LUSTERNIK-SCHNIRELMANN CATEGORY AND STRONG CATEGORY

BY

T. GANEA[1]

1. Introduction

The purpose of this note is to compare the following two numerical homotopy invariants of a topological space.

DEFINITION 1.1 The Lusternik-Schnirelmann category, cat B, of a topological space B is the least integer $k \geq 0$ with the property that B may be covered by $k + 1$ open subsets which are contractible in B; if no such integer exists, cat $B = \infty$.

DEFINITION 1.2. The strong category, Cat B, of a topological space B is the least integer $k \geq 0$ with the property that B has the homotopy type of a CW-complex which may be covered by $k + 1$ self-contractible subcomplexes; if no such integer exists, Cat $B = \infty$.

The first definition is classical; the second is the homotopy invariant version of an earlier definition due to Fox [3, §IV] and was introduced in [4]. Since a CW-pair has the homotopy extension property and since a CW-complex is locally contractible, the CW-complex, say B', described in 1.2 satisfies cat $B' \leq k$. Therefore, and since category is a homotopy type invariant, one has cat $B \leq$ Cat B for any space B; in particular, Cat $B = \infty$ if B fails to have the homotopy type of a CW-complex. Our main result is expressed by

THEOREM 1.3. *Let B be an $(n - 1)$-connected CW-complex with* cat $B \leq k$ $(k \geq 1, n \geq 2)$. *If* $\dim B \leq (k + 2)n - 3$, *then also* Cat $B \leq k$.

It is well known that cat $B \leq 1$ if and only if B is an H'-space, and it follows from 2.1 below that Cat $B \leq 1$ if and only if B has the homotopy type of a suspension. Hence, 1.3 may be considered as a generalization of the following result: *any $(n - 1)$-connected H'-space B of dimension $\leq 3n - 3$ has the homotopy type of a suspension.* Under the additional assumption that the homology of B is finitely generated, this last result was first proved in [1], and an example therein reveals that 1.3 yields the best possible result at least when $k = 1$. The proof to follow is essentially different from that given in [1]. In the final section, we show that our approach leads to a substantial simplification of the main geometric result in [6] which relates category to the differentials in certain spectral sequences.

The preceding two definitions, as stated in terms of coverings by certain subsets, do not dualize in the sense of [2]. Nevertheless, it is possible to dualize the main results of the paper. Thus, the dual of 2.2 below yields a satisfactory

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definition of cocategory [5, §6], and the dual of 2.1 may be used as an inductive definition of strong cocategory (omit 0-connectedness in both 2.1 and 2.2 when dualizing). Then, the dual of 1.3 extends to arbitrary values of k the fact, first proved in [8] and corresponding to the case $k = 1$, that any $(n - 1)$-connected H-space A with $\pi_q(A) = 0$ for $q \geq 3n$ has the homotopy type of a loop space. We will not give the details. The dual of 4.2 is equally valid and will be discussed elsewhere.

2. Alternative characterizations of category and strong category

A triple

$$A \xrightarrow{d} X \xrightarrow{f} C$$

of based spaces and based maps is a *cofibration* if d is an inclusion map with the based homotopy extension property and C results from X by shrinking the subset A to a point; f is the identification map. Let $k \geq 0$ be any integer.

PROPOSITION 2.1. *Let B be a 0-connected topological space. Then* Cat $B = 0$ *if and only if B is contractible, and* Cat $B \leq k + 1$ *if and only if there is a cofibration*

$$A \xrightarrow{d} X \xrightarrow{f} C$$

such that

(i) *C has the free homotopy type of B,*
(ii) *A and X have the based homotopy type of CW-complexes,*
(iii) *X is 0-connected and* Cat $X \leq k$.

