ON CHARACTERISTIC VECTOR FIELDS[1]

BY

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Introduction

In an attempt to formulate geometrically, and to generalize, the notion of convexity introduced by Hormander in [V, Chapt. VIII] for a partial differential operator we have found it necessary to consider the characteristic vector field of a first order partial differential equation in a more general context than usual. This paper is devoted to this formulation of characteristic vector fields.

The classical notion of a characteristic vector field can be formulated in the following way. Consider a $(p + 1)$-dimensional C^∞ real manifold M and the bundle $G_p(M)$ of p-planes over M, i.e. $G_p(M)$ consists of all (m, P) where m is any point of M and P is any p-dimensional subspace of the tangent space to M at m. One makes $G_p(M)$ into a C^∞ manifold, and a bundle over M, in a natural way. There are certain natural 1-forms on $G_p(M)$ that we call lift-forms and any two of these differ only by a factor which is a C^∞ function. If F is a real-valued C^∞ function defined on an open subset U of $G_p(M)$ and if we choose any one of these lift-forms, λ, then there is a natural way to associate with F a unique C^∞ vector field V, defined on U, which is called the characteristic vector field of F, relative to λ. If a different λ were used then a different vector field would be obtained but it would only differ by a factor which is a C^∞ function. Using V one can solve easily the non-characteristic Cauchy problem for the partial differential equation defined by F, and it is for this purpose that this V was introduced.

This is one way of phrasing the classical reduction of such a partial differential equation to an ordinary differential equation. Hormander, in [V], has also used this V for expressing a convexity condition, altho he has not described V in these terms. The aim of this paper is to express and understand V in a sufficiently general context to give a geometric interpretation to Hormander's convexity condition. We shall not discuss convexity in this paper, however.

We now explain briefly the more general context in which we shall consider this V. Consider a d-dimensional manifold M, where $d > p + 1$ and the bundle $G_p(M)$ of p-planes over M. So $G_p(M)$ now contains, in some sense, "many more" p-planes than in the previous case where $d = p + 1$. Let F be a C^∞ real-valued function defined on an open subset U of $G_p(M)$. There is no notion of characteristic vector field in this case. But if one considers a $1 : 1$ non-singular map ϕ of $R^{p+1} \to M$ it induces, in an obvious way (via its differential) a map ϕ' of $G_p(R^{p+1}) \to G_p(M)$. The image of $G_p(R^{p+1})$ will be a sub-

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manifold of $G_p(M)$ which we may consider as a "copy" of $G_p(R^{p+1})$ and F, considered only on this copy, will have a characteristic vector field on the copy. One can say this a little more carefully by considering $F \circ \phi'$, then its characteristic vector field W, then define $V = \phi'_* W$ (the choice of lift form must also enter when we come to a precise description of this). For every such ϕ we obtain (using the same F) such a V and, if two different ϕ agree on a neighborhood of a point of R^{p+1} then the corresponding V's will agree on a neighborhood. So the question arises: What infinitesimal data about ϕ at a point are needed to determine V? Then one wishes to express V in terms of the relevant infinitesimal data without reference to any such ϕ. The relevant infinitesimal data to determine V (still using a fixed F, and depending on certain consideration of lift forms) is an integrable point of $G_{p+1}(G_p(M))$, i.e. for each integrable point $(m, P, Q) \in G_{p+1}(G_p(M))$ we obtain a $V(m, P, Q)$ in the tangent space to $G_p(M)$ at (m, P), this V depending on the given F. So the object of this paper is first to give the needed information about integrable points of $G_{p+1}(G_p(M))$, and their relation to lift forms, and then to prove the existence of this V, depending on a given F defined on a subset of $G_p(M)$, which maps the manifold of these integrable points into the tangent bundle to $G_p(M)$. We mention that by using the bundle of lines over $G_p(M)$ instead of the tangent bundle we could state a theorem in which lift forms would not need to be mentioned, but that is an obvious consequence of our theorem.

1. Integrable points of $G_q(G_p(M))$, if $q > p$

In this section we discuss, and prove equivalent, several notions of integrability for elements of $G_q(G_p(M))$. In all this M will be a fixed manifold of dimension d and p, q will be integers with $0 < p \leq q \leq d$. The case where $p = q$ has been discussed in [I] and we make use of facts proved there, including, in particular, the lift forms discussed there.

