# SOME EXAMPLES OF SPECTRAL OPERATORS

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Examples of spectral operators will be produced by two processes:

(i) If $T$ is a given bounded linear operator in a Banach space $\mathfrak{M}$ with eigenvectors $\varphi_1, \varphi_2, \cdots$, and $\mathfrak{N}$ is the subspace of vectors $x = \sum c_i \varphi_i$ for which the series converges unconditionally, we can renorm $\mathfrak{N}$ (§2) to make $\mathfrak{N}$ a Banach space. Then $T$ restricted to $\mathfrak{N}$ becomes a spectral operator (§3). A similar procedure (sketched in §4) can be followed for unbounded operators.

(ii) If $T$ has the Fourier series as its eigenfunction expansion, then on a subspace of $L^p$ consisting of functions with lacunary Fourier series, $T$ is a spectral operator (§5).

## 1. Preliminaries

For our terminology on spectral operators and resolutions of the identity we refer the reader to [2], [3], where various properties of these objects will also be found. The notation $\sum x_i$ will mean $\sum_{i=1}^{\infty} x_i$, where the $x_i$ are vectors in a Banach space and convergence means convergence in norm.

By *unconditional convergence* of $\sum x_i$ to $x$, I mean that all rearrangements of the series converge to $x$; in other words, reordered convergence as defined in [1]; i.e.,

(B) $\sum x_{p(i)}$ converges to $x$ for each permutation $p(i)$ of the positive integers.

Following [1] we speak of *subseries convergence*, if

(D) $\sum x_{n(i)}$ converges for each increasing sequence $n(i)$ of positive integers,

and of *bounded-multiplier convergence*, if

(E) $\sum a_i x_i$ converges for each bounded sequence $(a_i)$ of real numbers.

We require

LEMMA 1.1. (B) $\Leftrightarrow$ (D) $\Leftrightarrow$ (E).

Day [1] proves that (D) $\Rightarrow$ (B), and the proof that (B) $\Rightarrow$ (D) is similar. Clearly (E) $\Rightarrow$ (D). The following proof that (D) $\Rightarrow$ (E) was kindly suplied by Professor Day. We first note that it is enough to consider non-negative sequences in (E).

If $\sum x_i$ is not bounded-multiplier Cauchy, then there is a sequence $(t_i)$, $0 \leq t_i \leq 1$ for all $i$, for which $\sum t_i x_i$ is not Cauchy. Hence there exists a convex neighborhood $U$ and disjoint blocks $B_j$ of terms such that $\sum_{i \in B_j} t_i x_i \notin U$. As $\sigma$ runs over the subsets of a set $B_j$, $\sum_{i \in \sigma} x_i$ runs over the

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corners of a parallelopiped which contains $\sum_{i \epsilon B_j} t_i x_i$. Therefore, for each $j$ there is a corner $\sum_{i \epsilon \sigma_j} x_i$ which does not belong to the convex neighborhood $U$. Hence the subseries of $\sum x_i$ which is made up of all terms in all the $\sigma_j$ is not convergent.

*Convention.* In expressions of the form

$$\sup_I (\cdots) \quad \text{or} \quad \sup_J (\cdots),$$

$J$ varies over *all subsets* of the natural numbers and $I$ varies over *all finite subsets.*

LEMMA 1.2. *If* $\sum x_n$ is *unconditionally convergent, then*

$$|||\, x \,||| \equiv \sup_J \| \textstyle\sum_J x_n \| = \sup_I \| \textstyle\sum_I x_n \| < \infty.$$

This follows from Lemma 1.1.

LEMMA 1.3. *If* $\sum y_n$ *is unconditionally convergent and* $(b_n)$ *is a bounded sequence, then* $\sum b_n y_n$ *is unconditionally convergent and*

$$|||\, \textstyle\sum b_n y_n \,||| \leq 2 \sup | b_n | \cdot ||| \textstyle\sum y_n |||.$$

*Proof.* If $x_n = b_n y_n$, then $\sum x_n$ satisfies (E) and so is unconditionally convergent.