Proof. The first statement is obvious. Let

$$\alpha : L \to A \quad \text{and} \quad \xi : X \to K$$

be based homotopy equivalences, where L and K are CW-complexes of which K is the union of $k + 1$ self-contractible subcomplexes K_i (we may assume ξ to be a based homotopy equivalence according to (ii) and [11, E, p. 333]). According to [9, Th. 2] we may assume L and the $(k + 2)$-ad $(K; K_0, \cdots, K_k)$ to be simplicial in the weak topology. Let M be the reduced mapping cylinder of a simplicial approximation $\phi : L \to K$ of $\xi \circ d \circ \alpha$; let $j : L \to M$ be the canonical inclusion and let J result from M by shrinking the subset $j(L)$ to a point. It follows from [11, Hilfssatz 7] that C has the based homotopy type of J, and the latter is a CW-complex consisting of $k + 2$ self-contractible subcomplexes: the reduced cone over L and the mapping cylinders of the maps $\phi_i : L_i \to K_i$ defined by ϕ, where $L_i = \phi^{-1}(K_i)$. Therefore, Cat $B \leq k + 1$. Conversely, suppose B has the free homotopy type of a CW-complex J which is the union of $k + 2$ self-contractible subcomplexes J_i. Let $X = \bigcup_{i=0}^{k} J_i$ and $A = X \cap J_{k+1}$. Since J is connected, we may obviously assume the J_i renumbered so that also X is connected; for the same reason, A is non-void. Obviously, Cat $X \leq k$ and, with $C = X/A$ resulting from X by shrinking A

to a point, the triple $A \to X \to C$ is a cofibration. The inclusion map $(X, A) \to (J, J_{k+1})$ induces a homeomorphism of C onto J/J_{k+1} and, since J_{k+1} is self-contractible, the latter has the free homotopy type of J and, hence, of B.

Let now B be an arbitrary topological space with base-point $*$. Define a sequence of fibrations

$$\mathfrak{F}_k : F_k \xrightarrow{i_k} E_k \xrightarrow{p_k} B \qquad \text{for} \quad k \geq 0$$

as follows. $\mathfrak{F}_0$ is the standard fibration $\Omega B \to PB \to B$, where PB is the space of all paths in B emanating from $*$, p_0 sends every path into its end-point, ΩB if the loop space, and i_0 the inclusion. Assuming $\mathfrak{F}_k$ to be defined, let $C_{k+1} = E_k \cup CF_k$ result from E_k by erecting a reduced cone over the subset F_k and let $r_{k+1} : C_{k+1} \to B$ extend p_k by mapping the cone into $*$. Then, convert r_{k+1} into a homotopically equivalent fibre map p_{k+1} with total space E_{k+1}, fibre $F_{k+1} = p_{k+1}^{-1}(*)$, and inclusion i_{k+1}; explicitly,

$$E_{k+1} = \{(x, \beta) \in C_{k+1} \times B^I \mid r_{k+1}(x) = \beta(0)\} \quad \text{and} \quad p_{k+1}(x, \beta) = \beta(1),$$

whereas the map $h_{k+1} : C_{k+1} \to E_{k+1}$, given by $h_{k+1}(x) = (x, \beta_x)$ with $\beta_x(s) = r_{k+1}(x)$ for all $s \in I$, is a homotopy equivalence satisfying $p_{k+1} \circ h_{k+1} = r_{k+1}$. This sequence is related to that giving the classifying space of a loop space.

PROPOSITION 2.2. *Let B be a based connected CW-complex. Then,* $\operatorname{cat} B \leq k$ *if and only if $\mathfrak{F}_k$ has a cross-section.*

Proof. It follows easily from [9] that F_k and E_k, hence also the reduced mapping cylinder M_k of i_k, have the based homotopy type of CW-complexes for any $k \geq 0$. Since $\operatorname{Cat} E_0 = 0$, consideration of the cofibrations $F_{k-1} \to M_{k-1} \to C_k$ reveals, by 2.1, that $\operatorname{Cat} E_k \leq k$ for any $k \geq 0$. The presence of a cross-section in $\mathfrak{F}_k$ implies that B is dominated by E_k so that $\operatorname{cat} B \leq k$. Conversely, we may assume that B is covered by $k + 1$ subcomplexes B_m, each of which contains $*$ and is contractible rel. $*$ in B. Let $A = \bigcup_{m=0}^{n-1} B_m$ and $D = A \cap B_n$, where $1 \leq n \leq k$, and let $j : A \cup B_n \to B$ be the inclusion. Since the subcomplex B_n is contractible rel. $*$ in B, there is a homotopy

$$j_t : A \cup B_n \to B \quad \text{with} \quad j_0 = j, j_1(B_n) = *, j_t(*) = *.$$

