ON QUASI PROJECTIVES

BY

L. E. T. Wu and J. P. Jans[1]

1. Introduction

An R module M is said to be quasi injective if every homomorphism

$$T \xrightarrow{f} M,$$

where T is a submodule of M, can be extended to an endomorphism of M. See [5], [6], [10] for properties and applications of quasi injective modules. Phrased in terms of diagrams, a module M is quasi injective if every diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & T & \xrightarrow{j} & M \\ & & {\scriptstyle f}\big\downarrow & & \\ & & M & & \end{array}$$

can be embedded in a commutative diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & T & \xrightarrow{j} & M \\ & & {\scriptstyle f}\big\downarrow & \swarrow {\scriptstyle f'} & \\ & & M & & \end{array}$$

where j is the natural injection of T into M.

In this paper we shall be concerned with a concept dual to quasi injectives. A module M is said to be quasi projective if every diagram

$$\begin{array}{ccccc} & & M & & \\ & & \big\downarrow{\scriptstyle f} & & \\ M & \underset{n}{\longrightarrow} & M/T & \longrightarrow & 0 \end{array}$$

can be embedded in a commutative diagram

$$\begin{array}{ccccc} & & M & & \\ & {\scriptstyle f'}\swarrow & \big\downarrow{\scriptstyle f} & & \\ M & \underset{n}{\longrightarrow} & M/T & \longrightarrow & 0 \end{array}$$

where n is the natural map of M on M/T.

From the duality of the definitions of quasi projective and quasi injective, it is easy to deduce a number of properties of quasi projectives from the dual

Received June 27, 1966.

[1] Supported by a National Science Foundation grant.

properties of quasi injectives. In Section 2, we list some of these properties supplying proof only where it isn't obvious how to dualize from the quasi injective case. In Section 3 we obtain a structure theorem for indecomposable finitely generated quasi projectives over semi-perfect rings. This structure theorem connects quasi projectives with the two sided ideal lattice of the ring. We can then obtain theorems relating quasi projectives and quasi injectives with the indecomposable module problem (which rings have many indecomposable modules? See [3], [8], [12]).

Throughout the paper we shall use the following notation: All modules are R-modules for a ring R. $Q(M)$ will denote the injective hull of M; that is,

$$0 \to M \xrightarrow{j} Q(M)$$

is exact, $Q(M)$ is injective and if $X \subseteq Q(M)$ such that $X \cap j(M) = 0$ then $X = 0$. We shall use $P(M)$ for the projective cover of M (if it exists);

$$P(M) \xrightarrow{\pi} M \to 0$$

is exact, $P(M)$ is projective and if $X \subseteq P(M)$ such that $X + \operatorname{Ker} \pi = P(M)$ then $X = P(M)$. Injective hulls always exist. See [4] for the injective hull and [1] for the projective cover. We shall denote by $E(M)$ the R-endomorphism ring of the R-module M. M then becomes an R-$E(M)$ bimodule.

2. Quasi injectives and quasi projectives

The following is a theorem proved by Johnson and Wong [10], and it is a fundamental tool in studying quasi injective modules.

THEOREM. *M is quasi injective if and only if M is an R-$E(Q(M))$ submodule of $Q(M)$.*

Since not every module has a projective cover, this theorem does not dualize completely. However, we do get the following propositions on quasi projectives by dualizing each half of the above theorem.

PROPOSITION 2.1. *If $0 \to T \to P \to M \to 0$ is exact with P projective and T is an R-$E(P)$ submodule of P then M is quasi projective.*

PROPOSITION 2.2 *If M is quasi projective and has a projective cover*

$$0 \to \operatorname{Ker} \pi \to P(M) \xrightarrow{\pi} M \to 0$$

then Ker π *is an R-$E(P(M))$ submodule of $P(M)$.*

Both of these propositions can be proved by dualizing the proofs of the two halves of the Johnson and Wong theorem. For an example of what we mean by dualizing a proof, we include below a proof of Proposition 2.2. Compare it to the proof of Theorem 1.1 of [10].

