak{p}$ implies ht $(\mathfrak{W} \cap A) = i$ (cf. [3, proof of Prop. 7-1]).

One sees that if $I' = \mathfrak{Q} \cap \mathfrak{Q}_1 \cap \cdots \cap \mathfrak{Q}_k$ is the irredundant decomposition of I' into primary ideals in $A[X]$, where $\mathfrak{Q}_\mathfrak{P} = I'_\mathfrak{P}$, then each prime ideal associated with $\mathfrak{Q}_j$ is an extension of a prime ideal of height i of A. Let $\mathfrak{P}_j$ be the radical of $\mathfrak{Q}_j$. Then our $A[X]$-sequence gives the element

$$l(A[X]_\mathfrak{P}/I'_\mathfrak{P}) \langle \mathfrak{P} \rangle + \sum_{j=1}^{k} l(A[X]_{\mathfrak{P}_i}/\mathfrak{Q}_{i\mathfrak{P}_i}) \langle \mathfrak{P}_i \rangle$$

in $R_i(A[X])$. Since $l(A[X]_\mathfrak{P}/I'_\mathfrak{P}) = l(A_\mathfrak{p}/IA_\mathfrak{p})$, the hypothesis on $C_{i-1}(A_\mathfrak{p})$ yields the proposition.

COROLLARY 7.2. *Let* dim $A = n < \infty$. *Suppose* $C_n(A_\mathfrak{p}) = 0$ *for each prime ideal* $\mathfrak{p}$ *of height* n. *Then* $C_{n+1}(A[X]) = 0$.

THEOREM 7.3. (cf. [2, Chap. VII, §3, n° 5, Cor. to Theorem. 2]). $C_\bullet(A) = 0$ *implies* $C_\bullet(A[X]) = 0$.

Remark. Theorem 7.3 does not hold for power series adjunction as Samuel's example in [10] shows.

COROLLARY 7.4. *If F is a field then* $C_\bullet(F[X_1, \cdots, X_n]) = 0$.

It is known from the theory of Krull domains that $C_1(A) \rightarrow C_1(A[X])$ is an isomorphism when A is a Krull domain [2, §1, n° 10, Prop. 18]. Although we have not been able to prove the complete analogue of this result for the groups C_i we do have the following:

THEOREM 7.5. *Let A contain on infinite field K. Then the homomorphism $C_i(A) \to C_i(A[X])$ is a monomorphism.*

Proof. It must be shown that if an element of $D_i(A[X])$ of the form $\sum_{j=1}^{s} n_j \langle \mathfrak{p}_j A[X] \rangle$, where each $\mathfrak{p}_j$ is a prime ideal of A with ht $\mathfrak{p}_j = i$, is in $R_i(A[X])$, then $\sum n_j \langle \mathfrak{p}_j \rangle$ is in $R_i(A)$.

Let $f_{1k}, \cdots, f_{ik}$, $k = 1, \cdots, m$, be the $A[X]$- sequences which, when multiplied by suitable integer coefficients, yield the relation $\sum n_j \langle \mathfrak{p}_j A[X] \rangle$. Using the fact that $X - \lambda$ and $X - \lambda'$ are relatively prime if $\lambda, \lambda' \in K$, $\lambda \neq \lambda'$, we see that for all but a finite number of elements λ of K, both $f_{1k}, \cdots, f_{ik}$, $X - \lambda$ and $X - \lambda, f_{1k}, \cdots, f_{ik}$ are $A[X]$-sequences. For it is no trouble to choose $X - \lambda$ such that the first is an $A[X]$-sequence, since no two $X - \lambda$ can be in the same associated prime ideal of $\sum f_{jk} A[X]$. Suppose that λ has been chosen so that $X - \lambda, f_{1k}, \cdots, f_{lk}$, $l < i$, is an $A[X]$-sequence. If, for any infinite number of λ,

$$A[X] \neq (X - \lambda)A[X] + \textstyle\sum_{j=1}^{l+1} f_{jk} A[X]$$

and $f_{l+1,k}$ is in some associated prime ideal of $(X - \lambda)A[X] + \sum_{j=1}^{l} f_{jk} A[X]$, then $f_{l+1,k}$ is in an infinite number of prime ideals of height $l + 1$ which contain the elements $f_{1k}, \cdots, f_{lk}$. That is, $f_{l+1,k}$ is in the radical of the ideal $\sum_{j=1}^{l} f_{jk} A[X]$ which contradicts the assumption that $f_{1k}, \cdots, f_{ik}$ is an $A[X]$-sequence.

Now let $\mathfrak{P}$ be a prime ideal of $A[X]$ of height i containing $I = \sum_{j=1}^{i} f_{jk} A[X]$ for some k, $1 \leq k \leq m$. If $\mathfrak{P}$ is of the form $\mathfrak{p}A[X]$ where ht $\mathfrak{p} = i$, $\mathfrak{p}$ a prime ideal of A then

$$\mathfrak{Q} = \mathfrak{P} + (X - \lambda)A[X]$$

is a prime ideal of height $i + 1$ containing $J = I + (X - \lambda)A[X]$. Now by the associative law for multiplicities [8], we have

$$\text{(a)} \quad l(A[X]_{\mathfrak{Q}}/J_{\mathfrak{Q}}) = \textstyle\sum_{\text{ht } \mathfrak{R}=i} l(A[X]_{\mathfrak{R}}/I_{\mathfrak{R}}) l(A[X]_{\mathfrak{Q}}/(\mathfrak{R}_{\mathfrak{Q}} + (X - \lambda)A[X]_{\mathfrak{Q}})).$$

At this point we restrict λ yet further so that if ht $\mathfrak{R} = i$ and $\mathfrak{R} \supseteq I$ but is not an extended ideal then $\mathfrak{R} \not\subseteq \mathfrak{P} + (X - \lambda) A[X]$.

Were this not possible, we would get

$$\mathfrak{R} \subseteq \textstyle\bigcap_{\lambda} \mathfrak{P} + (X - \lambda) A[X] = \mathfrak{P},$$

where the intersection extends over any infinite subset of K. Therefore, with the exception of a finite number of λ, $\mathfrak{R} \not\subseteq \mathfrak{P} + (X - \lambda) A[X]$. With this last restriction (a) becomes

$$\text{(b)} \qquad l(A[X]_{\mathfrak{Q}}/J_{\mathfrak{Q}}) = l(A[X]_{\mathfrak{p}A[X]}/I_{\mathfrak{p}A[X]}).$$

Finally we can show that the A-sequences $f_{1k}(\lambda), \cdots, f_{ik}(\lambda)$ multiplied by the same coefficients as $f_{1k}, \cdots, f_{ik}$ gives the original relation. For let

$\mathfrak{p}$ be a prime ideal of A, ht $\mathfrak{p} = i$. If $\mathfrak{p}$ is a $\mathfrak{p}_j$, $1 \leq j \leq s$, then

$$l(A_\mathfrak{p}/ \textstyle\sum_{j=1}^{i} f_{jk}(\lambda)A_\mathfrak{p}) = l(A[X]_\mathfrak{Q}/J_\mathfrak{Q}) = l(A[X]_{\mathfrak{p}A[X]}/I_{\mathfrak{p}A[X]})$$

by (b). ($\mathfrak{Q} = \mathfrak{p}A[X] + (X - \lambda)A[X]$.)