Suppose there is a based map $\gamma : A \to E_{n-1}$ satisfying $p_{n-1} \circ \gamma = j \mid A$; this certainly happens if $n = 1$ since $j \mid A$ is then nullhomotopic rel. $*$. Since p_{n-1} is a fibre map, there results a homotopy

$$\gamma_t : A \to E_{n-1} \quad \text{with} \quad \gamma_0 = \gamma, p_{n-1} \circ \gamma_t = j_t \mid A, \gamma_t(*) = *.$$

Then, $\gamma_1(D) \subset F_{n-1}$ and, since the reduced cone CF_{n-1} is contractible, the map $D \to F_{n-1}$ defined by γ_1 extends to a map $\beta : B_n \to CF_{n-1}$. The map

$$\phi : A \cup B_n \to E_{n-1} \cup CF_{n-1},$$

given by $\phi \mid A = \gamma_1$ and $\phi \mid B_n = \beta$, satisfies $r_n \circ \phi = j_1$ hence

$$p_n \circ h_n \circ \phi = j_1 ,$$

where $h_n : E_{n-1} \cup CF_{n-1} \to E_n$ is the homotopy equivalence described before 2.2. Since p_n is a fibre map, there results a based map $\Gamma : A \cup B_n \to E_n$ satisfying $p_n \circ \Gamma = j$, and the second part of 2.2 now follows by induction.

We shall also need a modification of the sequence of fibrations $\mathfrak{F}_k$. Let $N \geq 2$ be an arbitrary integer. Define new fibrations

$$\mathfrak{F}_k(N) : P_k \xrightarrow{i_k} Q_k \xrightarrow{p_k} B \qquad \text{for} \quad k \geq 0$$

as follows. $\mathfrak{F}_0(N)$ is the same as $\mathfrak{F}_0$. Assuming $\mathfrak{F}_k(N)$ to be defined, let A_k be the $(N-1)$-skeleton of the singular polytope of P_k and let $j_k : A_k \to P_k$ be the restriction of the canonical map. Let $R_{k+1} = Q_k \cup CA_k$ result by attaching to Q_k the reduced cone over A_k via the map $i_k \circ j_k$, and let $r_{k+1} : R_{k+1} \to B$ extend p_k by mapping the cone to the basepoint. Finally, $\mathfrak{F}_{k+1}(N)$ results by converting r_{k+1} into a homotopically equivalent fibre map p_{k+1}.

We denote by $H_q(X)$ the q-th reduced singular homology group of X with integral coefficients.

PROPOSITION 2.3. *Let B a 1-connected CW-complex. Then, for any $k \geq 0$, $\pi_1(Q_k) = 0$, Cat $Q_k \leq k$, $H_N(Q_k)$ is free and $H_q(Q_k) = 0$ if $q > N$. In case* $\dim B \leq N$, $\operatorname{cat} B \leq k$ *if and only if $\mathfrak{F}_k(N)$ has a cross-section.*

Proof. Introduce the cofibrations $A_{k-1} \to M_{k-1} \to R_k$, where M_k is the reduced mapping cylinder of $i_k \circ j_k$. Since Q_0 is contractible and $\dim A_{k-1} \leq N - 1$, the last two asserted properties of Q_k follow by induction using 2.1 and the exact homology sequence of a cofibration. Next, we prove that there are N-connected maps φ_k and ε_k such that the diagram

$$\begin{array}{rcccccc} \mathfrak{F}_k(N) : & P_k & \xrightarrow{i_k} & Q_k & \xrightarrow{p_k} & B \\ & \downarrow{\scriptstyle \varphi_k} & & \downarrow{\scriptstyle \varepsilon_k} & & \| \\ \mathfrak{F}_k : & F_k & \xrightarrow{i_k} & E_k & \xrightarrow{p_k} & B \end{array} \tag{1}$$

commutes. Let φ_0 and ε_0 be the identity maps. Suppose that $\pi_1(Q_k) = \pi_1(E_k) = 0$ and that (1) behaves as asserted for some $k \geq 0$. In the diagram

$$\begin{array}{ccccccc} A_k & \xrightarrow{i_k \circ j_k} & Q_k & \longrightarrow & Q_k \cup CA_k & \xrightarrow{r_{k+1}} & B \\ \downarrow{\scriptstyle \varphi_k \circ j_k} & & \downarrow{\scriptstyle \varepsilon_k} & & \downarrow{\scriptstyle \psi_{k+1}} & & \| \\ F_k & \xrightarrow{i_k} & E_k & \longrightarrow & E_k \cup CF_k & \xrightarrow{r_{k+1}} & B \end{array}$$