Let

$$0 \to \operatorname{Ker} \pi \to P(M) \xrightarrow{\pi} M \to 0$$

be the projective cover of M with M quasi projective and let $\gamma \in E(P(M))$. We shall show that $\gamma(\text{Ker } \pi) \subseteq \text{Ker } \pi$. Let $T = \text{Ker } \pi + \gamma(\text{Ker } \pi)$ and we see that since $\gamma(\text{Ker } \pi) \subseteq T$, γ induces $\bar{\gamma} : P(M)/\text{Ker } \pi \to P(M)/T$. This lifts to a map $\gamma' : M \to M/\pi(T)$ so we have the diagram

$$\begin{array}{ccc} & & M \\ & & \downarrow{\scriptstyle \gamma'} \\ M & \xrightarrow[n]{} & M/\pi(T) \to 0 \end{array}$$

which by the quasi projectivity of M can be embedded in the commutative diagram

$$\begin{array}{ccc} & & M \\ & {\scriptstyle \beta}\swarrow & \downarrow{\scriptstyle \gamma'} \\ M & \xrightarrow{n} & M/\pi(T) \to 0. \end{array}$$

Now using the projectivity of $P(M)$ we have the commutative diagram with $\alpha \in E(P(M))$

$$\begin{array}{ccc} P(M) & \xrightarrow{\pi} & M \to 0 \\ {\scriptstyle \alpha}\downarrow & & \downarrow{\scriptstyle \beta} \\ P(M) & \xrightarrow{\pi} & M \to 0. \end{array}$$

Now let $X = \{p \mid p \in P(M),\ \gamma(p) - \alpha(p) \in \text{Ker } \pi\}$ and we shall show $X = P(M)$. We do this by first showing $X + \text{Ker } \pi = P(M)$; then the fact that $P(M)$ is the projective cover of M implies $X = P(M)$.

Note that both α and γ induce maps from M to $M/\pi(T)$ where α induces α'

$$\begin{array}{ccccc} P(M) & \xrightarrow{\pi} & M & & \\ {\scriptstyle \alpha}\downarrow & & {\scriptstyle \beta}\downarrow & \searrow{\scriptstyle \alpha' = n\beta} & \\ P(M) & \xrightarrow[\pi]{} & M & \xrightarrow[n]{} & M/\pi T \end{array}$$

and γ induces

$$\begin{array}{ccccc} P(M) & \xrightarrow{\pi} & M & & \\ {\scriptstyle \gamma}\downarrow & & & \searrow{\scriptstyle \gamma'} & \\ P(M) & \xrightarrow[\pi]{} & M & \xrightarrow[n]{} & M/\pi(T). \end{array}$$

By chasing the diagram and using $n\beta = \gamma'$ we see that $\gamma' - \alpha' = 0$. It follows, therefore, that $(\gamma - \alpha)(P(M)) \subseteq T$. For each $p \in P(M)$, $\gamma(p) - \alpha(p) = k_1 + \gamma(k_2)$ with $k_i \in \text{Ker } \pi$ by the definition of T. It follows that $p - k_2 \in X$ and we have shown that $X + \text{Ker } \pi = P(M)$.

The fact that $P(M)$ is the projective cover of M implies that $X = P(M)$. Therefore $\gamma(\text{Ker } \pi) \subseteq \text{Ker } \pi$ since $\alpha(\text{Ker } \pi) \subseteq \text{Ker } \pi$. This completes the proof of Proposition 2.2.

As shown in [5], the class of quasi injectives is not closed under even finite direct sums. It is not hard to give examples showing that the class of quasi projectives also is not closed under finite direct sum. However, quasi projectives (and quasi injectives) are closed under taking direct summands and other operations.