If $\mathfrak{p}$ is no $\mathfrak{p}_j$, then

$$\begin{aligned} l(A_\mathfrak{p}/ \textstyle\sum f_{jk}(\lambda)A_\mathfrak{p}) &= l(A[X]_\mathfrak{Q}/J_\mathfrak{Q}) \\ &= \textstyle\sum_{\mathfrak{q}\subseteq\mathfrak{Q}, \text{ht } \mathfrak{q}=i} l(A[X]_\mathfrak{q}/I_\mathfrak{q})l(A[X]_\mathfrak{Q}/(\mathfrak{q}_\mathfrak{Q} + (X - \lambda)A[X]_\mathfrak{Q})). \end{aligned}$$

When the coefficients are multiplied and we sum, by assumption, the contribution from non-extended prime ideals will cancel, while for extended prime ideals $\mathfrak{q} \subseteq \mathfrak{p}A[X] + (X - \lambda)A[X]$, the contribution is $\sum_{\mathfrak{q}\subseteq\mathfrak{Q}} l(A[X]_\mathfrak{q}/I_\mathfrak{q})$.

Since $I \not\subseteq \mathfrak{q}$ implies a zero contribution, we again get that the sum over the extended $\mathfrak{q} \neq \mathfrak{p}_j A[X]$, $1 \leq j \leq s$ is zero, hence the result.

Corollary 7.6. *If A contains an infinite field, then $C_i(A[X]) = 0$ implies $C_i(A) = 0$.*

Remark. Both in the proof of Theorem 7.5 above and in the proof below of the corresponding fact for the groups W_i it would be sufficient to assume that $A/\mathfrak{m}$ is infinite for every maximal ideal $\mathfrak{m}$ of A.

We now treat the properties of the homomorphism $W_i(A) \to W_i(A[X])$.

Theorem 7.7. *The homomorphism $W_i(A) \to W_i(A[X])$ is an epimorphism for each i.*

Proof. Recall that the homomorphism $W_i(A) \to W_i(A[X])$ is given by $[A/\mathfrak{p}] \to [A[X]/\mathfrak{p}A[X]]$ where ht $\mathfrak{p} = i$. We need to show that the image of this homomorphism is all of $W_i(A[X])$.

Let $\mathfrak{P}$ be a prime ideal of $A[X]$, ht $\mathfrak{P} = i$. If ht $(\mathfrak{P} \cap A) = i$, then $\mathfrak{P} = (\mathfrak{P} \cap A)A[X]$, so $[A[X]/\mathfrak{P}]$ is an image. Therefore we may concern ourselves with those prime ideals $\mathfrak{P}$ of $A[X]$ with ht $(\mathfrak{P} \cap A) = i - 1$.

Let $\mathfrak{p} = \mathfrak{P} \cap A$. The ideal $\mathfrak{P}_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p}[X]$ is principal and non-zero in $A_\mathfrak{p}[X]/\mathfrak{p}A_\mathfrak{p}[X]$. Let $f \in A[X]$ be such that its image in $A_\mathfrak{p}[X]/\mathfrak{p}A_\mathfrak{p}[X]$ generates $\mathfrak{P}_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p}[X]$. Let $\mathfrak{a} = \mathfrak{p}A[X] + fA[X]$.

Let $\mathfrak{a} = \mathfrak{Q} \cap \mathfrak{Q}_1 \cap \cdots \cap \mathfrak{Q}_r \cap \mathfrak{S}_1 \cap \cdots \cap \mathfrak{S}_t$ be an irredundant decomposition of $\mathfrak{a}$ into primary ideals where $\mathfrak{Q}$ (resp. $\mathfrak{Q}_j$, $\mathfrak{S}_j$) has radical $\mathfrak{P}$ (resp. $\mathfrak{P}_j$, $\mathfrak{T}_j$) and such that ht $\mathfrak{P}_j = i$ and ht $\mathfrak{T}_j > i$. Then $\mathfrak{Q} = \mathfrak{P}$ and each $\mathfrak{P}_j = \mathfrak{p}_j A[X]$ where $\mathfrak{p}_j = \mathfrak{P}_j \cap A$. For if ht $\mathfrak{p}_j < i$, then $\mathfrak{p}_j = \mathfrak{p}$. Hence $(\mathfrak{P}_j)_\mathfrak{p} = \mathfrak{P}_\mathfrak{p}$ since both contain f. Hence

$$\mathfrak{P}_j = (\mathfrak{P}_j)_\mathfrak{p} \cap A[X] = \mathfrak{P}_\mathfrak{p} \cap A[X] = \mathfrak{P},$$

a contradiction.

In $K^0(\mathfrak{M}_{i-1}(A[X])/\mathfrak{M}_{i+1}(A[X]))$,

$$[A[X]/\mathfrak{a}] = [A[X]/\mathfrak{p}A[X] + fA[X]] = [A[X]/\mathfrak{p}A[X]] - [A[X]/\mathfrak{p}A[X]] = 0$$

since the sequence

$$0 \to A[X]/\mathfrak{p}A[X] \xrightarrow{f} A[X]/\mathfrak{p}A[X] \to A[X]/\mathfrak{a} \to 0$$

is exact. But also $[A[X]/\mathfrak{a}] = [A[X]/\mathfrak{P}] + \sum_{j=1}^{r} [A[X]/\mathfrak{Q}_j]$, so

$$[A[X]/\mathfrak{P}] = - \sum_{j=1}^{r} [A[X]/\mathfrak{Q}_j].$$

In $K^0(\mathfrak{M}_i(A[X])/\mathfrak{M}_{i+1}(A[X]))$,

$$[A[X]/\mathfrak{Q}_j] = l((A[X]/\mathfrak{Q}_j)_{\mathfrak{P}_j})[A[X]/\mathfrak{p}_j A[X]].$$

By combining these last two equations we get the result.

Remark. If $\dim A = n < \infty$, then $W_{n+1}(A[X]) = 0$.

COROLLARY 7.8. $W_\bullet(A) = 0$ *implies* $W_\bullet(A[X]) = 0$.

