COHOMOLOGY OF AFFINE ALGEBRAIC HOMOGENEOUS SPACES

BY

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1. Introduction

Let G be a connected algebraic linear group over an algebraically closed field F of characteristic 0, and let L be a fully reducible algebraic subgroup of G. Let R denote the F-algebra of all rational representative functions on G or, which is the same, the F-algebra of all polynomial functions of the algebraic variety structure of G. It is known that G/L has the structure of an affine algebraic variety, with the left L-fixed part R^L of R as the algebra of the polynomial functions. Here, R is viewed as a left L-module, the transform $x \cdot f$ of an element f of R by an element x of L being defined by $(x \cdot f)(y) = f(yx)$ for all elements y of G.

If L is a *maximal* fully reducible subgroup of G then G is a semidirect product $N \cdot L$, where N is the maximum unipotent normal subgroup of G. Hence R^L is then isomorphic with the algebra of all polynomial functions on N, which is an ordinary polynomial algebra $F[t_1, \cdots, t_m]$. Using the cohomological results of [5], we shall show here that, conversely, if R^L is an ordinary polynomial algebra then L is a *maximal* fully reducible subgroup of G.

The general case, where L is not necessarily connected, is reduced to the case where L is connected by an appropriate consideration of unramified extensions of affine algebras. Section 2 contains an exposition of the relevant known algebraic-geometric facts that is especially adapted to our present purpose. The main results are contained in Section 3, while Section 4 gives an illustrative example and an implication of the main result concerning the structure of the algebra of representative functions of a complex analytic linear group.

2. Unramified extensions

Let R and S be commutative rings with identity element, and with $S \subset R$. We say that R is *unramified* over S if, for every (unitary) R-module M, the only S-derivation of R into M is the 0-map. From the point of view of differential algebra, the significance of this notion may be illustrated as follows. Suppose that R is an integral domain, and that the field of fractions $[R]$ of R is separably algebraic over the field of fractions $[S]$ of S. Then every derivation σ of S is the restriction to S of one and only one derivation of $[R]$. The restriction of this derivation to R, followed by the canonical R-module epimorphism $[R] \rightarrow [R]/R$ is evidently an S-derivation of R into the R-module

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$[R]/R$. If R is unramified over S this is the 0-map, which means that our derivation of $[R]$ sends R into R. Thus, in this case, every derivation of S extends uniquely to a derivation of R.

We shall eventually be concerned only with the case where R and S are *affine algebras* over a field F, i.e., integral domains containing F and being finitely generated as F-algebras.

In discussing unramified ring extensions, it is convenient to introduce the *module of differentials*, which is defined as follows. Let R and S be commutative rings with identity element and with $S \subset R$. Let J denote the kernel of the multiplication map $R \otimes_S R \to R$. The module $D_S(R)$ of the S-differentials of R is defined as the R-module J/J^2. One has the canonical S-derivation $d : R \to D_S(R)$, where $d(x)$ is the canonical image in J/J^2 of the element $x \otimes 1 - 1 \otimes x$ of J for every x in R. One sees easily that the image of R in $D_S(R)$ is a system of R-module generators. Now if M is any R-module and if τ is any S-derivation of R into M then there is one and only one R-module homomorphism $\tau^* : D_S(R) \to M$ such that $\tau^* \circ d = \tau$. Hence R is unramified over S if and only if $D_S(R) = (0)$.