the first square commutes. Hence, it induces a map ψ_{k+1} yielding commutativity in the second and, obviously, also in the third square. According to [5, 1.1], F_k has the homotopy type of the join of $k + 1$ copies of ΩB. Therefore, and since $\pi_1(B) = 0$, F_k is certainly 0-connected; since $\varphi_k \circ j_k$ is $(N - 1)$-connected and $N \geq 2$, also A_k is 0-connected. Hence, it follows from [11, Hilfssatz 9] that

$$\pi_1(Q_k \cup CA_k) = \pi_1(E_k \cup CF_k) = 0.$$

Since $\varphi_k \circ j_k$ and ε_k are $(N - 1)$- and N-connected respectively, use of the 5-lemma in the first two squares reveals that ψ_{k+1} is homology, hence also homotopy, N-connected. When converting the maps r_{k+1} into homotopically equivalent fibre maps, ψ_{k+1} induces the desired map ε_{k+1} which, in turn, defines φ_{k+1} ; the connectivity of the latter follows from the 5-lemma applied for homotopy groups in (1) with k replaced by $k + 1$. Finally, any cross-section in $\mathfrak{F}_k(N)$ yields, by composition with ε_k , a cross-section in $\mathfrak{F}_k$; conversely, since ε_k is N-connected, any cross-section in $\mathfrak{F}_k$ lifts to a cross-section in $\mathfrak{F}_k(N)$ if $\dim B \leq N$, and the last statement in 2.3 follows from 2.2.

3. Creating cofibrations

We work with spaces of the based homotopy type of a CW-complex. If E is such a space, we denote by $\dim E$ the least of the dimensions of all CW-complexes in the homotopy type of E.

LEMMA 3.1. *Let the top row in the diagram*

$$(2)\qquad \begin{array}{ccccc} C & \xleftarrow{f} & X & \xleftarrow{d} & A \\ {\scriptstyle g}\uparrow & & \uparrow{\scriptstyle \gamma} & & \| \\ B & \xleftarrow{\varphi} & W & \xleftarrow{\alpha} & A \end{array}$$

be a cofibration and let g be any map. Let A, X, C, and B be 0-connected and let $\pi_1(B) = \pi_1(X) = 0$. Suppose that g is m-connected, f is c-connected, $\dim A \leq m + c - 1$, and $\dim B \leq m + c$ $(m \geq 2, c \geq 1)$. Then, there is a 1-connected space W and maps φ, γ, α such that

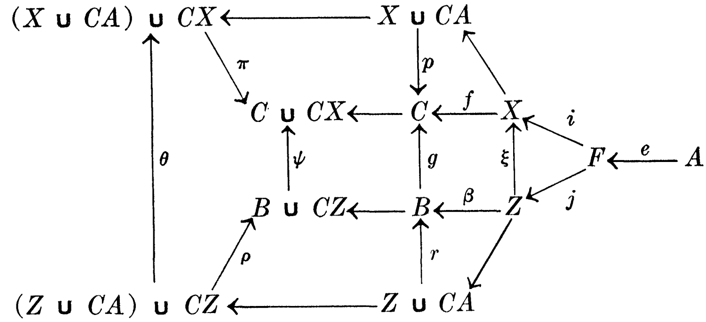
(i) *the diagram homotopy-commutes,*

(ii) *γ is m-connected and $\dim W \leq m + c$,*

(iii) *$\varphi \circ \alpha = *$ and the extension $W \cup CA \to B$ of φ which maps the cone into $*$ is a homotopy equivalence.*

Proof. Upon replacing B and X by homotopically equivalent spaces and retaining the notation, we may assume that g and f are fibre maps. We shall refer to the diagram

(3)