COROLLARY 7.9. $W_\bullet(F[X_1, \cdots, X_n]) = 0$ *where* F *is a field.*

We can also prove the analogue of Theorem 7.5 for the group $W_i(A)$.

THEOREM 7.10. *If A contains an infinite field, then the epimorphism $W_i(A) \to W_i(A[X])$ is an isomorphism.*

Proof. For simplicity, let $M[X] = A[X] \otimes_A M$ for an A-module M. Suppose M, N in $\mathfrak{M}_i(A)$ are such that $[M[X]] = [N[X]]$ in $W_i(A[X])$. By Lemma 2.1 of [6], there are objects U, V, W in $\mathfrak{M}_{i-1}(A[X])/\mathfrak{M}_{i+1}(A[X])$ and homomorphisms such that

$$0 \to U \to M[X] \oplus W \to V \to 0$$

and

$$0 \to U \to N[X] \oplus W \to V \to 0$$

are exact. Since A contains an infinite field, K, there is an element $f = X - \lambda$, $\lambda \in K$ which is outside of all the associated prime ideals of U, W and V. Hence, the objects U/fU, W/fW and V/fV are in $\mathfrak{M}_{i-1}(A)/\mathfrak{M}_{i+1}(A)$. Furthermore, by the serpent lemma [2, Chap. I, §1, n° 4, Prop. 2], the sequence

$$0 \to U/fU \to M \oplus (W/fW) \to V/fV \to 0$$

is exact since

$$\operatorname{Ker}(V \xrightarrow{f} V)$$

is zero. Likewise

$$0 \to U/fU \to N \oplus (W/fW) \to V/fV \to 0$$

is exact. Hence $[M] = [N]$ in $W_i(A)$.

COROLLARY 7.11. *If A contains an infinite field, then $W_i(A[X]) = 0$ implies $W_i(A) = 0$.*

8. From A/I to A

Throughout this section A denotes a locally Macaulay ring. Under certain circumstances, if I is an ideal of A of height k there is a homomorphism

$C_i(A/I) \to C_{i+k}(A)$. The following proposition is an instance of this

PROPOSITION 8.1. *Let I be an ideal generated by an A-sequence $x_1, \cdots, x_k$ of length k. Then there is a homomorphism $C_i(A/I) \to C_{i+k}(A)$.*

Proof. First we define a homomorphism $D_i(A/I) \to D_{i+k}(A)$ by the assignment $\langle\mathfrak{P}/I\rangle \to \langle\mathfrak{P}\rangle$ for a prime ideal $\mathfrak{P}$ of A of height $i + k$ containing I. This homomorphism is onto the subgroup of $D_{i+k}(A)$ generated by the prime ideals which contain I. Since (for $\mathfrak{p} = \mathfrak{P}/I$)

$$l_{(A/I)_\mathfrak{p}}((A/I)_\mathfrak{p}/ \textstyle\sum_{j=1}^{i} x_{j+k}(A/I)_\mathfrak{p}) = l_{A_\mathfrak{P}}(A_\mathfrak{P}/ \textstyle\sum_{j=1}^{i+k} x_j A_\mathfrak{P}),$$

it is clear that relations go to relations.

From the descriptions of the homomorphisms in Proposition 8.1 and the results of Section 5 we obtain the following results which are useful for computational purposes.

PROPOSITION 8.2. *Let u be an A-sequence. Then*

$$C_i(A/uA) \to C_{i+1}(A) \to C_{i+1}(A[u^{-1}]) \to 0$$

is exact.

COROLLARY 8.3. (a) *If $C_i(A/uA) = 0$, then*

$$C_{i+1}(A) \cong C_{i+1}(A[u^{-1}]).$$

(b) *If $C_{i+1}(A[u^{-1}]) = 0$, then $C_{i+1}(A)$ is generated by the set*

$$\{\text{cl}(\mathfrak{P}) \mid u \in \mathfrak{P},\ \ \text{ht}\,\mathfrak{P} = i + 1\}.$$

COROLLARY 8.4. $C_i(A) = 0$ *implies* $C_{i+1}(A[X]) \cong C_{i+1}([X, X^{-1}])$.

COROLLARY 8.5. *If A contains an infinite field and $C_i(A) = 0$, then*

$$C_{i+1}(A) \cong C_{i+1}(A[X, X^{-1}]).$$

As for the groups W_i, $i > 0$, we have the following:

PROPOSITION 8.6. *If I is an unmixed ideal of A of height k, then there is a homomorphism $W_i(A/I) \to W_{i+k}(A)$ induced by considering each A/I-module as an A-module. The image is generated by the set*

$$\{[A/\mathfrak{p}] : \text{ht}\,\mathfrak{p} = i + k,\ \ I \subseteq \mathfrak{p}\}.$$

Proof. Let $B = A/I$. The functor gotten from considering each B-module as an A-module induces functors

$$\mathfrak{N}_i = \mathfrak{M}_i(B) \to \mathfrak{M}_{i+k}(A) = \mathfrak{M}_i$$

for each i. These in turn induce group homomorphisms making the following diagram commutative:

$$\begin{array}{ccc} K^0(\mathfrak{N}_i/\mathfrak{N}_{i+1}) & \to & K^0(\mathfrak{N}_{i-1}/\mathfrak{N}_{i+1}) \\ \downarrow & & \downarrow e \\ K^0(\mathfrak{M}_{i+k}/\mathfrak{M}_{i+1+k}) & \to & K^0(\mathfrak{M}_{i-1+k}/\mathfrak{M}_{i+1+k}). \end{array}$$

As in such previous situations, e induces the desired homomorphism.

The following corollaries, direct analogues of the corollaries of Proposition 8.1, are listed here for the convenience of the reader.

COROLLARY 8.7. *Let u be an A-sequence. Then the sequence*

$$W_i(A/uA) \to W_{i+1}(A) \to W_{i+1}(A[u^{-1}]) \to 0$$

is exact.