Let P be a commutative ring with identity element containing S as a subring. Then the identification of $(P \otimes_S R) \otimes_P (P \otimes_S R)$ with $P \otimes_S (R \otimes_S R)$ gives rise to a canonical $P \otimes_S R$-module homomorphism of $P \otimes_S D_S(R)$ into $D_{P'}(P \otimes_S R)$, where P' denotes the canonical image of P in $P \otimes_S R$. Using the above universal property of the module of differentials, one shows readily that this is actually an isomorphism. In particular, suppose that R and S are algebras over a field F. Let K be an extension field of F, and put $P = K \otimes_F S$. Then $P \otimes_S R = K \otimes_F R$, so that the above becomes an isomorphism of $K \otimes_F D_S(R)$ onto $D_{K \otimes_F S}(K \otimes_F R)$. Hence *$R$ is unramified over S if and only if $K \otimes_F R$ is unramified over $K \otimes_F S$.*

We observe, for use below, that if F is algebraically closed and R is an affne algebra over F then $K \otimes_F R$ is an integral domain and hence an affine algebra over K. A proof of this well-known fact is easily extracted from [3, Ch. V]; see Corollary 2, p. 82 and Lemma 3, p. 84, noting that the field of fractions of R is separable over F.

The following result on polynomial algebras is vital for our present purpose.

Let S be the ordinary polynomial algebra $F[x_1, \cdots, x_n]$ over an algebraically closed field F of characteristic 0. Let R be an affine F-algebra containing S. Suppose that R is unramified over S and finitely generated as an S-module. Then $R = S$.

Our definition of unramified ring extension is in accord with the notions of algebraic geometry to the extent that our assumption that R be unramified over S implies that the affine algebraic variety determined by R is unramified over the affine algebraic variety determined by S, in the sense of Chapter VI of [2] (for example). Hence the above result is actually contained in known far more general results of algebraic geometry. In particular, it is contained in Th. 5, p. 88, of [1]. The following explicit application of the "Lefschetz Principle," which was pointed out to me by Chevalley, reduces the above result to the case where the base field F is the field C of the complex numbers, in which case the result follows almost immediately from the elementary facts of the topology of manifolds.

It is easy to see that we can find a subfield F_0 of F that is an algebraic closure of a finitely generated extension of the field Q of the rational numbers such that $R = F \otimes_{F_0} R_0$, where R_0 is an F_0-algebra that contains $F_0[x_1, \cdots, x_n]$ and is finitely generated as an $F_0[x_1, \cdots, x_n]$-module. Then R_0 is evidently an affine F_0-algebra. Since R is unramified over $S = F \otimes_{F_0} F_0[x_1, \cdots, x_n]$, we know from what we have seen above that R_0 is unramified over $F_0[x_1, \cdots, x_n]$.

Now we may regard F_0 as a subfield of the field C of the complex numbers. Since F_0 is algebraically closed, we know from the above that $C \otimes_{F_0} R_0$ is an affine C-algebra. Evidently, it contains $C[x_1, \cdots, x_n]$ and is finitely generated as a $C[x_1, \cdots, x_n]$-module. Moreover, since R_0 is unramified over $F_0[x_1, \cdots, x_n]$, we know that $C \otimes_{F_0} R_0$ is unramified over $C[x_1, \cdots, x_n] = C \otimes_{F_0} F_0[x_1, \cdots, x_n]$. If our result holds in the case where $F = C$ then we conclude that $C \otimes_{F_0} R_0 = C[x_1, \cdots, x_n]$, whence $R_0 = F_0[x_1, \cdots, x_n]$ and so $R = S$. Thus it suffices to prove the result in the case where $F = C$.

In that case, let V denote the affine algebraic variety whose points are the specializations of R into C and whose algebra of polynomial functions is R. There is an evident algebraic variety morphism μ of V onto the algebraic variety associated with $C[x_1, \cdots, x_n]$, which is simply C^n. The cohomomorphism of μ is the injection $C[x_1, \cdots, x_n] \to R$. Since R is integral over $C[x_1, \cdots, x_n]$, μ is surjective.