where $Z = \{(b, x) \in B \times X \mid g(b) = f(x)\}$, β and ξ are the projections, and F is the fibre of f with i as inclusion. Clearly, β is a fibre map with fibre F and inclusion given by $j(x) = (*, x)$. When converting f into a fibre map, the relation $f \circ d = *$ survives and yields a map e with $i \circ e = d$. Also ξ is a fibre map and has the same fibre as g. Hence, ξ is m-connected and, since $m \geq 2$, $\pi_1(Z) = 0$. In (3) the cones are attached in the obvious way and the unlabelled arrows denote inclusions. Let ψ be induced by g and ξ. Since F is the common fibre of f and β, the connectivities of f and ξ imply, by the relative Serre theorem (see for instance [10, 1.6]), that ψ is $(m + c + 1)$-connected. Let p and r extend f and β by mapping the cones to the base-points, let π and ρ be induced by p and r, and let θ be induced by ξ. Clearly, (3) commutes. Since the top row in (2) is a cofibration, p is a homotopy equivalence and, hence, so is π. Upon shrinking CZ and CX to the base-points, θ is converted into the identity map of the suspension ΣA and is, therefore, a homotopy equivalence. As a consequence, the connectivity of ψ implies that ρ is $(m + c)$-connected and hence, by the 5-lemma, that r is $(m + c)$-connected.[2] Since $H_{m+c}(B)$ is free, so is its subgroup $\operatorname{Im} \beta_*$ and, by [1, 2.1], there is a 1-connected CW-complex W with $\dim W \leq m + c$ and a map $w : W \to Z$ such that

$$w_* : H_q(W) \to H_q(Z) \quad \text{is isomorphic for} \quad q < m + c,$$

$$\beta_* \circ w_* : H_q(W) \to \operatorname{Im} \beta_* \quad \text{is isomorphic for} \quad q = m + c.$$

We replace W by a homotopically equivalent space so as to convert w into a fibre map. Then, since w is $(m + c - 1)$-connected whereas $\dim A \leq m + c - 1$, there is a map $\alpha : A \to W$ satisfying $w \circ \alpha = j \circ e$. Form $W \cup CA$ upon attaching the cone by means of α, and let

$$\eta : W \cup CA \to Z \cup CA \quad \text{and} \quad \tau : (W \cup CA) \cup CW \to (Z \cup CA) \cup CZ$$

be induced by w. By the 5-lemma, η induces isomorphisms of homology groups in dimensions $\leq m + c - 1$ and, therefore, so does $r \circ \eta$. Since τ is

[2] This could also be derived from [10, 2.4].

homotopically equivalent to the identity map of ΣA, τ_* is always isomorphic; since $\pi \circ \theta$ in (3) is a homotopy equivalence, ρ_* is always monomorphic and, hence, isomorphic in dimension $m + c$. Then, $r_* \circ \eta_*$ is isomorphic in dimension $m + c$ as shown by the 5-lemma in the diagram

$$\begin{array}{ccccccccc} 0 = H_{m+c}(A) & \to & H_{m+c}(W) & \to & H_{m+c}(W \cup CA) & \to & H_{m+c}((W \cup CA) \cup CW) & \to & H_{m+c-1}(W) \\ \downarrow & & \downarrow {\scriptstyle \beta_* \circ w_*} & & \downarrow {\scriptstyle r_* \circ \eta_*} & & \downarrow {\scriptstyle \rho_* \circ \tau_*} & & \downarrow {\scriptstyle w_*} \\ 0 & \longrightarrow & \operatorname{Im} \beta_* & \longrightarrow & H_{m+c}(B) & \longrightarrow & H_{m+c}(B \cup CZ) & \longrightarrow & H_{m+c-1}(Z). \end{array}$$

Since $\dim A \leq m + c - 1$, $H_q(W \cup CA) = 0$ for $q > m + c$ so that $r \circ \eta$ is a homotopy equivalence. To obtain the result, it only remains to set $\varphi = \beta \circ w$ and $\gamma = \xi \circ w$.

Proof of 1.3. We assume B to be an $(n - 1)$-connected CW-complex with $\operatorname{cat} B \leq k$ and $\dim B \leq (k+2)n - 3$ $(k \geq 1, n \geq 2)$ Let $N = (k+2)n - 3$ and introduce the diagram

$$\begin{array}{ccccc} Q_q & \xleftarrow{f_q} & Q_{q-1} & \longleftarrow & A_{q-1} \\ {\scriptstyle g_q} \uparrow & & \uparrow {\scriptstyle g_{q-1}} & & \| \\ B_q & \longleftarrow & B_{q-1} & \xleftarrow{\alpha_{q-1}} & A_{q-1} \end{array} \qquad (4) \quad (q \geq 1)$$

with top row taken from the definition of the fibrations $\mathfrak{F}_q(N)$ given in the preceding section; we may obviously regard the top row as a cofibration. According to [5, 1.1], the fibre F_{q-1} in the fibration $\mathfrak{F}_{q-1}$ has the homotopy type of the join of q copies of ΩB and is, therefore, $(qn - 2)$-connected. The map