COROLLARY 8.8. *Let u be an A-sequence.*

(a) *If $W_i(A/uA) = 0$, then $W_{i+1}(A) \cong W_{i+1}(A[u^{-1}])$*

(b) *If $W_{i+1}(A[u^{-1}]) = 0$, then $W_{i+1}(A)$ is generated by the set*

$$\{[A/\mathfrak{p}] : \text{ht}\, \mathfrak{p} = i + 1, \quad u \in p\}.$$

COROLLARY 8.9. (a) $W_i(A) = 0$ *implies*

$$W_{i+1}(A[X]) \cong W_{i+1}(A[X, X^{-1}]).$$

(b) *If A contains an infinite field, then $W_i(A) = 0$ implies*

$$W_{i+1}(A) \cong W_{i+1}(A[X, X^{-1}]).$$

9. Miscellaneous results

THEOREM 9.1. *If F is a field, then $C_\bullet(F[[X_1, \cdots, X_n]]) = 0$.*

Proof. Let $\mathfrak{P}$ be a prime ideal of $R_n = F[[X_1, \cdots, X_n]]$ with ht $\mathfrak{P} = i$. Let $f_1, \cdots, f_w$ be a set which generates $\mathfrak{P}$. We can find an automorphism σ of R_n so that each $f_1^\sigma, \cdots, f_w^\sigma$ is a polynomial in X_n, i.e., $\mathfrak{P}^\sigma$ har a set of generators in $F[[X_1, \cdots, X_{n-1}]][X_n]$.

So assume that $\mathfrak{P}$ is a prime ideal of height i in R_n which has a generating set in $R_{n-1}[X_n]$. Set $\mathfrak{p} = \mathfrak{P} \cap R_{n-1}[X_n]$. Then $\mathfrak{P} = \mathfrak{p}R_n$. Assuming, by induction, that $C_\bullet(R_{n-1}) = 0$, it follows that $C_\bullet(R_{n-1}[X_n]) = 0$. Setting $A = R_{n-1}[X_n]$, $B = R_n$, the situation can be summarized as follows. $\mathfrak{P}$ is a prime ideal of B, $\mathfrak{P} \cap A = \mathfrak{p}$ is such that $\mathfrak{p}B = \mathfrak{P}$ (it is easy to check that ht $\mathfrak{P}$ = ht $\mathfrak{p}$, since B is a flat A-module) and cl $(\mathfrak{p}) = 0$. Then under the homomorphism $C_i(A) \to C_i(B)$ of Theorem 5.1, cl $(\mathfrak{p}) \to$ cl $(\mathfrak{P})$. Therefore cl $(\mathfrak{P}) = 0$, so we are finished.

PROPOSITION 9.2. *If A is a complete discrete rank one valuation ring, then $C_\bullet(A[[X_1, \cdots, X_n]]) = 0$.*

Proof. Let π denote a generator of the maximal ideal of A, and let $\mathfrak{P}$ be a prime ideal of height i in $B = A[[X_1, \cdots, X_n]]$. If $\pi \in \mathfrak{P}$, then $\mathfrak{P}/\pi B$ is a prime ideal of height $i - 1$ in $A/\pi A[[X_1, \cdots, X_n]]$, so cl $(\mathfrak{P}) = 0$ by the previous theorem. Otherwise let $f_1, \cdots, f_k$ generate $\mathfrak{P}$. Since $\pi \notin \mathfrak{P}$, we may assume that no f_j is in πB. Applying the Theorem of Preparation in the form given in [2, Chap. 7, §3, n° 9, Prop. 6], one sees that it is again possible to choose a set of generators for $\mathfrak{P}^{\sigma}$ lying in $A[[X_1, \cdots, X_{n-1}]][X_n]$. The proof now concludes as in Theorem 9.1.

COROLLARY 9.3. *If A is a field or a complete discrete rank one valuation ring, then* $W_\bullet(A[[X_1, \cdots, X_n]]) = 0$.

PROPOSITION 9.4. *Let A be a semi-local ring with maximal ideals* $\mathfrak{m}(1), \cdots, \mathfrak{m}(k)$. *Assume* ht $\mathfrak{m}(j) = n$. *Then* $C_n(A) \cong \oplus_{j=1}^{k} C_n(A_{\mathfrak{m}(j)})$.

Proof. By Proposition 6.5, there is an epimorphism

$$C_n(A) \to \oplus_{j=1}^{k} C_n(A_{\mathfrak{m}(j)}).$$

To see that it is one-to-one, let $y_1, \cdots, y_n$ be an $A_{\mathfrak{m}(j)}$ sequence for some j. Let $x_1, \cdots, x_n$ be an A-sequence such that $\sum_{k=1}^{n} x_k A_{\mathfrak{m}(j)} = \sum_{k=1}^{n} y_k A_{\mathfrak{m}(j)}$. Set $I = \sum x_k A$. Choose $z_n \in A$ such that

$$z_n \equiv x_n \pmod{\mathfrak{m}(j) I_{\mathfrak{m}(j)} \cap A} \quad \text{and} \quad z_n \equiv 1 \pmod{\mathfrak{m}(k)}, \quad k \neq j$$

by the Chinese Remainder Theorem. It is easy to see that $x_1, \cdots, x_{n-1}, z_n$ is an A-sequence which yields the relation $l(A_{\mathfrak{m}(j)}/I_{\mathfrak{m}(j)}) \langle \mathfrak{m}(j) \rangle$ in $D_n(A)$.

PROPOSITION 9.5. *Suppose that $A_{\mathfrak{p}}$ is regular for every prime ideal $\mathfrak{p}$ of A,* ht $\mathfrak{p} = k$. *Then if $\mathfrak{p}(1), \cdots, \mathfrak{p}(r)$ are prime ideals of height k of A and $n_1, \cdots, n_r$ are non-negative integers, then there is an A-sequence $x_1, \cdots, x_k$ such that*

$$l(A_{\mathfrak{p}(i)}/ \textstyle\sum x_j A_{\mathfrak{p}(i)}) = n_i \quad \textit{for} \quad i = 1, 2, \cdots, r.$$

Proof. It is clear that for each i such that $1 \leq i \leq r$, there is an A-sequence $x_{1i}, \cdots, x_{ki}$ such that

$$l(A_{\mathfrak{p}(i)}/ \textstyle\sum_{j=1}^{k} x_{ji} A_{\mathfrak{p}(i)}) = n_i .$$

Set $I_i = \sum_{j=1}^{k} x_{ji} A$. Let S be the complement in A of $\mathfrak{p}(1) \cup \cdots \cup \mathfrak{p}(r)$ and in the semi-local ring A_S choose an A_S-sequence $y_1, \cdots, y_k$ such that

$$y_j \equiv x_{ji} \pmod{(\mathfrak{p}(i) I_i)_{\mathfrak{p}(i)} \cap A_S} \quad \text{for} \quad i = 1, \cdots, r.$$

Then an A-sequence $x_1, \cdots, x_k$ such that $\sum x_j A_S = \sum x_j A_S$ satisfies the requirements of the proposition.

PROPOSITION 9.6. *Let A and B be affine rings over a field k. Suppose that A is regular and $C_\bullet(K \otimes_k B) = 0$ for any field extension K of k. Then there is an epimorphism*

$$C_\bullet(A) \to C_\bullet(A \otimes_k B)$$

induced by $\langle \mathfrak{p} \rangle \to \langle \mathfrak{p} \otimes_k B \rangle$ for a prime ideal $\mathfrak{p}$ of A.