PROPOSITION 2.3. *If M is quasi projective and $M = S + T$ (R direct) then S and T are also quasi projective.*

Proof. We note that the same proof (with arrows reversed) works for quasi injectives.

Given the diagram

$$\begin{array}{ccc} & & S \\ & & \downarrow f \\ S & \xrightarrow{n} & S/X \to 0 \end{array}$$

it can be embedded in

$$\begin{array}{ccc} & & S + T \\ & & \downarrow f + i_T \\ S + T & \xrightarrow[n + i_T]{} & S/X + T \to 0. \end{array}$$

Then using the quasi projectivity of $S + T$, the above diagram can be embedded in a commutative diagram

$$\begin{array}{ccc} & & S + T \\ & \swarrow g & \downarrow f + i_T \\ S + T & \xrightarrow[n + i_T]{} & S/X + T \to 0. \end{array}$$

It is clear that $g \mid S = \bar{f}$ will be a lifting of f; this completes the proof.

The following theorem relates indecomposability of M, where M is quasi projective, with indecomposability of $P(M)$, its projective cover.

PROPOSITION 2.4. *If M is quasi projective and has a projective cover*

$$P(M) \xrightarrow{\pi} M \to 0$$

and if $P(M) = P_1 \oplus P_2$ (R-direct) then $M = M_1 \oplus M_2$ (R-direct) and

$$P_i \xrightarrow{\pi_i} M_i \to 0$$

is the projective cover of M_i where $\pi_i = \pi \mid P_i$.

Proof. The corresponding theorem for quasi injectives is proved dually.

The R-decomposition, $P(M) = P_1 \oplus P_2$, is achieved via two orthogonal idempotents E_1, $E_2 \in E(P(M))$, $P_i = P(M)E_i$. But since Ker π is an $E(P(M))$ submodule of $P(M)$, we have Ker π = (Ker $\pi)E_1 \oplus$ (Ker $\pi)E_2$,

R direct. Thus we can induce a decomposition in M by letting $M_i = P(M)E_i/(\text{Ker } \pi)E_i$ and it follows that

$$0 \to (\text{Ker } \pi)E_i \to P(M)E_i \xrightarrow{\pi_i} M_i \to 0$$

is exact and $M = M_1 \oplus M_2$, R direct.

If $X \subseteq P(M)E_1$ such that $X + \text{Ker } \pi_1 = P(M)E_1$ then

$$X + P(M)E_2 + \text{Ker } \pi = P(M)$$

so $X + P(M)E_2 = P(M)$. Therefore $X = P(M)E_1$ and it follows that

$$P(M)E_1 \xrightarrow{\pi_1} M_1 \to 0$$

is the projective cover of M_1. This completes the proof of Proposition 2.4.

We remark that if the decomposition $P(M) = P_1 \oplus P_2$ is non-trivial then so is $M = M_1 \oplus M_2$. For if $M_1 = 0$ then $P_1 \subseteq \text{Ker } \pi$ and $P_2 + \text{Ker } \pi = P(M)$ implies $P_2 = P(M)$ so $P_1 = 0$.

PROPOSITION 2.5. *If M has a projective cover and if M is quasi projective then $\oplus \sum^n M = M \oplus \cdots \oplus M$ (n copies) is also quasi projective.*

Proof. Again the dual proof works for quasi injectives. In that case the injective hull replaces the projective cover and one need not assume its existence.

Let

$$0 \to \text{Ker } \pi \to P(M) \xrightarrow{\pi} M \to 0$$

be the projective cover of M. The projective cover of $\oplus \sum^n M$ is $\oplus \sum^n P(M)$, the direct sum of n copies of $P(M)$ with appropriate projections [1]. By Proposition 2.1, it is sufficient to show the kernel of

