ON THE MULTIPLICATIVE EXTENSION PROPERTY II

BY

R. AND S. CLEVELAND

1. Introduction

Let B be a commutative Banach algebra. A subspace M of B has the multiplicative extension property (m.e.p.) if every linear functional on M of norm at most one is the restriction to M of a multiplicative linear functional on B. (See [1], [2] and [6].) Thus a subspace with the m.e.p. has the property that its conjugate space consists of scalar multiples of multiplicative functionals. This suggests the following generalization.

DEFINITION. A subspace M of B has the generalized m.e.p. if for some $\alpha > 0$, every linear functional on M of norm at most α is the restriction to M of a multiplicative linear functional on B. (In this case we say M has the α-m.e.p. If $\alpha = 1$, we just say M has the m.e.p.)

In [1] we considered certain examples of subspaces with the m.e.p. The purpose of this paper is to determine circumstances under which there exist subspaces with the generalized m.e.p. The basic general result is Theorem 2.2 which gives necessary and sufficient conditions for the existence of such a subspace. Theorem 2.3 gives a sufficient condition that we have found useful in construction of examples. These conditions were inspired by the construction in [1] of a subspace of the disc algebra with the m.e.p.

In Section 3 we investigate the generalized m.e.p. in function algebras on compact metric spaces and in the algebras $L^1(G)$, $M(G)$, and H^∞. We give an example of such a subspace of $M(G)$ that is different in nature from the original example of Hewitt and Kakutani [2].

The following notation is used throughout this paper. D is the open unit disc, Ω is the closed unit disc, and I is an index set. Ω^I denotes the functions on I into Ω with the product topology, and π_i is the projection of Ω^I into its i^{th} coordinate. δ_i is the element of Ω^I with $\delta_i(j) = 0$ if $i \neq j$ and $\delta_i(i) = 1$. If F is a subset of a Banach space E, c.l.s. F denotes the closed linear span of F.

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2. Generalities

Throughout this section B is a commutative Banach algebra with maximal ideal space $\mathfrak{M}_B = \mathfrak{M}$. We begin with a simple lemma giving conditions under which the generalized m.e.p. is preserved under isomorphisms.

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LEMMA 2.1. *Suppose M_1 has the α_1-m.e.p. in B_1, T is a continuous linear operator of M_1 into B, and η is a homomorphism of B into B_1 such that $\eta \circ T(f) = f$ for all $f \in M_1$. Then $T(M_1)$ has the α-m.e.p. in B, where $\alpha = \alpha_1/\| T \|$.*

Proof. Let $M = T(M_1)$ and $\alpha = \alpha_1/\| T \|$. If $L \in M^*$ and $\| L \| \leq \alpha$, then $\| T^*L \| \leq \alpha_1$, so there exists $h_1 \in \mathfrak{M}_{B_1}$ such that $T^*L(f) = \hat{f}(h_1)$ for all $f \in M_1$. Let $h = \eta^*(h_1)$. Then $h \in \mathfrak{M}_B$ and for $Tf \in M$,

$$L(Tf) = T^*L(f) = \hat{f}(h_1) = (\eta \circ Tf)^\wedge(h_1) = (Tf)^\wedge(h).$$

THEOREM 2.2. *B has an infinite-dimensional subspace with the generalized m.e.p. if and only if the following conditions hold.*

(a) *There is a subset $X \subset \mathfrak{M}$, an infinite set I, and a continuous map φ of X onto a compact convex symmetric subset of Ω^I which contains the functions δ_i, $i \in I$.*

(b) *If $M^\wedge = $ c.l.s. $\{\pi_i \circ \varphi : i \in I\}$, there is a continuous operator $T : M^\wedge \to B$ such that $(Tg)^\wedge \mid X = g$ for all $g \in M^\wedge$.*

Proof. Suppose M_1 is an infinite-dimensional subspace of B with the α-m.e.p. Let

$$\{f_i : i \in I\}$$

be a maximal topologically free subset of M_1 (i.e. $f_i \notin$ c.l.s. $\{f_j : i \neq j \in I\}$) such that $\| f_i \| = 1/\alpha$ for all $i \in I$ (see [8, Propositions 1 and 2] and [9, p. 221]). Clearly I is an infinite set. Let

$$M = \text{c.l.s. } \{f_i : i \in I\}$$

and let

$$X = \{m \in \mathfrak{M} : |f(m)| \leq \alpha \| f \| \quad \text{for all} \quad f \in M\}.$$

Then X is a subset of S_α, the closed ball of radius α in M^*.