Proof. $\otimes$ means $\otimes_k$ throughout this proof.

Note that the hypothesis $C_\bullet(K \otimes B) = 0$ implies, in particular, that $\mathfrak{p} \otimes B$ is a prime ideal of $A \otimes B$ for each prime ideal $\mathfrak{p}$ of A.

Now let $\mathfrak{P}$ be a prime ideal of height i of $A \otimes B$; we will proceed, by induction on $k = i - \text{ht}(\mathfrak{P} \cap A)$, to show that $\text{cl}(\mathfrak{P})$ is in the subgroup generated by $\{\text{cl}(\mathfrak{p} \otimes B)\}$ where $\text{ht}\,\mathfrak{p} = i$, $\mathfrak{p}$ a prime ideal of A. If $k = 0$, there is nothing to prove, and the induction is on its way.

Assume now that $\text{ht}\,\mathfrak{p} = j < i$, where $\mathfrak{p} = \mathfrak{P} \cap A$. Choose an A-sequence $x_1, \cdots, x_j$ such that $\sum x_l A_\mathfrak{p} = \mathfrak{p}A_\mathfrak{p}$ (this is possible since A is regular). Let $I = \sum x_l A = \mathfrak{p} \cap \mathfrak{r}$ where $\mathfrak{p}_1, \cdots, \mathfrak{p}_v$ are the prime ideals of height j of A containing $\mathfrak{r}$. Let S (resp. T) be the complement of

$$\mathfrak{p} \text{ (resp., } \mathfrak{p} \cup \mathfrak{p}_1 \cup \cdots \cup \mathfrak{p}_v) \quad \text{in} \quad A.$$

Since $A_T/I_T = A_T/\mathfrak{p}_T \oplus A_T/\mathfrak{r}_T$, let e_1 and e_2 denote elements of A_T which map onto $(0, 1)$ and $(1, 0)$ respectively. Let $e_i = f_i/t$ for suitable

$$f_i \in A, \qquad t \in T.$$

Consider the ring $A_S \otimes B$. Since $\mathfrak{P} \cap A = \mathfrak{p}$, $\mathfrak{P}$ extends to a prime ideal $\mathfrak{P}'$ in $A_S \otimes B$, and the image, $\mathfrak{P}''$, of $\mathfrak{P}'$ in

$$(A_S \otimes B)/(I_S \otimes B) = (A_S/I_S) \otimes B = (A/\mathfrak{p})_S \otimes B$$

is such that $\text{cl}(\mathfrak{P}'') = 0$ (since $(A/\mathfrak{p})_S$ is a field). Thus there are $((A/I) \otimes B)_S$-sequences $y_{j+1,m}, \cdots, y_{i,m}$, which (when multiplied by suitable coefficients) display the fact that $\text{cl}(\mathfrak{P}'') = 0$. Choose $A/I \otimes B$-sequences $z'_{j+1,m'}, \cdots, z'_{i,m'}$ which generate the same ideal as the corresponding sequences of y's at S and let $z_{j+1,m}, \cdots, z_{i,m}$ be preimages in $A \otimes B$. By the construction $x_1, \cdots, x_j, z_{j+1,m}, \cdots, z_{i,m}$ is an $A \otimes B$-sequence for each m.

We now show that $x_1, \cdots, x_j, z_{j+1,m}, \cdots, f_1 + f_2 z_{im}$ is an $A_T \otimes B$ sequence. It is only necessary to show that if $\mathfrak{W}$, say, is a prime ideal of height $i - 1$ of $A_T \otimes B$ which contains the first $i - 1$ terms of this sequence, then $f_1 + f_2 z_{im} \notin \mathfrak{W}$. Suppose the contrary. Since $f_1 f_2 \in I \subseteq \mathfrak{W}$, and $z_{im} \notin \mathfrak{W}$, by assumption, we see that both f_1 and $f_2 \in \mathfrak{W}$. But $\mathfrak{W} \supseteq \mathfrak{p}$ or $\mathfrak{W} \supseteq \mathfrak{p}_n$ for some $n = 1, 2, \cdots, v$ and we get $\mathfrak{W} \cap A \supsetneqq \mathfrak{p}$ or $\mathfrak{W} \cap A \supsetneqq \mathfrak{p}_n$ for some n. But then $\mathfrak{W}(A_T \otimes B) = A_T \otimes B$, a contradiction.

Finally, starting at the $(j + 1)^{\text{th}}$ element, choose an $A \otimes B$-sequence $x_1, \cdots, x_j, x_{j+1,m}, \cdots, x_{im}$ which generates the same ideal in $A_T \otimes B$ as does the sequence $x_1, \cdots, x_j, z_{j+1,m}, \cdots, z_{im}$ for each m. By reasoning similar to the above, it is seen that if $\mathfrak{Q}$ is a prime ideal of $A \otimes B$, $\text{ht}\,\mathfrak{Q} = i$, and $\mathfrak{Q}$ contains $\{x_1, \cdots, x_{im}\}$, then $\mathfrak{Q} \cap A = \mathfrak{p}$ or $\text{ht}\,\mathfrak{Q} \cap A > j$.

It is now a straightforward exercise to show that the element of $D_i(A \otimes B)$ obtained from the last sequences with the same coefficients is

$$\langle \mathfrak{P} \rangle + \sum\nolimits_{\text{ht}(\mathfrak{Q} \cap A) > j} m_\mathfrak{Q} \langle \mathfrak{Q} \rangle$$

and the induction hypothesis finishes the proof.

Remark. In the following section this result will enable us to conclude that (for example) $C_\bullet(B_{2l+1} \otimes_\mathbf{C} B_{2k+1}) = 0$ where B_n denotes the affine coordinate ring of the complex n-sphere.

PROPOSITION 9.7. *Let A be a one-dimensional domain such that the integral closure A' of A in the field of quotients of A is a finitely generated A-module. Then $C_1(A)$ is finitely generated over a homomorphic image of $C_1(A')$. In particular, if A' is a principal ideal domain, then $C_1(A)$ is finitely generated.*

Proof. A' is a Dedekind domain. If $\mathfrak{f}$ denotes the conductor of A' over A, then $\mathfrak{f} \neq (0)$ and $A_\mathfrak{p}$ is integrally closed if and only if $\mathfrak{p} \not\supseteq \mathfrak{f}$. Let $\mathfrak{p}_1, \cdots, \mathfrak{p}_k$ be the prime ideals of A containing $\mathfrak{f}$ and choose $0 \neq x$ in $\mathfrak{p}_1 \cdots \mathfrak{p}_k$. Then everything follows from the exact sequence

$$C_0(A/xA) \to C_1(A) \to C_1(A[x^{-1}]) \to 0$$

by noting that $A[x^{-1}] = A'[x^{-1}]$.