$$\oplus \sum^n P(M) \xrightarrow{+\Sigma\pi} \oplus \sum^n M \to 0$$

is an $E(\oplus \sum^n P(M))$ submodule of $\oplus \sum^n P$. The endomorphism ring of $\oplus \sum^n P(M)$ can be viewed as the total $n \times n$ matrix ring over $E(P(M))$. Also the kernel of the above map is $\oplus \sum^n \text{Ker } \pi$. Since it is clear that any map $f_{ij} : P_i(M) \to P_j(M)$ (from the i^{th} copy of $P(M)$ to the j^{th}) must carry $\text{Ker } \pi_i$ into $\text{Ker } \pi_j$ (because $\text{Ker } \pi$ is an $E(P(M))$ module), it follows that $\oplus \sum^n \text{Ker } \pi$ is an $E(\oplus \sum^n P(M))$ submodule of $\oplus \sum^n P(M)$. This concludes the proof of the proposition.

We define a quasi projective cover in a manner analogous to the projective cover.

DEFINITION.

$$QP(M) \xrightarrow{\bar{\pi}} M \to 0$$

will be called the quasi projective cover of M provided

(1) $QP(M)$ is quasi projective;

(2) if $X + \text{Ker } \bar{\pi} = QP(M)$ then $X = QP(M)$;

(3) if $0 \neq T \subseteq \text{Ker } \bar{\pi}$ then $QP(M)/T$ is not quasi projective.

Note how conditions (1) and (2) are almost the conditions for a projective cover. We have the following existence theorem for quasi projective covers.

PROPOSITION 2.6. *If M has a projective cover*

$$P(M) \xrightarrow{\pi} M \to 0$$

then it has a quasi projective cover

$$QP(M) \xrightarrow{\bar{\pi}} M \to 0$$

which is unique up to isomorphism over the identity on M.

Proof. In the light of Propositions 2.1 and 2.2 it is clear how to construct a quasi projective cover out of a projective cover. Let X be the (unique) maximal R-$E(P(M))$ submodule contained in Ker π. Existence is assured by Zorn's lemma and uniqueness follows from the fact that the sum of two R-$E(P(M))$ submodules contained in Ker π is again contained in Ker π.

Now let $QP(M) = P(M)/X$ map onto M by the induced map $\bar{\pi}$. By Proposition 2.1, $QP(M)$ is quasi projective. If $\bar{Y} + \text{Ker } \bar{\pi} = QP(M)$ then $Y + \text{Ker } \pi = P(M)$ where Y is the pre-image in $P(M)$ of $\bar{Y}$. It follows that $Y = P(M)$ and $\bar{Y} = QP(M)$. Condition (3) for a quasi projective cover is satisfied by the maximality of X.

Now we test uniqueness. Suppose now that M has another quasi projective cover

$$Z \xrightarrow{\delta} M \to 0,$$

satisfying conditions (1), (2), and (3). Since $P(M)$ is projective we have the commutative diagram

$$\begin{array}{ccc} Z & \xrightarrow{\delta} & M \to 0 \\ {\scriptstyle\mu}\uparrow & & \uparrow{\scriptstyle i_M} \\ P(M) & \xrightarrow[\pi]{} & M \to 0. \end{array}$$

Since Im μ + Ker $\delta = Z$, it follows from condition (2) that μ is an epimorphism and Ker $\mu \subseteq$ Ker π. Therefore the map

$$P(M) \xrightarrow{\mu} Z \to 0$$

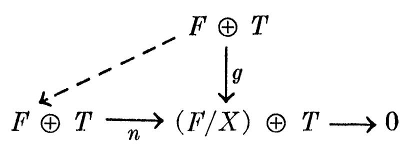
is the projective cover of Z and by Proposition 2.2, Ker μ is an $E(P(M))$ submodule of $P(M)$. By the choice of X above we see $X \supseteq$ Ker μ. If X properly contained Ker μ we would have the condition $0 \neq \mu(X) \subseteq$ Ker δ with $Z/\mu(X) = P(M)/X = QP(M)$ which is quasi projective. But this would contradict condition (3) for $Z \to M \to 0$ to be a quasi projective cover. Therefore $X =$ Ker μ and μ induces an isomorphism $\bar{\mu}$ so that the following diagram is commutative.

$$\begin{array}{ccc} Z & \to & M \to 0 \\ {\scriptstyle\bar{\mu}}\uparrow & & \uparrow{\scriptstyle i_M} \\ QP(M) & \to & M \to 0. \end{array}$$

This completes the proof of Proposition 2.6.