For $L \in S_\alpha$ and $i \in I$, let $\Phi(L)(i) = L(f_i)$. Then Φ is continuous in the weak* topology of S_α into Ω^I. Hence $\Phi(S_\alpha)$ is compact. Let $K = \Phi(X)$ and $\varphi = \Phi \mid X$.

By the Hahn-Banach theorem, M has the α-m.e.p. in B. Therefore $K = \Phi(S_\alpha)$, and it is easy to check that K is compact, convex, and symmetric in Ω^I. Since $\{f_i : i \in I\}$ is topologically free, for each $i \in I$ there is an $L \in S_\alpha$ such that $L(f_i) = \alpha \| f_i \| = 1$ and $L(f_j) = 0$ for $j \neq i$. Thus $\delta_i \in K$ for all $i \in I$, and (a) holds.

If $f \in M$, we can choose $x_0 \in X$ such that $|f(x_0)| = \alpha \| f \|$. Thus

$$\| f \| \leq (1/\alpha) \| \hat{f} \mid X \|.$$

Also $\hat{f}_i \mid X = \pi_i \circ \varphi$ for each $i \in I$. Let $\nu(f) = \hat{f} \mid X$ for $f \in M$, and let $M^\wedge = \nu(M)$. Then ν maps M bicontinuously onto $M^\wedge$. Taking $T = \nu^{-1}$, we have (b).

Conversely, suppose (a) and (b) hold. Let $K = \varphi(X)$ and $M = T(M^\wedge)$. Then M is bicontinuously isomorphic to M and is the closed linear span of the linearly independent family

$$\{T(\pi_i \circ \varphi) : i \in I\}.$$

Write $f_i = T(\pi_i \circ \varphi)$.

Since K is a compact convex symmetric subset of the Banach space of bounded functions on I, it follows from the result of Phelps [6] that the space of all continuous linear functions on K has the m.e.p. in $C(K)$. If

$$\mathfrak{L} = \text{c.l.s.}\ \{\pi_i \mid K : i \in I\},$$

then $\mathfrak{L}$ has the m.e.p. in $C(K)$, and $M^\wedge = \{l \circ \varphi : l \in \mathfrak{L}\}$. If $l \in \mathfrak{L}$,

$$\| l \|_K = \sup \{| l \circ \varphi(x)| : x \in X\} = \| l \circ \varphi \|_x$$

because φ maps X onto K. Thus $\tau : l \rightarrow l \circ \varphi$ is an isometry of $\mathfrak{L}$ onto $M^\wedge$. Suppose λ is a linear functional on $M^\wedge$ with $\| \lambda \| \leq 1$. Then $\tau^*\lambda \in \mathfrak{L}^*$ and $\| \tau^*\lambda \| \leq 1$, so there is some $k \in K$ such that $(\tau^*\lambda)(l) = l(k)$ for all $l \in \mathfrak{L}$. Because φ is onto we can choose $x \in X$ such that $\varphi(x) = k$. Thus $\lambda(l \circ \varphi) = (\tau^*\lambda)(l) = l(k) = l \circ \varphi(x)$ for all $l \in \mathfrak{L}$. Hence $M^\wedge$ has the m.e.p. in $C_b(X)$, the bounded continuous functions on X.

Now $T : M^\wedge \rightarrow B$, and we have a homomorphism η of B into $C_b(X)$ defined by $\eta(f) = \hat{f} \mid X$, so that $\eta \circ Tf = f$ for all $f \in M^\wedge$. Thus by Lemma 2.1, M has the α-m.e.p. in B, where $\alpha = 1/\| T \|$.

To apply Theorem 2.2 to a given Banach algebra, one must not only be able to construct the subspace $M^\wedge$ with the m.e.p. in $C(X)$ for some $X \subset \mathfrak{M}$, but also to obtain a linear operator $T : M^\wedge \rightarrow B$ which is continuous. In Theorem 2.3 we have an alternative method that shifts the emphasis to a better choice of X.

