A NOTE ON REFLECTION MAPS

BY

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1. Introduction

Let $\mathcal{C}$ be a category. Our terminology for categories will be based on [2] and [8]. Thus a full subcategory $\mathcal{B}$ of $\mathcal{C}$ is *reflective* if the functor which injects $\mathcal{B}$ into $\mathcal{C}$ has an adjoint (or, in the language of [2], a left-adjoint) from $\mathcal{C}$ onto $\mathcal{B}$. Equivalently, $\mathcal{B}$ is reflective in $\mathcal{C}$ iff every object A of $\mathcal{C}$ has a *reflection map*, that is a morphism $e : A \to B$ where $B \in \mathcal{B}$ and such that whenever $f : A \to B'$ is a morphism with $B' \in \mathcal{B}$ then there is a unique $g : B \to B'$ such that $f = ge$ (see [2] or [8]). Reflection maps coincide with the *front adjunctions* of [6].

In this paper we shall consider the problem of determining when a given morphism $e : A \to B$ is a reflection map. We shall also consider the more general problem of determining when a class of morphisms, $\{e_i : A_i \to B_i\}$, is contained in the class of reflection maps associated with a full reflective subcategory. We have obtained results in the case in which the class $\{e_i : A_i \to B_i\}$ is a set and also the case in which every e_i is an epimorphism. In Section 4, theorem 1.1 is used to settle in the negative a question implicitly raised in [6] as to whether every reflection map is an epimorphism in the category of Hausdorff spaces and maps. Another example discusses the problem of obtaining a "universal covering space" for any Hausdorff space with base with base point. Throughout this paper we shall assume that every indexed collection of objects of $\mathcal{C}$ has a product and a coproduct. We shall also assume that every morphism f of $\mathcal{C}$ can be factored as $f = me$ where m is a monomorphism and e is an epimorphism. (This factorization and the assumption of coproducts are needed in the proofs of 2.1 and 2.2.)

In addition to the above terminology, we shall also assume that the reader is familiar with the terms *subobject*, *quotient object*, *well-powered* and *co-well-powered* as defined in [2]. We shall also make use of bicategories in the sense of Isbell for which the relevant definitions and known results shall be introduced as needed.

We can now state our main result.

Theorem 1.1. *Let $\mathcal{C}$ be well-powered. The set of morphisms $\{e_i : A_i \to B_i\}$ is contained in the class of reflection maps associated with a full reflective subcategory iff every morphism $f : A_i \to B_j$ can be factored so that $f = ge_i$ for a unique morphism $g : B_i \to B_j$.*

Corollary. *Let $\mathcal{C}$ be well-powered. A morphism $e : A \to B$ is a reflection*

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map iff every morphism $f : A \rightarrow B$ *can be factored so that* $f = ge$ *for a unique morphism* $g : B \rightarrow B$.

We have obtained a better result for the following special type of reflective subcategory.

DEFINITION. A full subcategory $\mathcal{B}$ of $\mathcal{C}$ is *epi-reflective*, in $\mathcal{C}$, if $\mathcal{B}$ is reflective and if all of the associated reflection maps are epimorphisms.

THEOREM 1.2. *Let* $\mathcal{C}$ *be co-well-powered. A class of epimorphisms*

$$\{e_i : A_i \rightarrow B_i\}$$

is contained in the class of reflection maps associated with an epi-reflective subcategory iff every morphism $f : A_i \rightarrow B_j$ *can be factored so that* $f = ge_i$ *for at least one morphism* $g : B_i \rightarrow B_j$.

Added in proof. The above corollary has also been obtained by J. R. Isbell, Bull. Amer. Math. Soc., 72, p. 644.

2. Epi-reflective subcategories

In what follows, the class of all epimorphisms of $\mathcal{C}$ shall be denoted by $E_{\mathcal{C}}$, or simply E, if there is no danger of confusion.