Remark 1. We do not know of a more general theorem on the existence of quasi projective covers. However, they exist more generally than projective covers as the following examples show. Modules over the ring Z of integers (that is abelian groups) only have projective covers when they are free [1]. However, finite abelian groups always have quasi projective covers. To show this it is sufficient to consider only the case of p-primary abelian groups p a prime since there are only trivial homomorphisms between primary groups with different primes. If $P = \oplus \sum_{i=1}^{n} C_i$ is a direct sum of cyclic groups of order p^{r_i} with $t = \max r_i$, then P becomes a $Z/(p^t)$ module. In the category of $Z/(p^t)$ modules it has a projective cover and a quasi projective cover by Proposition 2.6. It is not hard to see that its quasi projective cover in the category of $Z/(p^t)$ modules is also its quasi projective cover in the category of Z modules. In fact, in this case its projective cover in the $Z/(p^t)$-category is its quasi projective cover in the Z-category. It follows, since projective $Z/(p^t)$ modules are free, that finite quasi projective p-primary abelian groups are direct sums $\oplus \sum_{i=1}^{n} C_i$ where each C_i has order p^t.

However, in the following we show that a finitely generated abelian group A which is not finite does not have a quasi projective cover unless A is free. First note that if $A = F \oplus T$ with F free ($\neq 0$) and T finite ($\neq 0$), then F contains a subgroup X such that there is a non-trivial homomorphism $f : T \to F/X$ which gives a homomorphism $g : F \oplus T \to (F/X) \oplus T$ where $g(x, t) = (f(t), 0)$. Now it is clear that in the following diagram the dotted line cannot be filled with an endomorphism of A



because elements of T must go into T under every homomorphism. The above argument shows that no group of the form $F \oplus T$ (F, $T \neq 0$) can be quasi projective. It follows that the only possible quasi projective covers for groups of the form $F \oplus T$ are the free groups. But then the same argument that shows such groups don't have projective covers also shows they don't have quasi projective covers [1].

Remark 2. We should note that we do not know if some form of Propositions 2.2, 2.4, 2.5, or 2.6 can be proved without the assumption of the existence of a projective cover, at least in some weak form as with the example above, of finite abelian groups.

3. Quasi projectives over perfect rings

Since most of our results in Section 2 depended upon the existence of a projective cover, in this section we shall begin by restricting our attention to rings all of whose finitely generated left modules have projective covers. Such rings were defined as semi-perfect rings by Bass and have been studied by Bass in [1]. In addition to being able to use our results of Section 2, we can also use the following characterization of left semi-perfect rings which appeared in [1]:

THEOREM (Bass). *For a ring R with Jacobson radical N the following are equivalent:*
1. *R is left semi-perfect.*
2. *R/N is semi-simple Artinian and idempotents can be lifted modulo N.*

By property 2 of the above theorem, many of the standard arguments for rings with minimum condition carry over to left semi-perfect rings. In the light of Proposition 2.3, in studying finitely generated quasi projectives over semi-perfect rings, it is enough to consider indecomposable quasi projectives. The following Theorem 3.1 gives a characterization of these.

We remark that finitely generated modules over semi-perfect rings are the direct sum of a finite number of indecomposable modules (same proof as with minimum condition) and that the Krull Schmidt theorem holds for finitely generated projectives over semi-perfect rings. We do not know if the uniqueness part of the Krull Schmidt theorem holds for non-projectives over semi-perfect rings, even in the finitely generated case.