Let p be a point of V, and let T_p denote the tangent space to V at p. The elements of T_p may be identified with the C-differentiations $R \to C$, i.e., with the C-derivations $R \to C$, where C is viewed as an R-module, the endomorphism of C corresponding to an element f of R being the multiplication by $f(p)$. The differential of μ at p is the map of T_p into the tangent space to C^n at $\mu(p)$ that sends each element of T_p onto its restriction to $C[x_1, \cdots, x_n]$. An element of T_p that is annihilated by the differential of μ is therefore a $C[x_1, \cdots, x_n]$-derivation of R into C. Hence the fact that R is unramified over $C[x_1, \cdots, x_n]$ implies that the differential of μ at p is injective. Since the dimension of T_p is at least equal to the dimension n of V, we conclude that the differential of μ is bijective at every point p of V.

Viewing V and C^n as complex analytic manifolds and μ as a complex analytic map, we see from the last result and from the fact that the inverse image in V of each point of C^n is non-empty and finite, that μ is a topological covering map $V \to C^n$. Since C^n is simply connected, μ must therefore be a homeomorphism, whence it is clear that we must have $R = C[x_1, \cdots, x_n]$.

3. Affine algebraic homogeneous spaces

Let G be a connected algebraic linear group over an algebraically closed field F of characteristic 0, and let L be a fully reducible algebraic subgroup of G. Let R denote the F-algebra of all polynomial functions on G. Then the left L-fixed part R^L of R is the algebra of all polynomial functions on the affine algebraic variety G/L (see [5, Th. 5.1]). If τ is an element of the R-module T_R of all F-derivations of R and x is an element of G then we define the transform $x\cdot\tau$ as an element of T_R by $(x\cdot\tau)(f) = x\cdot\tau(x^{-1}\cdot f)$. Similarly, we define a right action of G on T_R by $(\tau\cdot x)(f) = \tau(f\cdot x^{-1})\cdot x$, where $(f\cdot x)(y) = f(xy)$. The Lie algebra G° of G may be identified with the right G-fixed part ${}^G T_R$ of T_R. Clearly, G° is stable under the left action $\tau \to x\cdot\tau$, and this is the adjoint action of G on G°. The R-module of all F-derivations $R^L \to R$ is canonically isomorphic with $R \otimes_F (G^\circ/L^\circ)$, and T_{R^L} is canonically isomorphic with the left L-fixed part $(R \otimes_F (G^\circ/L^\circ))^L$ [5, Cor. 2.1].

For every commutative F-algebra P with identity element, we denote by $A(T_P)$ the complex of the P-valued differential forms for P, and we refer the reader to [5, Section 3] for the details of the definition. We recall from there that the right action of G on T_{R^L}, which is defined exactly as the right action of G on T_R was defined above, dualized to a left action of G on $A(T_{R^L})$ with respect to which $A(T_{R^L})$ is a G-module complex over F. By [5, Th. 3.1], *the cohomology space $H(A(T_{R^L})^G)$ of the G-fixed part of this complex is isomorphic, as a graded vector space over F, with the L-fixed part $H(G^\circ, L^\circ)^L$ of the relative Lie algebra cohomology space $H(G^\circ, L^\circ)$ for (G°, L°) in F.* The key facts leading to this result are the relations between derivations and Lie algebra elements that we described above.

It is known that L is contained in a *maximal* fully reducible algebraic subgroup K of G, and that G is a semidirect product $N\cdot K$, where N is the maximum unipotent normal subgroup of G. For details and references concerning this, see [4, Section 3]. Since G is connected, so is K. By [5, Th. 4.1], there is an isomorphism of graded vector spaces

$$H(K^\circ, L^\circ) \otimes_F H(G^\circ, K^\circ) \to H(G^\circ, L^\circ)$$

which is obtained by putting together the canonical homomorphism

$$H(G^\circ, K^\circ) \to H(G^\circ, L^\circ),$$

a linear and degree-preserving pre-inverse of the restriction homomorphism

$$H(G^\circ, L^\circ) \to H(K^\circ, L^\circ)$$

(which is surjective because G° is the semidirect sum $K^\circ + N^\circ$) and the cup product

$$H(G^\circ, L^\circ) \otimes_F H(G^\circ, L^\circ) \to H(G^\circ, L^\circ).$$

Moreover, since L is fully reducible, the L-modules involved here are semisimple, whence one sees easily that the map $H(K^\circ, L^\circ) \to H(G^\circ, L^\circ)$ may be chosen so that it is an L-module homomorphism. Since $H(G^\circ, K^\circ)$ is trivial as a K-module and hence as an L-module, the above isomorphism then induces an isomorphism

$$H(K^\circ, L^\circ)^L \otimes_F H(G^\circ, K^\circ) \to H(G^\circ, L^\circ)^L.$$