$$j_{q-1} : A_{q-1} \to P_{q-1}$$

in the definition of $\mathfrak{F}_q(N)$ is $(N - 1)$-connected, and

$$\varphi_{q-1} : P_{q-1} \to F_{q-1}$$

in (1) is N-connected. Therefore, A_{q-1} is $(qn - 2)$-connected if $q \leq k$, and P_{q-1} is $(qn - 2)$-connected if $q \leq k + 1$. By 2.3, $\pi_1(Q_{q-1}) = \pi_1(Q_q) = 0$; therefore, f_q is certainly $(n - 1)$-connected if $1 \leq q \leq k$. Let $c = n - 1$. Since $\operatorname{cat} B \leq k$, 2.3 yields a cross-section $g : B \to Q_k$ in $\mathfrak{F}_k(N)$, and the connectivity of P_k readily implies that g is $((k + 1)n - 2)$-connected. Let $m = (k+1)n - 2$. Starting with $B_k = B$ and $g_k = g$, consecutive application of 3.1 in (4) yields a sequence of spaces

$$B = B_k \leftarrow B_{k-1} \leftarrow \cdots \leftarrow B_1 \leftarrow B_0$$

in which every B_q has the homotopy type of $B_{q-1} \cup CA_{q-1}$ so that, by 2.1,

$$\operatorname{Cat} B_q \leq \operatorname{Cat} B_{q-1} + 1 \quad \text{and} \quad \operatorname{Cat} B \leq \operatorname{Cat} B_1 + k - 1.$$

We now prove that $\operatorname{Cat} B_1 \leq 1$. Consider (4) with $q = 1$. Convert α_0 into a fibre map and let L be its fibre with inclusion $l : L \to A_0$. The map g_0, given by 3.1, is m-connected and Q_0 is contractible; therefore, B_0 is $(m - 1)$-connected. Since A_0 is $(n - 2)$-connected and $m - 1 > n - 2$, L is $(n - 2)$-connected. By [5, 2.1], the map $A_0 \cup CL \to B_0$, which extends α_0 by mapping the cone into $*$, is $(m + n - 1)$-connected and, by the 5-lemma, the resulting map $\phi : \Sigma L \to B_1$ is homology $(m + n - 1)$-connected. Since $\dim B_1 \leq m + n - 1$, it follows from [1, 2.1] that there is a connected CW-complex L_0 and a map $\lambda : L_0 \to L$ such that $\phi \circ \Sigma\lambda : \Sigma L_0 \to B_1$ induces isomorphisms of homology groups in all dimensions. Since $\pi_1(\Sigma L_0) = \pi_1(B_1) = 0$, $\phi \circ \Sigma\lambda$ is actually a homotopy equivalence, and 1.3 is proved.

Remark 3.2. The space E_1 in $\mathfrak{F}_1$ has the homotopy type of $PB \cup C\Omega B$. If B has the homotopy type of a CW-complex, we may shrink the contractible subspace PB to a point without altering the homotopy type of E_1. The resulting space is $\Sigma\Omega B$ and p_1 is, then, equivalent to the map $R : \Sigma\Omega B \to B$ given by $R\langle s, \omega\rangle = \omega(s)$. Suppose now that B is an H'-space with comultiplication $\tau : B \to B \vee B$ satisfying

$$J \circ \tau \simeq \Delta,$$

where $J : B \vee B \to B \times B$ is the inclusion of $(B \times *) \cup (* \times B)$ in the Cartesian product and $\Delta : B \to B \times B$ is the diagonal map. Then, as is well known, $\operatorname{cat} B \leq 1$ with a homotopy cross-section $\Gamma : B \to \Sigma\Omega B$ satisfying

$$R \circ \Gamma \simeq 1 \quad \text{and} \quad \tau \simeq (R \vee R) \circ \sigma \circ \Gamma,$$

where $\sigma : \Sigma\Omega B \to \Sigma\Omega B \vee \Sigma\Omega B$ is the comultiplication given in any suspension by

$$\begin{aligned} \sigma\langle s, y\rangle &= (\langle 2s, y\rangle, *) && \text{if} \quad 0 \leq 2s \leq 1, \\ &= (*, \langle 2s - 1, y\rangle) && \text{if} \quad 1 \leq 2s \leq 2. \end{aligned}$$