Remark. If A has only one prime ideal $\mathfrak{p}$ such that $A_\mathfrak{p}$ is not integrally closed and $C_1(A') = 0$ then $C_1(A) \cong C_1(A_\mathfrak{p})$. For example

$$C_1(\mathbf{Z}[\sqrt{-3}]) = \mathbf{Z}/2\mathbf{Z} \quad \text{and} \quad C_1(\mathbf{R}[x, y]) = \mathbf{Z}/2\mathbf{Z} \quad (x^2 + y^2 = 0).$$

10. Examples

First we give an example of a domain A such that $C_2(A) \neq W_2(A)$. (Note that since A is a domain, $C_1(A) = W_1(A)$.) Let $B = \mathbf{Z}[\sqrt{-3}]$, $B' = \mathbf{Z}[\frac{1}{2}(1 + \sqrt{-3})]$. Set $A = B[X]$. The integral closure of A in $\mathbf{Q}(\sqrt{-3})(X)$ is $B'[X] = A'$. Let $\mathfrak{m}$ be the ideal of A generated by $\{2, 1 + \sqrt{-3}, X\}$ and $\mathfrak{n}$ the ideal of A' generated by $\{\frac{1}{2}(1 + \sqrt{-3}), X\}$. It is clear that $\mathfrak{n}$ is the only maximal ideal of A' lying over $\mathfrak{m}$ and also that $[A'/\mathfrak{n} : A/\mathfrak{m}] = 2$. We know from Theorem 7.7, that $W_2(A) = 0$.

PROPOSITION 10.1. $C_2(A) \neq 0$. *In fact* $C_2(A_\mathfrak{m}) \neq 0$.

Proof. Applying formula 8 of [12, p. 299], we obtain the equation

$$[A'_\mathfrak{n} : A_\mathfrak{m}]e_{A_\mathfrak{m}}(f_1, f_2 \mid A_\mathfrak{m}) = [A'_\mathfrak{n}/\mathfrak{n}A'_\mathfrak{n} : A_\mathfrak{m}/\mathfrak{m}A_\mathfrak{m}]e_{A'_\mathfrak{n}}(f_1, f_2 \mid A'_\mathfrak{n})$$

where f_1, f_2 is an A-sequence. That is $e_{A_\mathfrak{m}}(f_1, f_2 \mid A_\mathfrak{m}) = 2e_{A'_\mathfrak{n}}(f_1, f_2 \mid A'_n)$ which establishes the assertion.

Remark. This establishes, by the way, that $C_1(A) \neq 0$. From the remark in Section 9 above we can conclude that $C_1(A) \cong \mathbf{Z}/2\mathbf{Z}$.

Let now A_n denote the coordinate ring of the real affine n-sphere; i.e.

$$A_n = \mathbf{R}[X_0, X_1, \cdots, X_n]/(X_0^2 + X_1^2 + \cdots + X_n^2 - 1) = \mathbf{R}[x_0, x_1, \cdots, x_n].$$

We proceed to compute $C_\bullet(A_n)$ for $n = 1, 2, 3$.

PROPOSITION 10.2. *For any n, $C_n(A_n) \cong \mathbf{Z}/2\mathbf{Z}$.*

Proof. Let $\mathfrak{m}$ be the maximal ideal generated by $x_0 - 1, x_1, x_2, \cdots, x_n$ of A_n. It will be shown that $\mathrm{cl}(\mathfrak{m}) \neq 0$, while $2\,\mathrm{cl}(\mathfrak{m}) = 0$. The latter follows immediately by noticing that

$$\mathfrak{q} = (x_0 - 1)A_n + x_1 A_n + \cdots + x_{n-1} A_n$$

is primary of length 2 for $\mathfrak{m}$.

Suppose $f_1, \cdots, f_n$ is an A_n-sequence and let $\Gamma_1, \cdots, \Gamma_k$ be the irreducible curves defined by $f_1 = 0, \cdots, f_n = 0$ in real affine $n + 1$ space. Let $\Gamma_1', \cdots, \Gamma_k'$ be the closures of $\Gamma_1, \cdots, \Gamma_k$ in real projective $n + 1$ space. Consider the intersection of Γ_1', say, with the projective closure of the n sphere in *complex* projective $n + 1$ space. There will be an even number of intersections (properly counted). The complex points fall into conjugate pairs; therefore there are an even number of real points of intersection (properly counted) and all of these lie in the finite part of $n + 1$ space since the n-sphere is bounded for real points.

The upshot is that in the relation going with the A_n-sequence $f_1, \cdots, f_n$, the sum of the coefficients on the maximal ideals $\mathfrak{m}'$ such that $A_n/\mathfrak{m}'$ is $\mathbf{R}$ is divisible by 2. This demonstrates that $\mathrm{cl}(\mathfrak{m}) \neq 0$.

We conclude by showing that $\mathrm{cl}(\mathfrak{m})$ generates $C_n(A_n)$. Clearly if $\mathfrak{m}'$ is another maximal ideal such that $A_n/\mathfrak{m}' \cong \mathbf{R}$, then $\mathrm{cl}(\mathfrak{m}) + \mathrm{cl}(\mathfrak{m}') = 0$. If $\mathfrak{n}$ is a maximal ideal such that $A_n/\mathfrak{n} \cong \mathbf{C}$ then let α_i be the residue of x_i modulo $\mathfrak{n}$, and note that the equations of the line joining $(\alpha_0, \cdots, \alpha_n)$ to $(\bar{\alpha}_0, \cdots, \bar{\alpha}_n)$ form an A_n-sequence displaying the relation $\mathrm{cl}(\mathfrak{n}) = 0$.

PROPOSITION 10.3. (i) $C_1(A_1) = \mathbf{Z}/2\mathbf{Z}$. (ii) $C_1(A_2) = 0, C_2(A_2) = \mathbf{Z}/2\mathbf{Z}$. (iii) $C_1(A_3) = 0, C_2(A_3) = 0, C_3(A_3) = \mathbf{Z}/2\mathbf{Z}$.

Proof. Since A_n is a UFD for $n \geq 2$, the only group remaining to be found is $C_2(A_3)$.