Via the isomorphism between differential forms cohomology and relative Lie algebra cohomology given above, and the similar one in which K takes the place of L, this gives us an isomorphism of graded vector spaces

$$H(K^\circ, L^\circ)^L \otimes_F H(A(T_{RK})^G) \to H(A(T_{RL})^G).$$

It was shown in [5, pp. 273–274] that $A(T_{RL})$, and similarly $A(T_{RK})$, is *rationally injective* as a G-module, in the sense of [4]. Because of the semi-direct product decomposition $G = N \cdot K$, the F-algebra R^K is isomorphic with the algebra of all polynomial functions on N. Hence R^K is an ordinary polynomial algebra over F. This implies, by the elementary algebraic version of the Poincaré Lemma, that the cohomology of $A(T_{RK})$ is trivial, so that this complex, augmented by the injection $F \to R^K$, is a rationally injective *resolution* of the trivial G-module F, in the sense of [4].

Now let us suppose that R^L is also an ordinary polynomial algebra. Then the G-module complex $A(T_{RL})$, augmented by the injection $F \to R^L$, is also a rationally injective resolution of the trivial G-module F. This implies, by the elementary facts concerning rational group cohomology (see [4]), that the cohomology spaces $H(A(T_{RK})^G)$ and $H(A(T_{RL})^G)$ are isomorphic as graded vector spaces. In fact, each is isomorphic with the rational cohomology space of G for the trivial G-module F. If we combine this fact with the above tensor product isomorphism, we see that we must then have $H^n(K^\circ, L^\circ)^L = (0)$ for every positive n.

The next lemma will show that this implies that $L = K$, provided that L is connected, or at least *unimodular*, in the sense below.

If V is a vector space over F then we shall denote by $E^m(V)$ the homogeneous component of degree m of the exterior F-algebra built over V. With representations of groups and Lie algebras on V, we associate representations on $E^m(V)$ in the canonical fashion. Let s denote the dimension of the Lie algebra P° of an algebraic linear group P over F. Then $E^s(P^\circ)$ is 1-dimensional, so that the representation of P on $E^s(P^\circ)$ that is obtained from the adjoint representation of P on P° yields a homomorphism γ of P into the multiplicative group F^* of F. We call γ the *adjoint character* of P, and we shall say that P is *unimodular* if γ is trivial.

LEMMA 3.1. *Let Q be a unimodular algebraic linear group over a field F of characteristic 0, and let P be a fully reducible algebraic subgroup of Q. Let γ denote the adjoint character of P. Then γ is trivial on the connected component P_1 of the identity in P, and $H^d_\gamma(Q^\circ, P^\circ) \neq (0)$, where d is the dimension of Q°/P° and $H^d_\gamma(Q^\circ, P^\circ)$ denotes the characteristic P-submodule of $H^d(Q^\circ, P^\circ)$ that belongs to γ.*

Proof. Since P is fully reducible, P_1 is generated by its center and its commutator subgroup. Hence it is clear that γ is trivial on P_1.