THEOREM 2.3. *Let $X \subset \mathfrak{M}$ and suppose there is a continuous map φ of X onto Ω^I such that for each $i \in I$ there is some $f_i \in B$ with $\hat{f}_i \mid X = \pi_i \circ \varphi$, and $\| f_i \| \leq 1/\alpha \| \hat{f}_i \mid X \|$. Then* c.l.s. $\{f_i : i \in I\}$ *has the α-m.e.p. in B.*

Proof. Let $M =$ c.l.s. $\{f_i : i \in I\}$. Suppose $L \in M^*$ and $\| L \| \leq \alpha$. Then $| L(f_i)| \leq 1$ for all $i \in I$. If $\sigma \in \Omega^I$ is such that $\sigma(i) = L(f_i)$, then for $x \in \varphi^{-1}(\{\sigma\})$ we have $\hat{f}_i(x) = \sigma(i) = L(f_i)$ for all $i \in I$. Since this holds for the generators of M, $L(f) = \hat{f}(x)$ for all $f \in M$.

It should be noted that the functions $p_i = \pi_i \circ \varphi$ in the above proof have the property that for any finite $J \subseteq I$ and any complex numbers α_j, $j \in J$,

$$\| \textstyle\sum_{j \in J} \alpha_j p_j \| = \textstyle\sum_{j \in J} | \alpha_j |$$

Thus the mapping $p_i \rightarrow f_i$ has an extension to all of $M^\wedge =$ c.l.s. $\{p_i : i \in I\}$ that is still of norm at most $1/\alpha$. Thus condition (b) of Theorem 2.2 is also satisfied under the hypotheses of Theorem 2.3.

3. Examples

In this section we consider certain standard Banach algebras. The question is whether or not a given algebra has a subspace with the generalized m.e.p

THEOREM 3.1. *If X is an uncountable compact metric space and A is any uniformly closed subalgebra of $C(X)$ that separates points and contains the constants, then A contains an infinite-dimensional subspace with the m.e.p.*

Proof. By Pełczyński [5] there exists a closed uncountable subset S_0 of X and a linear operator of extension $T_1 : C(S_0) \to A$ with $\| T_1 \| = 1$ (and $(T_1 f) | S_0 = f$). Also S_0 contains a subset P homeomorphic to the Cantor ternary set. By the theorem of Pełczyński in [4], there is a linear operator of extension $T_2 : C(P) \to C(S_0)$ with $\| T_2 \| = 1$. Thus $T = T_1 T_2$ is a linear operator of extension of $C(P)$ into A. Let I be any countable index set and let φ be a continuous map of P onto Ω^I. By Theorem 2.3 c.l.s. $\{\pi_i \circ \varphi : i \in I\} = M_1$ has the m.e.p. in $C(P)$. Since T is an isometry of M_1 into A and since the homomorphism $\eta(F) = F \mid P$, for $F \in A$, satisfies $(\eta \circ T)f = f$ for $f \in M_1$, we see by Lemma 2.1 that $T(M_1)$ has the m.e.p. in A.

This last theorem includes Theorem 4 of [1] as a special case. This provides subspaces of the disc algebra $A(D)$ with the m.e.p. Evidently, these same spaces serve as subspaces of $H^\infty(D)$, the bounded analytic functions in D, with the m.e.p. One may ask whether there are any subspaces of $H^\infty(D)$ that are different from these. We show next that there are such subspaces.

THEOREM 3.2. *There exist infinite-dimensional subspaces M of $H^\infty(D)$ with the m.e.p. such that $M \cap A(D) = (0)$.*

Proof. Let ψ be the homeomorphism of D into the fiber $\mathfrak{M}_1$ as constructed in [3, p. 166 ff.]. The map $f \to \hat{f} \circ \psi$ is a homomorphism of H^∞ onto itself that maps $A(D)$ into the constant functions. There is associated with ψ a function $h \in H^\infty$ with the property that $\hat{h} \circ \psi(\lambda) = \lambda$ for all $\lambda \in D$.