The following treatment of epi-reflective subcategories is for the most part known and given in [4] and [7]. The key definition is:

DEFINITION. A monomorphism f of $\mathcal{C}$ is an *extremal monomorphism* if $f = hg$ and $g \in E$ imply that g is an equivalence.

We shall let $M^{\#}_{\mathcal{C}}$ (or simply $M^{\#}$ if there is no danger of confusion) denote the class of all extremal monomorphisms of $\mathcal{C}$.

Remark. It is easily checked that if $rd = 1$, an identity morphism, then $d \in M^{\#}$. For if $d = hg$ and $g \in E$ then $rhg = 1$ and so $grhg = g$. Hence grh is also an identity as $g \in E$. Thus g is an equivalence as $g^{-1} = rh$. It follows that $M^{\#}$ contains all equivalences.

The following two results are proven in [7].

PROPOSITION 2.1. *If* $he = mg$ *where* $e \in E$ *and* $m \in M^{\#}$ (as in Figure 1), *there then exists a morphism* r *such that* $re = g$ *and* $mr = h$.

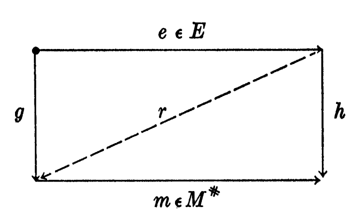

FIGURE 1

THEOREM 2.2. *If $\mathcal{C}$ is either well-powered or co-well-powered then* $(M^{\#}, E)$ *is a bicategory structure in the sense of Isbell on* $\mathcal{C}$. *This means that* $M^{\#}$ *is closed under composition and that every morphism f can be factored as* $f = f_1 f_0$ *where* $f_1 \in M^{\#}$ *and* $f_0 \in E$. *Moreover this factorization is unique to within equivalences in the sense that if* $f = hg$ *with* $h \in M^{\#}$ *and* $g \in E$ *then there is an equivalence e such that* $h = f_1 e$ *and* $eg = f_0$.

COROLLARY. *Let* $f_i : X_i \to Y_i$ *be an indexed subset of* $M^{\#}$. *Let* $X = \prod X_i$ *and* $Y = \prod Y_i$ *be products and let* $p_i : X \to X_i$ and $\bar{p}_i : Y \to Y_i$ *denote the pojections. Let* $f : X \to Y$ *be the morphism determined by* $\bar{p}_i f = f_i p_i$. *Then* $f \in M^{\#}$.

Proof. Let $f = f_1 f_0$ where $f_1 \in M^{\#}$ and $f_0 \in E$. In view of Proposition 2.1, there is, fos each i, a morphism r_i such that $r_i f_0 = p_i$ and $f_i r_i = \bar{p}_i f_1$. Let r be the morphism for which $p_i r = r_i$ for all i. Then $r f_0 = 1_X$ since $p_i r f_0 = p_i$ for all i. By a previous remark, this implies that $f_0 \in M^{\#}$ and so $f = f_1 f_0 \in M^{\#}$ as $M^{\#}$ is composition closed.

DEFINITION. X is an *$M^{\#}$-subobject* of Y if $M^{\#} \cap \text{Hom}(X, Y) \neq \emptyset$.

DEFINITION. Let $\mathcal{B}$ be a full subcategory of $\mathcal{C}$. Then $\mathcal{C}$ is *co-well-powered relative to* $\mathcal{B}$ if each object X of $\mathcal{C}$ has no more than a set of quotient objects in $\mathcal{B}$. (Extending a standard abuse of language, we shall say that a quotient object, which is an equivalence class of epimorphisms, is "in $\mathcal{B}$" if it has at least one representative member $e : X \to Y$ for which Y is an object of B.)

Remark. In view of Theorem 2.2, we shall assume from now on that $(M^{\#}, E)$ is a bicategory structure on $\mathcal{C}$.