Since P is fully reducible, P° is a direct P-module summand of Q°. Let π be a P-module projection of Q° onto a P-module complement for P° in Q°. Let p and q denote the dimensions of P° and Q°, respectively, so that $d = q - p$. We choose a non-zero element u of $E^p(P^\circ)$ and define the d-dimensional cochain f for Q° in $E^q(Q^\circ)$ by

$$f(\sigma_1, \cdots, \sigma_d) = \pi(\sigma_1) \cdots \pi(\sigma_d)u,$$

where the product is taken within the exterior algebra built over Q°, after identifying $E^p(P^\circ)$ with its canonical image in $E^p(Q^\circ)$. Using that π is a P-module homomorphism and that P_1 acts trivially on $E^p(P^\circ)$ as well as on $E^q(Q^\circ)$, one sees directly that f is fixed under the canonical action of P_1 on the space of cochains for Q° in $E^q(Q^\circ)$. Moreover, $f(\sigma_1, \cdots, \sigma_d)$ is evidently 0 whenever one of the σ_i's belongs to P°. Thus f is a *relative* cochain for (Q°, P°) in $E^q(Q^\circ)$. Since the dimension d of f is equal to the dimension of Q°/P°, it follows that f is necessarily a cocycle. If x is any element of P we have

$$\begin{aligned}(x\cdot f)(\sigma_1, \cdots, \sigma_d) &= x\cdot f(x^{-1}\cdot\sigma_1, \cdots, x^{-1}\cdot\sigma_d)\\ &= \pi(\sigma_1) \cdots \pi(\sigma_d)(x\cdot u) = \gamma(x)f(\sigma_1, \cdots, \sigma_d),\end{aligned}$$

which shows that the cohomology class of f is an element of $H^d_\gamma(Q^\circ, P^\circ)$; note that the Q-module $E^q(Q^\circ)$ may be identified with the trivial Q-module F, because Q is unimodular.

Now we claim that if g is any $(d - 1)$-dimensional relative cochain for (Q°, P°) in $E^q(Q^\circ)$ then the coboundary of g is 0. In order to verify this, choose an F-basis $(\varepsilon_1, \cdots, \varepsilon_d)$ of $\pi(Q^\circ)$, and let v be the product $\varepsilon_1 \cdots \varepsilon_d$ in $E^d(Q^\circ)$. Then vu is a non-zero element of $E^q(Q^\circ)$, so that we may write each $g(\sigma_1, \cdots, \sigma_{d-1})$ in the form $g^*(\sigma_1, \cdots, \sigma_{d-1})vu$, with $g^*(\sigma_1, \cdots, \sigma_{d-1})$ in F. Now it suffices to show that $(\delta g)(\varepsilon_1, \cdots, \varepsilon_d) = 0$. A direct computation with the explicit coboundary formula, using that the exterior multiplication by u annihilates every element of P°, shows that

$$(\delta g)(\varepsilon_1, \cdots, \varepsilon_d) = \textstyle\sum_{i=1}^d (-1)^{i-1} g^*(\varepsilon_1, \cdots, \hat{\varepsilon}_i, \cdots, \varepsilon_d)v(\varepsilon_i\cdot u).$$

Here, $\varepsilon_i\cdot u$ is the transform of u by ε_i with respect to the action of Q° on $E^p(Q^\circ)$ that corresponds to the adjoint representation of Q°. Hence it is seen that $\varepsilon_i\cdot u$ is a sum of products each of which has one factor in $\pi(Q^\circ)$, whence $v(\varepsilon_i\cdot u) = 0$.

This proves our above assertion, so that we may now conclude that the cohomology class of f is not 0. Lemma 3.1 is therefore established. Note that the first part implies that *every fully reducible connected algebraic linear group is unimodular.*

Now we are in a position to prove the main result.

THEOREM 3.2. *Let G be a connected algebraic linear group over the algebraically closed field F of characteristic* 0. *Let L be a fully reducible algebraic subgroup of G. Then the algebra R^L of all polynomial functions on the affine algebraic variety G/L is an ordinary polynomial algebra if and only if L is maximal fully reducible in G.*

Proof. Let K be a maximal fully reducible subgroup of G containing L. It is known that if $L = K$ then R^L is a polynomial algebra. Now suppose that R^L is a polynomial algebra. If L is connected it follows then from the above that $L = K$. In fact, since both K and L are then unimodular, Lemma 3.1 says that $H^d(K^\circ, L^\circ)^L \neq (0)$, where d is the dimension of K°/L°. By what we had seen earlier, this forces $d = 0$, so that $L^\circ = K^\circ$ and so $L = K$.