Let M_1 be an infinite-dimensional subspace of $A(D)$ with the m.e.p. Let $T : M_1 \to H^\infty$ be the map $T(F) = F \circ h$. Then T is an isometry of M_1 into H^∞ such that $(TF)^\wedge \circ \psi = F$ for all $F \in M_1$. Then, by Lemma 2.1, the space $M = T(M_1)$ has the m.e.p. in H^∞. Also $M \cap A(D) = (0)$ because TF is mapped by ψ into the non-constant $F \in M_1$, if $F \neq 0$. In fact, if $F \in M_1$ and $F \neq 0$, there must exist for every $\zeta \in D$ a linear functional $L_\zeta \in M_1^*$ such that $L_\zeta(F) = \zeta \| F \|$, and $\| L \| \leq 1$. Since M_1 has the m.e.p. there is some $z_\zeta \in \Omega$ such that $F(z_\zeta) = \zeta \| F \|$. Thus M_1 cannot contain any non-zero constants, and M cannot contain any non-zero elements of $A(D)$.

Next we apply the theorems of §2 to algebras of measures on groups.

THEOREM 3.3. *Let G be a locally compact abelian group. Then a necessary and sufficient condition that $L^1(G)$ have a subspace with the generalized m.e.p. is that G not be compact.*

Proof. Suppose G is compact and M is a subspace of $L^1(G)$ with the α-m.e.p. Let $f \in M$, $\| f \| = 1$. If Γ is the dual of G, then Γ is discrete and $\hat{f}$ vanishes at infinity on Γ. Thus there is a finite subset F of Γ such that $| \hat{f}(\gamma) | < \alpha/2$ for

$\gamma \notin F$. Let L be a linear functional on M with $\| L \| = \alpha = L(f)$. For each $t \in (\frac{1}{2}, 1)$, tL is a linear functional on M, $\alpha/2 \leq \| tL \| \leq \alpha$, and, if $t_1 \neq t_2$, $t_1 L \neq t_2 L$. Since M has the α-m.e.p. there corresponds to each $t \in (\frac{1}{2}, 1)$ a $\gamma_t \in \Gamma$ such that $tL(g) = \hat{g}(\gamma_t)$ for $g \in M$, and this correspondence is 1-1. In particular, $\hat{f}(\gamma_t) = \alpha t > \alpha/2$, so that $\gamma_t \in F$ for all $t \in (\frac{1}{2}, 1)$. But this says that F is uncountable, which is a contradiction.

If G is not compact, then the dual, Γ, of G contains a perfect Helson set K which is homeomorphic to the Cantor ternary set [7, Theorem 5.6.6]. If I is countable, there is a continuous map φ of K onto Ω^I. Because K is a Helson set, there exists for each $i \in I$ an $f_i \in L^1(G)$ with $\hat{f}_i \mid K = \pi_i \circ \varphi$ and $\| f_i \| \leq \beta \| \hat{f}_i \mid K \|$, where β is a constant depending only on K. By Theorem 2.3, c.l.s. $\{f_i : i \in I\}$ has the $1/\beta$ m.e.p.

Now we pose the same question for $M(G)$. If G is discrete, then $M(G) = L^1(G)$, and the question of existence of subspaces with the generalized m.e.p. is answered by Theorem 3.3. If G is not discrete, the answer is always affirmative—there always exist subspaces with the m.e.p. This is the result of Hewitt and Kakutani [2]. A different result in the same vein is the following.

THEOREM 3.4. *If G is a compact abelian group with an uncountable dual Γ, then there exists an infinite dimensional subspace of $M(G)$ with the α-m.e.p. for some $\alpha > 0$ and having the property that every linear functional of norm at most α is a continuous character of G.*

Proof. By modifying the procedure of [7, 5.7.6] one can obtain an uncountable Sidon set E in Γ. Since E is discrete and uncountable, there is a continous map φ of E onto Ω^I, where I is countable. The functions $\pi_i \circ \varphi$ are continuous and bounded on E. Since E is a Sidon set there exists for each $i \in I$ a measure $\mu_i \in M(G)$ such that

$$\hat{\mu}_i \mid E = \pi_i \circ \varphi \quad \text{and} \quad \| \mu_i \| \leq B \| \hat{\mu}_i \mid E \|_\infty ,$$

where B is a constant depending only on E. By Theorem 2.3, $M =$ c.l.s. $\{\mu_i : i \in I\}$ has the $1/B$-m.e.p. in $M(G)$. Moreover, if $L \in M^*$ and $\| L \| \leq 1/B$, there exists a $\gamma \in E$ such that $L(\mu) = \hat{\mu}(\gamma)$ for all $\mu \in M$.

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University of Illinois
Urbana, Illinois