THEOREM 3.1. *If M is a finitely generated indecomposable quasi projective over a left semi-perfect ring R, then*

$$M = Re/J \cap Re$$

where e is an indecomposable idempotent and J a two sided ideal of R.

Proof. By Proposition 2.4, if M is indecomposable, so is its projective cover $P(M)$. Conversely, the projective cover of a direct sum is the direct sum of the projective covers. Thus M is indecomposable if and only if its projective cover is indecomposable.

It is known [1] that P is an indecomposable finitely generated projective over a left semi-perfect ring R if and only if $P = Re$ where e is an indecomposable idempotent. Using Propositions 2.1 and 2.2, we know that Re/L is an indecomposable quasi projective if and only if L is an R-$E(Re)$ submodule. So to complete the proof of the theorem it is sufficient to show that the R-$E(Re)$ submodules of Re are of the form $Re \cap J$ where J is a two sided ideal, and conversely, $Re \cap J$ is an R-$E(Re)$ submodule.

We note first that $E(Re)$ can be realized as right multiplications by elements of the subring eRe. Certainly any right multiplication by an element of eRe gives an element of $E(Re)$. Conversely, if $f \in E(Re)$, then $ef(e) = f(e^2) = er_0 e \in eRe$ and it follows that $f(xe) = xf(e) = xe(er_0 e)$. Thus, f is given by a right multiplication by $er_0 e$.

Now suppose $L = Le$ is an R-eRe submodule of Re. Let $J = Le + LeR(1 - e)$, where the sum is direct as left R modules by the orthogonality of e and $1 - e$. The following strings of containments show that J is a two sided ideal:

$$LeR = LeRe + LeR(1 - e) \subseteq J$$

and

$$LeR(1 - e)R = LeR(1 - e)Re + LeR(1 - e)\, R(1 - e)$$
$$\subseteq LeRe + LeR(1 - e) \subseteq J.$$

Note that $J \cap Re = L$.

Conversely, if J is two sided then $J = Je + J(1 - e)$ is a decomposition of J into left ideals, where $Je = J \cap Re$. Now form $JeeRe \subseteq JRe = Je$. Thus $J \cap Re$ is an R-eRe submodule of Re.

This completes the proof of Theorem 3.1.

In the following theorem we obtain a connection between quasi projectives and the indecomposable problem. The indecomposable problem is "Which rings have infinitely many non-isomorphic indecomposable modules?" It is usually asked about rings with minimum condition and in the following theorem we shall deal with such rings.

THEOREM 3.2. *If R is a ring with minimum condition of left ideals then R has an infinite number of non-isomorphic quasi projective indecomposable left modules if and only if the two sided ideal lattice of R is infinite.*

Proof. First decompose the identity of R into indecomposable orthogonal idempotents $1 = e_1 + \cdots + e_n$ and fix this decomposition. Let I be the set of ideals of R, ε the finite set of idempotents $e_1, \cdots, e_n$ and let Q be the set of equivalence classes (under isomorphism) of indecomposable quasi projective R-modules. Since R has only a finite number of indecomposable projectives [1] and each indecomposable quasi projective is realized as a factor of one of these, it is clear that the equivalence classes of indecomposable quasi projectives form a set.

Now define the function $F : I \times \varepsilon \to Q, F(J, e_i) =$ the class of $Re_i/J \cap Re_i$. We first show that the function F is onto Q. If M is indecomposable quasi projective then M has one of the Re_i as projective cover [1], $Re_i \to M \to 0$. By Theorem 3.1, $M = Re_i/J \cap Re_i$ and $F(J, e_i)$ is the class of M. It follows that if Q is infinite so is I, because ε is finite.