THEOREM 2.3. *Let* $\mathcal{C}$ *be co-well-powered relative to the full subcategory* $\mathcal{B}$. *Let* $\mathcal{B}$ *be closed under the formation of products and* $M^{\#}$*-subobjects. Then* $\mathcal{B}$ *is epi-reflective in* $\mathcal{C}$.

Proof. The argument used by Isbell in his proof of Freyd's theorem in [4], p. 1276, is applicable here.

Proof of Theorem 1.2. Let $\mathcal{B}$ be the full subcategory of $\mathcal{C}$ consisting of all objects X such that for every morphism $f : A_i \to X$ there is a morphism $g : B_i \to X$ with $g e_i = f$. Note that g is uniquely determined since $e_i \in E$.

It is easily verified that $\mathcal{B}$ is closed under the formation of products and (in view of Proposition 2.1) of $M^{\#}$-subobjects. Since $\mathcal{C}$ is co-well-powered, $\mathcal{B}$ is epi-reflective.

3. The span of a set of objects

DEFINITION. Let S be a set of objects of $\mathcal{C}$. Let $\mathcal{B}$ be the full subcategory consisting of all $M^{\#}$-subobjects of products of indexed families of objects of S. Then $\mathcal{B}$ is the *span* of S in the category $\mathcal{C}$.

In the rest of this section, $\mathcal{B}$ shall be assumed to be the span of a set S, of objects.

PROPOSITION 3.1. *$\mathcal{B}$ is closed under the formation of products and $M^{\#}$-subobjects and $\mathcal{C}$ is co-well-powered relative to $\mathcal{B}$. Thus $\mathcal{B}$ is epi-reflective in $\mathcal{C}$.*

Proof. $\mathcal{B}$ is obviously closed under the formation of products of $M^{\#}$-subobjects in view of the corollary to Theorem 2.2 and the fact that $M^{\#}$ is composition closed.

Let X be a given object of $\mathcal{C}$. Let U be the set $\bigcup\{\text{Hom }(X, Y) \mid Y \in S\}$ and let Q be the family of quotient objects of X which lie in $\mathcal{B}$. To prove Q is a set we shall show that the following function $\phi : 2^U \to Q$ is onto: if

$$\{f_\alpha : X \to Y_\alpha\} \in 2^U,$$

let $f : X \to \prod Y_\alpha$ be determined by $p_\alpha f = f_\alpha$, and define $\phi(\{f_\alpha\}) = f_0$, the epimorphism part of the factorization $f = f_1 f_0$ for which $f_1 \in M^{\#}$.

Let $g : X \to Z$ represent an element of Q. Since $Z \in \mathcal{B}$ there is an extremal monomorphism $h : Z \to P = \prod\{Y_i \mid i \in I\}$ where $Y_i \in S$ for all $i \in I$. Write $f = hg$ and $f_i = p_i f$ where p_i is the i-th projection. Define an equivalence relation on the set I by $i \sim j$ iff $f_i = f_j$ and let $\{I_\alpha \mid \alpha \in A\}$ be the set of equivalence classes. Let $\pi : I \to A$ be the natural projection for which $\pi_i = \alpha$ iff $i \in I_\alpha$ and let $s : A \to I$ be any choice function, i.e. a function such that $\pi \circ s =$ identity on A. Define $Y_\alpha = Y_{s\alpha}$, $T = \prod\{Y_\alpha \mid \alpha \in A\}$, $p_\alpha : T \to Y_\alpha$ the projections and $f_\alpha = f_{s\alpha}$. Then π, s determine maps $d : T \to P$, $r : P \to T$ by $p_i d = p_{\pi i}$, $p_\alpha r = p_{s\alpha}$. Since $\pi \circ s$ is the identity, $rd = 1_T$. Thus $d \in M^{\#}$. Let $\bar{f} : X \to T$ be defined by $p_\alpha \bar{f} = f_\alpha$, and let $\bar{f} = \bar{h}\bar{g}$ be a factorization with $\bar{g} \in E$ and $\bar{h} \in M^{\#}$. Then $p_i d\bar{f} = p_{\pi i}\bar{f} = f_{\pi i} = f_i = p_i f$ for all $i \in I$ so that $d\bar{f} = f$. Substituting, we have $f = (d\bar{h})\bar{g}$, $\bar{g} \in E$, $d\bar{h} \in M^{\#}$ (as $M^{\#}$ is composition closed), hence there is an equivalence e such that $eg = \bar{g}$. Thus $\bar{g}$ represents the same element of Q as g. Furthermore $\{f_\alpha\} \in 2^U$, by definition of $\sim$, and $\phi(\{f_\alpha\}) = \bar{g}$.