From the proof quoted for Lemma 1.1, we see that if $0 \leq b_n \leq 1$ and

$(*)$ $b_n$ has only a finite number of non-zero values

then $\| \sum b_n y_n \| \leq ||| \sum y_n |||$. Moreover, this is true without $(*)$, since $\sum b_n y_n$ converges. It follows that

$$\| \textstyle\sum b_n y_n \| \leq 2 \sup | b_n | \cdot ||| \textstyle\sum y_n |||$$

for each bounded real sequence $(b_n)$, and hence that

$$|||\, \textstyle\sum b_n y_n \,||| \leq 2 \sup | b_n | \cdot ||| \textstyle\sum y_n |||$$

for these sequences.

LEMMA 1.4. *If* $\sum x_n$ *is unconditionally convergent, then for each* $\varepsilon > 0$ *there exists* $K > 0$ *such that*

$$\| \textstyle\sum_I x_n \| < \varepsilon, \quad \textit{if } i > K \textit{ for all } i \textit{ in } I.$$

For if not, choose a sequence $I_1, I_2, \cdots$ of disjoint finite sets of natural numbers such that $\| \sum_{I_r} x_n \| > \varepsilon$ for each $r$; this gives a non-convergent subseries of $\sum x_n$.

We can restate this as

LEMMA 1.5. *If* $\sum x_n$ *is unconditionally convergent to* $x$, *then*

$$|||\, \textstyle\sum_K^\infty x_n \,||| \to 0;$$

*i.e.,* $||| x - \sum_1^K x_n ||| \to 0$, *as* $K \to +\infty$.

## 2. The space $\mathfrak{N}$

Let $\mathfrak{M}$ be a Banach space and let $(\varphi_n)$ be a linearly independent sequence in $\mathfrak{M}$ such that no $\varphi_n$ is in the closed span of $\{\varphi_i : i \neq n\}$. Equivalently, there exists a sequence $(\psi_n)$ in $\mathfrak{M}^*$ such that $\psi_n(\varphi_m) = \delta_{mn}$. Thus, if $x = \sum c_i \varphi_i$, we must have $c_i = \psi_i(x)$.

DEFINITION. *$\mathfrak{N}$ is the set of vectors $x$ such that $\sum \psi_i(x)\varphi_i$ is unconditionally convergent to $x$, with the norm*

$$||| x ||| = \sup_I \| \textstyle\sum_I \psi_i(x)\varphi_i \|.$$

(By Lemma 1.2, the right side is finite.)

THEOREM 2.1. *$\mathfrak{N}$ is a Banach space with the norm $|||\cdot|||$.*

*Proof.* It is sufficient to show that $\mathfrak{N}$ is complete. Let $x_n \in \mathfrak{N}$ $(n \geq 1)$ and $||| x_n - x_m ||| \to 0$. Then $\| x_n - x_m \| \to 0$ so that there exists an $x$ such that $\| x_n - x \| \to 0$. Thus, for each finite set of integers $I$, putting

$$E(I)x = \textstyle\sum_I \psi_i(x)\varphi_i,$$

we have

$$\begin{aligned} \| E(I)x_n - E(I)x \| &\leq \textstyle\sum_I \| \psi_i(x_n - x)\varphi_i \| \\ &\leq \textstyle\sum_I \| \psi_i \| \, \| x_n - x \| \, \| \varphi_i \| \to 0. \end{aligned}$$

Since we have $||| x_n - x_m ||| < \varepsilon$ for $m, n > K(\varepsilon)$, then for each $I$,

$$\| E(I)x_n - E(I)x_m \| < \varepsilon \quad \text{for } m, n > K(\varepsilon).$$

Thus $\| E(I)x_n - E(I)x \| \leq \varepsilon$ for $n > K(\varepsilon)$ and hence

$$||| x_n - x ||| \leq \varepsilon \quad \text{for } n > K(\varepsilon).$$

LEMMA 2.2. $\mathfrak{M}^* \subseteq \mathfrak{N}^*$.