It remains to be proved only that if R^L is an ordinary polynomial algebra then L must be connected. Let L_1 denote the connected component of the identity in L. We claim that R^{L_1} is unramified over R^L. In order to see this, consider the module $D_{R^L}(R^{L_1})$ of the R^L-differentials of R^{L_1}. Let A denote its annihilator in R^{L_1}. The left action of L on R^{L_1} reduces to an action of the finite group L/L_1, and R^L is the L/L_1-fixed part of R^{L_1}. In particular, this shows that R^{L_1} is finitely generated as an R^L-module, so that

$$R^{L_1} = R^L f_1 + \cdots + R^L f_p,$$

where the f_i's are elements of R^{L_1} that are integral over R^L. Let $m_i(x)$ denote the monic minimum polynomial for f_i in the polynomial ring $R^L[x]$. Then, if $m_i'(x)$ denotes the formal derivative of $m_i(x)$ in $R^L[x]$, it is easily seen that the product $m_1'(f_1) \cdots m_p'(f_p)$ is a non-zero element of A. Thus $A \neq (0)$.

Since the intersection of the family of all maximal ideals of R^{L_1} is (0), there must therefore be a maximal ideal P of R^{L_1} such that A is not contained in P. Now the right action of G on R stabilizes R^L and R^{L_1}, and the induced action of G on R^{L_1} corresponds to the transitive action of G on G/L_1. It follows that G stabilizes A and acts transitively on the set of the maximal ideals of R^{L_1}. Hence we conclude that A is not contained in any maximal ideal of R^{L_1}, which means that $A = R^{L_1}$, whence $D_{R^L}(R^{L_1}) = (0)$.

Thus R^{L_1} is unramified over R^L. Hence we conclude from the main result of Section 2 that we must have $R^L = R^{L_1}$, so that $L_1 = L$. This completes the proof of Theorem 3.2.

4. Illustration

If A and B are affine algebras over a field F, with $A \subset B$, and if B is unramified over A and finitely generated as an A-module, then let us say that B is an *unramified module-finite* extension of A. If F is algebraically closed and of characteristic 0, and if A is the algebra R^L of polynomial functions on a homogeneous affine variety G/L as above, then our result shows that *if the cohomology of the A-valued differential forms for A is trivial and if A has no proper unramified module-finite extensions then A is an ordinary polynomial* *algebra over F*. It would be interesting to know whether or not this is a characterization of the ordinary polynomial algebras among *arbitrary* (or at least among the regular) affine algebras over F.

The following example shows that the condition that the differential forms cohomology be trivial does *not* suffice for the conclusion that A is a polynomial algebra. Let G be the group $SL(2, C)$ of all 2 by 2 matrices of determinant 1 over the field C of the complex numbers. Let L be the fully reducible algebraic subgroup of G that is generated by the matrices of the form $\left(\begin{smallmatrix} a & 0 \\ 0 & a^{-1} \end{smallmatrix}\right)$, where a ranges over all non-zero complex numbers, and the matrix $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. Then L_1 is the group of the matrices $\left(\begin{smallmatrix} a & 0 \\ 0 & a^{-1} \end{smallmatrix}\right)$ and L/L_1 is of order 2. Let R be the algebra of all polynomial functions of G. Noting that G is fully reducible, we may apply [5, Th. 3.2], which gives that $H(A(T_{RL}))$ is isomorphic with $H(G^\circ, L^\circ)^L$. Now this relative Lie algebra cohomology space is easily computed. One finds, first, that $H^1(G^\circ, L^\circ) = (0)$, while $H^2(G^\circ, L^\circ)$ is 1-dimensional. Now the action of L on $H(G^\circ, L^\circ)$ goes via the group L/L_1 of order 2 and hence is determined by the action of the single element $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$, which is easily seen to act on $H^2(G^\circ, L^\circ)$ so as to send each cohomology class u onto $-u$. Hence we have $H^n(G^\circ, L^\circ)^L = (0)$ for every $n > 0$. Thus R^L has trivial differential forms cohomology. On the other hand, R^L cannot be a polynomial algebra, because this would imply that G/L is homeomorphic with C^2, while the canonical map $G/L_1 \to G/L$ is a non-trivial covering.