Proof of Theorem 1.1. We shall let S be the set $\{B_i\}$ and let $\mathcal{B}$ be the span of S in $\mathcal{C}$. Then $\mathcal{B}$ is epi-reflective. In view of Lemma 3.3 of [7], it is easily seen that $\mathcal{B}$ is well-powered and satisfies all of the other assumptions that we have made for $\mathcal{C}$. Thus Theorem 2.2 is applicable and there exists a bicategory structure $(M^{\#}_{\mathcal{B}}, E_{\mathcal{B}})$ on the category $\mathcal{B}$.

We shall let $\mathcal{A}$ be the span of S in the category $\mathcal{B}$. Thus an object of $\mathcal{A}$ is an $M^{\#}_{\mathcal{B}}$-subobject of a product of members of S. Since $\mathcal{A}$ is epi-reflective in $\mathcal{B}$, by Proposition 3.1, it easily follows that $\mathcal{A}$ is a reflective subcategory of $\mathcal{C}$.

It remains to show that each $e_i : A_i \to B_i$ is a reflection map for $\mathcal{A}$. Let e_i be fixed. For simplicity let $e = e_i$, $A = A_i$ and $B = B_i$. Let $g : A \to Z$ be given where $Z \in \mathcal{A}$. We must show that there is a unique morphism $s : B \to Z$ for which $se = g$.

We shall prove uniqueness first. Assume that $re = se$ where $r, s : B \to Z$. Since $Z \in \mathfrak{A}$ there is a monomorphism $m : Z \to P = \prod \{Y_t\}$ such that $m \in M^{\#}_{\mathfrak{B}}$ and $Y_t \in S$ for all t. Let $p_t : P \to Y_t$ denote the projection morphism for each t. Then $p_t\, mre = p_t\, mse$ for all t. This implies that $p_t\, mr = p_t\, ms$, for all t, in view of the hypothesis about unique factorizations (note that $Y_t = B_j$ for some j since $Y_t \in S$). This last equation implies that $mr = ms$ and so $r = s$ as m is a monomorphism.

Before proving the existence of s, we shall make two observations. Let $m : Z \to P$ be as above. We factor $e = e_1 e_0$ so that $e_0 : A \to C$ is in $E_{\mathfrak{C}}$ and $e_1 : C \to B$ is in $M^{\#}_{\mathfrak{C}}$. Note that $C \in \mathfrak{B}$. The above uniqueness argument can easily be modified so as to prove that $e_1 \in E_{\mathfrak{B}}$.

We also claim that $m \in M^{\#}_{\mathfrak{C}}$ as well as $M^{\#}_{\mathfrak{B}}$. We factor $m = m_1 m_0$ so that $m_1 \in M^{\#}_{\mathfrak{C}}$ and $m_0 \in E_{\mathfrak{C}}$. Since m_0 is obviously a morphism of $\mathfrak{B}$ it follows that $m_0 \in E_{\mathfrak{B}}$. This implies that m_0 is an equivalence as $m = m_1 m_0 \in M^{\#}_{\mathfrak{B}}$. Hence $m_0 \in M^{\#}_{\mathfrak{C}}$ and so $m \in M^{\#}_{\mathfrak{C}}$ as $M^{\#}_{\mathfrak{C}}$ is composition closed.