*Proof.* This is obvious since $|||\cdot||| \geq \|\cdot\|$.

## 3. Behaviour of $T$ in $\mathfrak{N}$

With $\mathfrak{M}$, $\varphi_n$ and $\psi_n$ as before, let $T$ be a bounded linear operator in $\mathfrak{M}$ having the $\varphi_n$ as eigenvectors with eigenvalues $\lambda_n$ (which are not necessarily distinct).

THEOREM 3.1. *$T\mathfrak{N} \subseteq \mathfrak{N}$ and $T$ is bounded in $\mathfrak{N}$.*

*Proof.* Since $T$ is bounded, then $|\lambda_i| \leq \| T \|$. Thus, by (1.3), if $x = \sum \psi_i(x)\varphi_i$ is in $\mathfrak{N}$, then $Tx = \sum \lambda_i \psi_i(x)\varphi_i$ will be in $\mathfrak{N}$. Also by (1.3), $T$ is bounded in $\mathfrak{N}$ with bound at most $2 \sup |\lambda_n|$.

To show that $T$ is a spectral operator in $\mathfrak{N}$, write

$$E(\tau)x = \textstyle\sum_{\lambda_i \in \tau} \psi_i(x)\varphi_i$$

for each $x = \sum \psi_i(x)\varphi_i$ in $\mathfrak{N}$ and each subset $\tau$ of the complex plane. Clearly $E(\tau)$ is a projection operator in $\mathfrak{N}$.

THEOREM 3.2. *$E(\tau)$ is a resolution of the identity and is countably additive on the Boolean algebra $\mathcal{B}$ of all subsets of the plane.*

*Proof.* The properties ($\alpha$) of [2, p. 324] are obvious with the exception of

$$(*) \qquad \| E(\sigma) \| \leq K.$$

To prove ($*$) with $K = 2$, note that

$$||| E(\tau)x ||| = \sup_I \| \textstyle\sum_{i \in I, \lambda_i \in \tau} \psi_i(x)\varphi_i \| \leq 2 ||| x |||.$$

By countable additivity I mean that $E(\tau_n)x \to E(\tau)x$ for all $x$ in $\mathfrak{N}$ if $\tau_n \uparrow \tau$. (This implies Dunford's property ($\varepsilon$).) In fact, (1.5) gives

$$||| E(\tau)x - E(\tau_n)x ||| \to 0.$$

THEOREM 3.4. *$T$ is a spectral operator in $\mathfrak{N}$.*

*Proof.* ($\mathcal{B}$ can consist of all subsets of the plane, or of all Borel subsets and $\Gamma$ can be the whole of $\mathfrak{N}^*$.) The fact that $T$ commutes with all $E(\tau)$ is obvious, but the property

$$(3.5) \qquad \sigma(T; E(\tau)\mathfrak{N}) \subseteq \bar{\tau}$$

must be proved. Let $\mu \notin \bar{\tau}$. For $x$ in $E(\tau)\mathfrak{N}$; i.e., $x = \sum_{\lambda_i \in \tau} \psi_i(x)\varphi_i$, define

$$Sx = \sum (\lambda_i - \mu)^{-1} \psi_i(x)\varphi_i .$$

Since $|(\lambda_i - \mu)^{-1}| \leq d^{-1}$ where $d$ is the distance from $\mu$ to $\tau$, then $Sx$ exists. By (1.3),

$$||| Sx ||| \leq 2\, d^{-1} ||| x |||.$$

Thus $S$ is bounded and

$$S(T - \mu I)x = (T - \mu I)Sx = x$$

on $E(\tau)\mathfrak{N}$.