Suppose now that $\dim B \leq 3n - 3$ and consider (4) with $q = 1$, $B_1 = B$, A_0 the $(3n - 4)$-skeleton of ΩB, $Q_1 = \Sigma A_0$, and g_1 resulting, as in the proof of 2.3, by compressing $\Gamma : B \to \Sigma\Omega B$ into ΣA_0. Let R_0 be the restriction of R to ΣA_0 so that $R_0 \circ g_1 \simeq 1$. In the diagram

$$\begin{array}{ccccc} \Sigma L & \xrightarrow{\ \phi\ } & B & \xrightarrow{\ g_1\ } & \Sigma A_0 \\ \downarrow{\scriptstyle\sigma} & & \downarrow{\scriptstyle\tau} & & \downarrow{\scriptstyle\sigma} \\ \Sigma L \vee \Sigma L & \xrightarrow{\ \phi \vee \phi\ } & B \vee B & \underset{R_0 \vee R_0}{\overset{g_1 \vee g_1}{\rightleftarrows}} & \Sigma A_0 \vee \Sigma A_0 \end{array}$$

the maps ϕ and l defined at the end of the proof of 1.3 satisfy

$$g_1 \circ \phi \simeq \Sigma l.$$

Since Σl commutes with σ and since g_1 is the compression of Γ one has

$$\begin{aligned} \tau \circ \phi &\simeq (R_0 \vee R_0) \circ \sigma \circ g_1 \circ \phi \\ &\simeq (R_0 \vee R_0) \circ (g_1 \vee g_1) \circ (\phi \vee \phi) \circ \sigma \simeq (\phi \vee \phi) \circ \sigma. \end{aligned}$$

As a consequence, the homotopy equivalence $\phi \circ \Sigma\lambda : \Sigma L_0 \to B$ is *primitive* with respect to comultiplication in the H'-space B and the suspension ΣL_0. Hence, 1.3 generalizes the full result of [1, Th. A].

We close this section by deriving from 2.2 and 1.3 a very short proof of a result first obtained for category in [7] and extended to strong category in [4].

COROLLARY 3.3. *If B is an $(n-1)$-connected CW-complex with* $\dim B \leq r$, *then* $\operatorname{Cat} B \leq r/n$ $(n \geq 2)$.

Proof. Let k be the largest integer $\leq r/n$. Since F_k in $\mathfrak{F}_k$ is $((k+1)n-2)$-connected and $\dim B \leq (k+1)n-1$, $\mathfrak{F}_k$ has a cross-section so that $\operatorname{cat} B \leq k$. Since $(k+1)n-1 \leq (k+2)n-3$, $\operatorname{Cat} B \leq k$.

4. Remarks on a spectral sequence

It has already been observed in [12] that $\operatorname{cat} B \leq k$ if and only if the k-th fibration in a certain sequence has a cross-section, and the spectral sequence associated with these fibrations has been investigated in [12] and [6]. All the results contained in [6, §1 and §2] automatically transfer to the homology spectral sequence arising from the sequence of fibrations $\mathfrak{F}_k$ defined above. Here, we shall only illustrate the advantage of the fibration-cofibration approach used in the definition of the $\mathfrak{F}_k$'s by giving a simple proof of a geometric result (Corollary 4.3 below) which immediately implies, as in [6, Th. 2.1], that $d^r = 0$ if $r > \operatorname{cat} B$; the result is equivalent to [6, Lemma 2.2] of which the proof in [6] is quite intricate.

LEMMA 4.1. *Let the top row in the diagram*

$$\begin{array}{ccccc} & & C & \xleftarrow{f} & X & \xleftarrow{d} & A \\ & & \big\downarrow{\scriptstyle e} & & & & \\ B & \underset{g}{\overset{p}{\rightleftarrows}} & E & \xleftarrow{i} & F & & \\ & & \big\downarrow{\scriptstyle j} & & & & \\ & & E \cup CF & & & & \end{array}$$

be a cofibration and let the second row be a fibration with a cross-section g; let j be the inclusion and let e be any map. If