Finally we shall prove the existence of s. Let $g : A \to Z$ and $m : Z \to P$ be as above. In view of the factorization hypothesis there is morphism $h_t : B \to Y_t$ such that $h_t\, e = p_t\, mg$ for each t. Let $h : B \to P$ be the morphism for which $p_t\, h = h_t$ for all t. Clearly, $(he_1)e_0 = mg$. In view of Proposition 2.1, there is a morphism $r : C \to Z$ such that $re_0 = g$ and $mr = he_1$. Applying Proposition 2.1 to the diagram formed by m, r, h and e_1 in the category $\mathfrak{B}$, we see that there is a morphism $s : B \to Z$ such that $se_1 = r$ and $ms = h$ (since $e_1 \in E_{\mathfrak{B}}$ and $m \in M^{\#}_{\mathfrak{B}}$). Obviously s is the desired morphism. ∎

4. Examples

4.1. In [6], the question is raised as to whether every full reflective subcategory is epi-reflective in the category of Hausdorff spaces and maps. We shall resolve this question by exhibiting a reflection map which is not an epimorphism. (Note that $f : X \to Y$ is an epimorphism in this category iff $f(X)$ is dense in Y.)

Our construction depends upon the existence (proven in [1]) of two infinite Hausdorff spaces, X_1 and X_2 which admit no non-trivial continuous functions $f : X_i \to X_j$ where i and j are 1 or 2. (The trivial functions are the constants and the identity functions, in case $i = j$.)

Using these spaces, we first choose $x_1 \in X_1$ and $x_2 \in X_2$. We let $B = X_1 \times X_2$ and let A be the subset $X_1 \times \{x_2\} \cup \{x_1\} \times X_2$. We give A the relative topology. Let $e : A \to B$ be the obvious injection. Then e is not an epimorphism but is a reflection map in view of Theorem 1.1.

4.2. It is well known (e.g. see [3]) that every Hausdorff space that is *well-connected* (i.e. connected, locally arcwise connected and semi-locally simply connected) admits a universal covering space. In the category of well-connected Hausdorff spaces with base points, the universal covering maps are coreflection maps (or the duals of reflection maps) which show that the subcategory of simply connected spaces is coreflective. It would be interesting to know whether this coreflection can be extended to the category of all Hausdorff spaces with base points. In other words, is the class of all universal covering maps a subclass of the class of coreflection maps for some coreflective subcategory?

The dual of Theorem 1.1 only shows that any *set* of universal covering maps is such a subclass. The dual of Theorem 1.2 is not immediately applicable since the universal covering maps are not always monomorphisms. To remedy this flaw we pass to the category, $\mathcal{C}$, of arcwise connected Hausdorff spaces with base points. This category has products and coproducts and is well-powered. (For if $f : (A, a) \to (B, b)$ is a monomorphism then for each $x \in A$ we can choose $g_x : [0, 1] \to A$ such that $g_x(0) = a$ and $g_x(1) = x$. Since f is left-cancellable, g_x, and hence x, is determined by $fg_x : [0, 1] \to B$. Hence the cardinal of A is bounded by the cardinal of $B^{[0,1]}$.)

It follows that Theorem 1.2 is applicable to $\mathcal{C}$ since the universal covering maps are monomorphisms in $\mathcal{C}$. Thus there is at least one full coreflective subcategory $\mathcal{B}$ of $\mathcal{C}$ such that the coreflection maps for $\mathcal{B}$ include all of the universal covering maps. Thus every object $X \in \mathcal{C}$ has a coreflection, which can be regarded as a pseudo-universal covering space. (Incidentally we note that the restriction to arc-wise connected spaces is somewhat artificial since one can always work with the arc-component of the base point of a given Hausdorff space.)

The following questions remain: Is $\mathcal{B}$ uniquely determined? Is every member of $\mathcal{B}$ simply connected? We plan to discuss this example further in another paper.

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