Consider

$$A_3[T, T^{-1}] = \mathbf{R}[x_0, x_1, x_2, x_3, T, T^{-1}] = \mathbf{R}[y_0, y_1, y_2, y_3, T, T^{-1}] = B,$$

say, where $y_0^2 + y_1^2 + y_2^2 + y_3^2 - T^2 = 0$. Setting $U = T - y_0$, and $V = T + y_0$, we can write this last relation as

$$y_1^2 + y_2^2 + y_3^2 = UV.$$

We have the exact sequence $C_1(B/UB) \rightarrow C_2(B) \rightarrow C_2(B[U^{-1}]) \rightarrow 0$. Now

$$B/UB \cong \mathbf{R}[y_1, y_2, y_3, V, T^{-1}]$$

where $y_1^2 + y_2^2 + y_3^2 = 0$; but

$$C_1(\mathbf{R}[y_1, y_2, y_3]) = 0$$

[11, p. 36, example 3], so we get $C_1(B/UB) = 0$, hence $C_2(B) \cong C_2(B[U^{-1}])$. But

$$B[U^{-1}] \cong \mathbf{R}[y_1, y_2, y_3, U, U^{-1}, T^{-1}]$$

where y_1, y_2, y_3, U are algebraically independent over $\mathbf{R}$. Therefore $C_2(B[U^{-1}]) = 0$, so $C_2(B) = 0$. Since

$$C_2(B) = C_2(A_3[T, T^{-1}]) \cong C_2(A_3)$$

by Corollary 8.5, we are done.

Considering A_3, we have $0 = C_1(A_3) = C_2(A_3)$, while $C_3(A_3) \cong \mathbf{Z}/2\mathbf{Z}$. This gives at once that $0 = W_0(A_3) = W_1(A_3) = W_2(A_3)$ and $W_3(A_3) \cong \mathbf{Z}/2\mathbf{Z}$ by Proposition 3.2. We now note that Proposition 4.4 is best possible by sketching a proof of the fact that $K^0(\mathfrak{M}_0(A_3)) \cong \mathbf{Z}$. Recall that it suffices to show that if $[A_3/\mathfrak{m}]$ is the class of the A_3-module $A_3/\mathfrak{m}$ in $K^0(\mathfrak{M}_0)$, then $[A_3/\mathfrak{m}] = 0$.

To show this it is sufficient to take a projective resolution of $A_3/\mathfrak{m}$. If one can be found with all the projectives free, then $[A_3/\mathfrak{m}] = 0$ as a rank count will show.

Now the homological dimension of $A_3/\mathfrak{m}$ is 3, and a free resolution of $A/\mathfrak{m}$ is

$$0 \longrightarrow A_3^4 \xrightarrow{p_3} A_3^7 \xrightarrow{p_2} A_3^4 \xrightarrow{p_1} A_3 \xrightarrow{\varepsilon} A_3/m \longrightarrow 0,$$

where the homomorphisms are to be given. ε is the augmentation.

$$p_1(a, b, c, d) = ax_1 + b\,x_2 + c\,x_3 + d(x_0 - 1).$$

$$\begin{aligned} p_2(a, b, c, d, e, f, g) = (&-ax_2 - bx_3 - c(x_0 - 1) + g\,x_1,\\ &ax_1 - dx_3 - e(x_0 - 1) + gx_2,\\ &bx_1 + dx_2 - f(x_0 - 1) + gx_3,\\ &cx_1 + ex_2 + fx_3 + g(x_0 + 1)). \end{aligned}$$

p_3 is the injection of the kernel of p_2 into A_3^7, so we must show that the kernel is free. It is projective and has rank 4, so any 4 elements which generate it will be a basis. A straightforward calculation shows that $\operatorname{Ker} p_2$ is generated by the eight vectors

$$\begin{aligned} v_1 &= (x_1, 0, 0, -x_3, -x_0 - 1, 0, x_2)\\ v_2 &= (0, x_0 - 1, -x_3, 0, 0, x_1, 0)\\ v_3 &= (0, x_1, 0, x_2, 0, -x_0 - 1, x_3)\\ v_4 &= (-x_0 + 1, 0, x_2, 0, -x_1, 0, 0)\\ v_5 &= (x_3, -x_2, 0, x_1, 0, 0, 0)\\ v_6 &= (0, 0, x_1, 0, x_2, x_3, x_0 - 1)\\ v_7 &= (x_2, x_3, x_0 + 1, 0, 0, 0, -x_1)\\ v_8 &= (0, 0, 0, x_0 - 1, -x_3, x_2, 0). \end{aligned}$$

Now let $e_1 = v_1 + v_2$, $e_2 = v_3 + v_4$, $e_3 = v_5 + v_6$, $e_4 = v_7 + v_8$. Then e_1, e_2, e_3, e_4 generate $\operatorname{Ker} p_2$. For

$$\begin{aligned} 2v_2 &= (x_0 - 1)e_1 + x_1 e_2 - x_2 e_3 + x_3 e_4,\\ -2v_3 &= x_1 e_1 - (x_0 + 1)e_2 + x_3 e_3 + x_2 e_4,\\ 2v_6 &= x_2 e_1 + x_3 e_2 + (x_0 - 1)e_3 - x_1 e_4 \end{aligned}$$

and

$$-2v_8 = -x_3 e_1 + x_2 e_2 + x_1 e_3 + (x_0 - 1)e_4 .$$

Hence the other v_j may be obtained as well.

Let B_n denote the affine coordinate ring of the complex n-sphere, i.e.,

$$B_n = \mathbf{C}[X_0, X_1, \cdots, X_n]/(X_0^2 + \cdots + X_n^2 - 1) = \mathbf{C}[x_0, \cdots, x_n].$$

We compute $C_\bullet(B_n)$ for all n (the results of the computations show that $W_\bullet(B_n) = C_\bullet(B_n)$).

Proposition 10.4. *Let F be a field such that $i = \sqrt{-1} \in F$ and the characteristic of F is not 2. Let $D_n = F[x_0, \cdots, x_n]$ where $\sum_0^n x_j^2 = 1$. If n is odd, then $C_\bullet(D_n) = 0$.*

Proof. Let $n = 2k + 1$. We go by induction on k. If $k = 0$, then $D_n = F[x_0, x_1]$ with $x_0^2 + x_1^2 = 1$. Set $u = x_0 + ix_1$, $v = x_0 - ix_1$ to transform D_n into $F[u, v]$ where $uv = 1$. Thus

$$C_1(D_n) = C_1(F[u, u^{-1}]) = 0.$$

Suppose $C_\bullet(D_{2k-1}) = 0$ for $k = l - 1$. D_{2k+1}, by a change of variable, can be transformed into

$$A = F[y_0, y_1, \cdots, y_{2k}, y_{2k+1}]$$

where $y_0 y_1 + \cdots + y_{2k} y_{2k+1} = 1$. Consider the exact sequence

$$C_\bullet(A/y_0 A) \to C_{\bullet+1}(A) \to C_{\bullet+1}(A[y_0^{-1}]) \to 0.$$

We have

$$A/y_0 A \cong F[y_1, y_2, y_3, \cdots, y_{2k}, y_{2k+1}]$$

where $y_2 y_3 + \cdots + y_{2k} y_{2k+1} = 1$, so

$$A/y_0 A \cong D_{2k-1}[y_1],$$

hence $C_\bullet(A/y_0 A) = 0$ by induction. Furthermore

$$A[y_0^{-1}] = F[y_0^{-1}, y_0, y_2, y_3, \cdots, y_{2k}, y_{2k+1}]$$

where $y_0, y_2, \cdots, y_{2k+1}$ are algebraically independent over F, so $C_{\bullet+1}(A[y_0^{-1}]) = 0$ also. $C_0(A) = 0$ since A is a domain. Hence $C_\bullet(A) = 0$.