We can show that $T$ is a scalar type operator.

THEOREM 3.6. *$T = \int \lambda E(d\lambda)$ in the uniform operator topology.*

*Proof.* Let $L_1, L_2, \cdots, L_n$ be disjoint sets with diameter less than $\varepsilon$ whose union is the disc of radius $\| T \|$. Choose $\xi_i$ in $L_i$. Then

$$\begin{aligned} ||| Tx - \textstyle\sum \xi_i E(L_i)x ||| &= ||| \textstyle\sum \lambda_j \psi_j(x)\varphi_j - \sum_i \sum_{\lambda_j \in L_i} \xi_i \psi_j(x)\varphi_j ||| \\ &= ||| \textstyle\sum_i \sum_{\lambda_j \in L_i} (\lambda_j - \xi_i)\psi_j(x)\varphi_j ||| \\ &\leq 2 \sup |\lambda_j - \xi_i| \cdot ||| \textstyle\sum_j \psi_j(x)\varphi_j ||| \qquad \text{(by (1.3)} \\ &\leq 2\varepsilon ||| x |||. \end{aligned}$$

## 4. Unbounded operators

Let $\mathfrak{M}$, $\varphi_n$ and $\psi_n$ be as before. Let $T$ be a closed linear operator in $\mathfrak{M}$ for which the $\varphi_n$ are eigenvectors with the eigenvalues $\lambda_n$; let $|\lambda_n| \to \infty$. As before we can define $E(\tau)$ and show that it is a uniformly bounded resolution of the identity. Since $\mathfrak{N}$ is not necessarily invariant under $T$ we define a new operator $S$ in $\mathfrak{N}$ by

$$D(S) = \{x : x \in D(T) \cap \mathfrak{N} \quad \text{and} \quad Tx \in \mathfrak{N}\}$$

$$Sx = Tx \quad \text{for } x \text{ in } D(S).$$

Since $T$ is closed in $\mathfrak{M}$, $S$ is closed in $\mathfrak{N}$. If $\lambda$ is not equal to some $\lambda_i$, then $(S - \lambda I)^{-1}$ is closed. If $x$ is a finite linear combination of the $\varphi_i$, i.e., $x = \sum_1^n \psi_i(x)\varphi_i$, then

$$(S - \lambda I)^{-1} x = \sum_1^n (\lambda_i - \lambda)^{-1}\psi_i(x)\varphi_i$$

and $|||(S - \lambda I)^{-1}x||| \leq 2 \sup |(\lambda_i - \lambda)^{-1}| \cdot |||x|||$. Since $(S - \lambda I)^{-1}$ is closed, bounded and defined on a dense subset of $\mathfrak{N}$ it is bounded on $\mathfrak{N}$ to $\mathfrak{N}$. By §3, $(S - \lambda I)^{-1}$ is a scalar type spectral operator. Thus we can regard $S$ as a spectral operator. (For the definition of [4], $(S - \lambda I)^{-1}$ must be compact, which is true since, by (3.6), $(S - \lambda I)^{-1}$ can be approximated by finite-dimensional operators.)

## 5. Spaces of lacunary series

Fix a sequence $n_1 < n_2 < \cdots$ of positive integers such that $n_{r+1}/n_r > Q > 1$ for some $Q$. Consider the subspace $\mathfrak{C}$ of $L^p$ consisting of functions of the form

$$\sum_1^\infty (a_r \cos n_r x + b_r \sin n_r x).$$

From [5, Theorem V.8.20] it follows that the $L^r$ norm on $\mathfrak{C}$ is equivalent to the $L^2$ norm (for $r > 1$). Since the Fourier series is unconditionally convergent in $L^2$, it is unconditionally convergent in $\mathfrak{C}$. Thus any operator in $L^r$ having the functions $\cos n_r x$ and $\sin n_r x$ as eigenfunctions will be a spectral operator in $\mathfrak{C}$.

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