$$e \circ f \simeq g \circ p \circ e \circ f,$$

then

$$j \circ e \simeq j \circ g \circ p \circ e.$$

Proof. For any space V, let $+$ and $-$ denote track addition and subtraction in the group $\pi(\Sigma A, V)$, and let τ denote the operation of $\pi(\Sigma A, V)$ on the set $\pi(C, V)$ which is associated with the given cofibration [11, 4.3]. To simplify notations, we omit the circle when composing maps. Since $ef \simeq gpef$, by [11, 4.5] there is a map $\varepsilon : \Sigma A \to E$ such that

$$e \simeq \varepsilon \top gpe, \tag{5}$$

hence, since $pg = 1$,

$$pe \simeq p\varepsilon \top pe. \tag{6}$$

Since $pg = 1$, one also has $p(\varepsilon - gp\varepsilon) \simeq 0$ and there results a map $\varphi : \Sigma A \to F$ such that

$$\varepsilon \simeq i\varphi + gp\varepsilon. \tag{7}$$

Therefore,

$$e \simeq (i\varphi + gp\varepsilon) \top gpe \qquad \text{by (5) and (7),}$$

so that

$$\begin{aligned} je &\simeq j(i\varphi + gp\varepsilon) \top jgpe && \text{by naturality,} \\ &\simeq (ji\varphi + jgp\varepsilon) \top jgpe && \\ &\simeq jgp\varepsilon \top jgpe, && \text{since } ji \simeq 0, \\ &\simeq jg(p\varepsilon \top pe) && \text{by naturality,} \\ &\simeq jgpe && \text{by (6).} \end{aligned}$$

We now go back to the definition of $\mathfrak{F}_k$ and introduce the composite

$$j_k : E_k \to E_k \cup CF_k \to E_{k+1},$$

where the first map is the inclusion and the second is the homotopy equivalence h_{k+1} ; let

$$j_m^n = j_{n-1} \circ \cdots \circ j_m : E_m \to E_n$$

for $n > m$ and $j_m^m = 1$.

THEOREM 4.2. *Let B be a connected CW-complex. If* cat $B \leq k$ *with cross-section $g : B \to E_k$, then $j_m^n \simeq j_k^n \circ g \circ p_m$ for all $n \geq k + m$.*

Proof. The statement is obviously true if $m = 0$ since both sides are then defined on E_0 which is contractible. Suppose the statement to be true for some $m \geq 0$ and let $n \geq k + m + 1$. The first row in the diagram

$$\begin{array}{ccccc} & & E_{m+1} & \xleftarrow{j_m} & E_m & \xleftarrow{i_m} & F_m \\ & & \Big\downarrow{\scriptstyle j_{m+1}^{n-1}} & & & & \\ B & \underset{j_k^{n-1} \circ g}{\overset{p_{n-1}}{\rightleftarrows}} & E_{n-1} & \xleftarrow{i_{n-1}} & F_{n-1} & & \\ & & \Big\downarrow{\scriptstyle j_{n-1}} & & & & \\ & & E_n & & & & \end{array}$$

may be considered as a cofibration, the second is a fibration and, since $n-1 \geq k$, $p_{n-1} \circ \dot{j}_k^{n-1} \circ g = p_k \circ g = 1$. One has

$$\dot{j}_k^{n-1} \circ g \circ p_{n-1} \circ \dot{j}_{m+1}^{n-1} \circ \dot{j}_m = \dot{j}_k^{n-1} \circ g \circ p_m \simeq \dot{j}_m^{n-1} = \dot{j}_{m+1}^{n-1} \circ \dot{j}_m,$$

where the equalities are obvious, and the equivalence is valid since $n-1 \geq k+m$ and since the statement is true for m. Hence, by 4.1,

$$\dot{j}_{m+1}^{n} = \dot{j}_{n-1} \circ \dot{j}_{m+1}^{n-1} \simeq \dot{j}_{n-1} \circ \dot{j}_k^{n-1} \circ g \circ p_{n-1} \circ \dot{j}_{m+1}^{n-1} = \dot{j}_k^{n} \circ g \circ p_{m+1}.$$

COROLLARY 4.3. *Under the same assumptions, the map*

$$\zeta = \dot{j}_k^{q-1} \circ g \circ p_q : E_q \to E_{q-1}$$

satisfies $\zeta \circ \dot{j}_{q-r}^{q} \simeq \dot{j}_{q-r}^{q-1}$ *if* $q \geq r > k$.

Remark. Here cat $* = 0$ whereas in [6] cat $* = 1$.

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UNIVERSITY OF WASHINGTON
SEATTLE, WASHINGTON