To show the converse we first cite a theorem proved in [2] for rings with minimum condition: If $Re/L = Re'/L'$ then these isomorphisms can be realized by right multiplication by elements α and β of R. In particular, there exist elements $\alpha, \beta \in R$ such that $L \cong L\alpha = L'$ and $L' \cong L'\beta = L$. Applying this to isomorphic quasi projectives: if $Re_i/Je_i \cong Re_j/Ie_j$ where I, J are ideals, then $Je_i\, \alpha = Ie_j$ and $Ie_j\, \beta = Je_i$. Using the fact that I and J are ideals, we have $Je_i \subseteq I$ and $Ie_j \subseteq J$.

Now, if $Re_i/Je_i \cong Re_i/Ie_i$ for $i = 1, \cdots, n$ then $\sum Je_i = J \subseteq I$ and $\sum Ie_i = I \subseteq J$ or $I = J$. Stated contrapositively, if $I \neq J$ then the two sets $F(J \times \varepsilon)$ and $F(I \times \varepsilon)$ are distinct subsets of Q. It follows that if I is infinite then Q is also infinite. This completes the proof of Theorem 3.2.

If we specialize to finite dimensional algebras over a field we obtain the following corollary:

COROLLARY 3.3. *If R is a K algebra, K a field and $[R:K]$ finite then the following are equivalent:*

1. *R has an infinite number of indecomposable quasi projective left modules.*
2. *R has an infinite number of indecomposable quasi projective right modules.*
3. *R has an infinite number of indecomposable quasi injective right modules.*
4. *R has an infinite number of indecomposable quasi injective left modules.*
5. *R has an infinite ideal lattice.*

Proof. The functor $(\cdot)^* = \mathrm{Hom}_K(\cdot\,, K)$ gives a perfect duality from the category of finitely generated left R modules to finitely generated right R modules, and similarly right to left. This functor carries indecomposable quasi projectives to indecomposable quasi injectives and the reverse. It follows that 1 and 3 are always equivalent and similarly 2 and 4. Theorem 3.2 establishes the equivalence of 1 and 5. But since 5 is left-right symmetric, it also establishes the equivalence of 2 and 5. Thus they are all equivalent.

REFERENCES

1. H. BASS, *Finitistic dimension and a homological generalization of semi-primary rings*, Trans. Amer. Math. Soc., vol. 95 (1960), pp. 466–488.
2. R. BRAUER, *Some remarks on associative rings and algebras*, National Research Council Publication no. 502 (1957), pp. 4–11.
3. C. W. CURTIS AND J. P. JANS, *On algebras with a finite number of indecomposable modules*, Trans. Amer. Math. Soc., vol. 114 (1965), pp. 122–132.
4. B. ECKMANN AND A. SCHOPF, *Über injektive Moduln*, Arch. Math., vol. 4 (1953), pp. 75–78.
5. C. FAITH AND Y. UTUMI, *Quasi-injective modules and their endomorphism rings*, Arch. Math., vol. 15 (1964), pp. 166–174.
6. C. FAITH, *Lectures on injective modules and quotient rings*, Lecture notes from Rutgers University, New Brunswick, N. J., 1965.
7. N. JACOBSON, *Lectures in abstract algebra*, Princeton, N. J., Van Nostrand, 1951.
8. J. P. JANS, *On the indecomposable representations of algebras*, Ann. of Math., vol. 66 (1957), pp. 418–429.
9. ———, *Rings and homology*, New York, Holt, Rinehart and Winston, 1964.
10. R. E. JOHNSON AND E. T. WONG, *Quasi-injective modules and irreducible rings*, J. London Math. Soc., vol. 36 (1961), pp. 260–268.
11. H. NAGAO AND T. NAKAYAMA, *On the structure of M_0- and M_u- modules*, Math. Z., vol. 59 (1953), pp. 164–170.
12. H. TACHIKAWA, *A note on algebras of unbounded representation type*, Proc. Japan Acad., vol. 36 (1960), pp. 59–61.

UNIVERSITY OF CALIFORNIA
SANTA CRUZ, CALIFORNIA
UNIVERSITY OF WASHINGTON
SEATTLE, WASHINGTON