Proposition 10.5. *Let n be even, say $n = 2k$. Then $C_i(B_{2k}) = 0$ for $i \neq k$, while $C_k(B_{2k}) \cong \mathbf{Z}$.*

Proof. For $k = 0$, $B_0 = \mathbf{C}[x_0]$, $x_0^2 = 1$. Thus $B_0 = \mathbf{C} \oplus \mathbf{C}$, so $C_0(B_0) \cong \mathbf{Z}$.

Now consider B_{2k} for $k > 0$. By the usual change of variable, transform B_{2k} into $A = \mathbf{C}[y_0, y_1, \cdots, y_{2k}]$ where $y_0^2 + y_1 y_2 + \cdots + y_{2k-1} y_{2k} = 1$. Using the exact sequence

$$C_i(A/y_1 A) \to C_{i+1}(A) \to C_{i+1}(A[y_1^{-1}] \to 0$$

we compute, as above, that $C_{i+1}(A[y_1^{-1}]) = 0$, so we have

$$C_i(A/y_1 A) \rightarrow C_{i+1}(A) \rightarrow 0,$$

exact. But $A/y_1 A \cong D_{2k-2}[y_2]$. Hence if $i \neq k - 1$, $C_{i+1}(A) = 0$ since $C_i(D_{2k-2}) = 0$. Also we know that $C_k(A)$ is a cyclic group. This yields at once that $W_i(B_{2k}) = 0$ if $i \neq k$, and $W_k(B_{2k}) = C_k(B_{2k})$ is cyclic.

We will now establish the proposition fully (in light of Corollary 4.2) by showing that the rational rank of $K^0(\mathfrak{M}_0(B_{2k}))$ is at least 2.

Let X denote the complex projective $2k$ sphere, X' the intersection of X with the hyperplane at infinity. Then X' is the complex projective $(2k - 1)$-sphere and $X - X'$ is the affine $2k$-sphere.

With these X', X, $X - X'$, apply the exact sequence

$$K(X') \rightarrow K(X) \rightarrow K(X - X') \rightarrow 0$$

of Grothendieck groups [1, Prop. 7, p. 115].

We know that the homomorphism $A(Y) \rightarrow K(Y)$ has torsion kernel [5, p. 151], where here $A(Y)$ denotes the Chow ring of Y. Supplying the computations of [7, Theorem 1, p. 238] we find that the rational rank of $K(X - X')$ is indeed at least 2. Now $K(X - X') = K^0(\mathfrak{M}_0(D_{2k}))$, so we are done.

Remark. We are indebted to K. Mount for suggestions which led to our computations above. The referee has suggested the following theorem and its proof. Let $K^i(A)$ denote $K^i(\mathfrak{M}_0(A))$ for $i = 1, 2$.

THEOREM. *$K^0(B_k) \cong K^0(B_{k-2})$ for $k \geq 2$. In particular*

$$K^0(B_k) \cong K^0(B_0) \cong \mathbf{Z} \oplus \mathbf{Z} \quad k \text{ even}$$

$$K^0(B_k) \cong K^0(B_1) \cong \mathbf{Z} \quad k \text{ odd}$$

Proof. Let $u = x_{k-1} + i\,x_k$, $\bar{u} = x_{k-1} - i\,x_k$. The following sequence is exact

$$K^1(B_k) \rightarrow K^1(B_k[u^{-1}]) \rightarrow K^0(B_k/uB_k) \rightarrow K^0(B_k) \rightarrow K^0(B_k[u^{-1}]) \rightarrow 0.$$

As before $B_k[u^{-1}] = \mathbf{C}[X_1, \cdots, X_{k-1}, X_{k-1}^{-1}]$, so

$$K^0(B_k[u^{-1}]) \cong \mathbf{Z} \quad \text{and} \quad K^1(B_k[u^{-1}]) \cong \mathbf{C}^* \times \mathbf{Z}$$

(see Theorems 1 and 2 of H. Bass, A. Heller and R. G. Swan, *The Whitehead group of a polynomial extension*, Publ. math. I. H. E. S., n° 22, Paris (1964)). From this it follows that $K^0(B_k) \cong K^0(B_k/uB_k)$. But $B_k/uB_k \cong B_{k-2}[\bar{u}]$, so $K^0(B_k) \cong K^0(B_{k-2})$.

The calculation of $K^0(B_1)$ is implied by Proposition 10.4 and that of $K^0(B_0)$ is in the proof of Proposition 10.5.

The full conclusion of Proposition 10.5 now follows as above.

11. Concluding remarks

Theorem 9.2 would be quite powerful if we had the analogue of Mori's lemma for the groups W_i—we could then conclude that $W_\bullet(A) = 0$ for every unramified regular local ring A. Concerning a regular local ring A, the following questions merit consideration.

Question 11.1 Does $W_\bullet(\hat{A}) = 0$ imply $W_\bullet(A) = 0$ ($\hat{A}$ denotes the completion of A)?

Question 11.2. Is $W_\bullet(A) = 0$?

Question 11.3. Is $(i-1)!\, W_i(A) = 0$ (cf. [5, p. 150])?

A generalization of Question 11.3 which the computation of $W_\bullet(A_3)$ and the results in [5] suggest is

Question 11.4. Suppose A is a regular ring and $K^0(\mathfrak{M}_0(A)) = \mathbf{Z}$. Is $(i-1)!\, W_i(A) = 0$?

In the geometric setting, both C_i and W_i are concerned with chains—but if we restrict A to be, say, the coordinate ring of a non-singular affine variety, then C_i and W_i both derive from the group of cycles.

The group $W_i(A)$, where A is a regular ring, seems to be the analogue of the i^{th} component of the Chow ring (cf. [9, Theorem 10]); in general there is probably no possibility of making $W_\bullet(A)$ into a graded ring. Question 11.4 above is one of many leading to an investigation of how serious the loss of the ring structure is.

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