Finally, we point out a bearing of Theorem 3.2 on the structure of the algebra $\mathcal{R}(G)$ of all complex analytic representative functions of a faithfully representable complex analytic group G. It has been shown in [7] that G can be endowed with affine algebraic variety structures that are compatible with the structure of G as a complex analytic manifold and are such that the translation action of G on itself from the right is an action by automorphisms of the algebraic variety. It has been shown there also that the corresponding algebras of polynomial functions may be identified with the left stable *basic subalgebras* of $\mathcal{R}(G)$, i.e., with the left stable subalgebras B such that the elements of exp (Hom (G, C)) are free over B and, together with B, generate all of $\mathcal{R}(G)$.

These structures all arise in the following way. For every such algebraic variety structure, there is an algebraic group hull G^* of G and a fully reducible (abelian and connected) algebraic subgroup L of G^* such that $G^* = LG$ and $G^* \cap L = (1)$, and the algebraic variety structure of G is obtained by transporting the canonical algebraic variety structure of $L\backslash G^*$. The corresponding basic subalgebra B is then canonically isomorphic with the algebra of all polynomial functions on $L\backslash G^*$.

Changing sides appropriately in Theorem 3.2, we conclude that B can be an ordinary polynomial algebra over C only if L is maximal fully reducible in G^*. But a maximal fully reducible subgroup of G^* must contain a maximal reductive subgroup of G. Hence, if B is a polynomial algebra then G has no non-trivial reductive subgroups. By the structure theory of complex analytic linear groups, this implies that G is simply connected and solvable.

Conversely, if G is simply connected and solvable then the basic subalgebras of $\mathcal{R}(G)$ are ordinary polynomial algebras over C. For this, we recall that all basic subalgebras of $\mathcal{R}(G)$ are isomorphic as C-algebras, and that the standard construction of basic subalgebras yields a polynomial algebra in the case where G is simply connected and solvable (see [6, Section 3]). Thus we have the result that *if G is a faithfully representable complex analytic group then the basic subalgebras of $\mathcal{R}(G)$ are polynomial algebras over C if and only if G is simply connected and solvable.*

REFERENCES

1. S. ABHYANKAR, *Tame coverings and fundamental groups of algebraic varieties*, Amer. J. Math., vol. 81 (1959), pp. 46–94.
2. C. CHEVALLEY, *Fondements de la géométrie algébrique*, Faculté des Sciences de Paris, 1957/1958, mimeographed notes.
3. ———, *Algebraic functions of one variable*, Amer. Math. Soc. Math. Surveys, no. VI, 1951.
4. G. HOCHSCHILD, *Cohomology of algebraic linear groups*, Illinois J. Math., vol. 5 (1961), pp. 492–519.
5. G. HOCHSCHILD AND B. KOSTANT, *Differential forms and Lie algebra cohomology for algebraic linear groups*, Illinois J. Math., vol. 6 (1962), pp. 264–281.
6. G. HOCHSCHILD AND G. D. MOSTOW, *On the algebra of representative functions of an analytic group*, Amer. J. Math., vol. 83 (1961), pp. 111–136.
7. ———, *Affine embeddings of complex analytic homogeneous spaces*, Amer. J. Math., vol. 87 (1965), pp. 807–839.

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