DEFINITION. An $(m, P, Q) \in G_q(G_p(M))$ is *integrable* iff $\pi_* Q$ has dimension q and Q contains a p-plane Q_0 such that (m, P, Q_0) is an integrable element of $G_p(G_p(M))$, in the sense of [I]. We denote the set of integrable elements of $G_q(G\ (M))$ by $I_{q,p}(M)$. (This π is the projection map of $G_p(M) \to M$.)

In [I] we denoted the integrable elements of $G_p(G_p(M))$ by $IG_p^2(M)$ but here we shall write $I_{p,p}(M)$. We recall from [I] that one definition of an integrable element (m, P, Q_0) of $G_p(G_p(M))$ was the following: $\pi_* Q_0$ has dimension p and all C^∞ lift forms λ and their $d\lambda$ vanish on Q_0. We shall characterize, in Lemma 1.1 below, the integrable elements of $G_q(G_p(M))$ in terms of lift forms but for this we need a definition. Consider any $(m, P, Q) \in G_q(G_p(M))$. We define

$$L(Q) = [\lambda \mid Q : \lambda \text{ is a lift form at } (m, P)]$$

i.e. $L(Q)$ consists of all restrictions of all lift forms to Q. We recall from [I] that a lift form λ at (m, P) is a linear function on the tangent space to $G_p(M)$

at (m, P) with the property that $\lambda(t) = 0$ if $\pi_* t \in P$. So $L(Q)$ is the set of all restrictions of such λ to Q (it would be more explicit to write $L(m, P, Q)$).

LEMMA 1.1. *Let $(m, P, Q) \in G_q(G_p(M))$. Then (m, P, Q) is integrable iff all the following three conditions hold:*

(i) *$\pi_* Q$ has dimension q,*

(ii) $\dim L(Q) = q - p$,

(iii) *if α is any C^∞ lift form and s, t are any tangent vectors to $G_p(M)$ at (m, P) then $\lambda(s) = \lambda(t) = 0$ for all $\lambda \in L(Q)$ implies $d\alpha(s, t) = 0$.*

Proof. First suppose (m, P, Q) is integrable. Then (i) holds by definition. Let Q_0 be an integrable p-dimensional subspace of Q. Because each $\lambda \in L(Q)$ vanishes on Q_0 each such λ gives rise to a unique linear function on Q/Q_0, thus giving us an isomorphism A of $L(Q)$ into the dual space of Q/Q_0. We now show

(a) if $t' \in Q/Q_0$, with $t' \neq 0$, then there exists a λ' in $AL(Q)$ such that $\lambda'(t') \neq 0$.

This clearly implies $AL(Q)$ is the dual space to Q/Q_0, so $AL(Q)$ has dimension $q - p$, and because A is an isomorphism this will imply that $L(Q)$ has dimension $q - p$, proving (ii). To prove (a) it is sufficient to prove

(a′) if $t \in Q$, $t \notin Q_0$, then there exists a $\lambda \in L(Q)$ such that $\lambda(t) \neq 0$.

If $t \in Q$, $t \notin Q_0$, then, because π_* has dimension q, π_* is an isomorphism on Q, so $\pi_* t \notin P$. Because the set of lift forms at (m, P) is the annihilator of $\pi^{-1}P$ this shows some lift form does not annihilate t, proving (a). Hence (ii) is proved. We have incidentally proved that

(b) $Q_0 = [t \in Q \mid \lambda(t) = 0 \text{ for all } \lambda \in L(Q)]$.

Because of (b) and the characterization of integrable elements of $G_p(G_p(M))$ by lift forms (see [I]) we have (iii).

Now suppose (i), (ii), (iii) hold. So $\pi_* Q$ has dimension q. Define Q_0 by (b). Because $\dim Q = q$ and $\dim L(Q) = q - p$ we see that $\dim Q_0 = p$. Because all lift forms at (m, P) vanish on Q_0, by definition, and because we are assuming (iii) we see, by the characterization of integrable elements in terms of lift forms, that (m, P, Q_0) is integrable.

In [I] the term "integrable" was used for those elements of $G_p(G_p(M))$ that were capable of being "second order tangent" to a p-dimensional submanifold of M. We shall show below that the elements of $I_{q,p}(M)$ are the elements of $G_q(G_p(M))$ that are capable of being "second order tangent" but in a more complicated sense. This requires some coordinate calculations whose purpose is the following. If ϕ is a non-singular C^∞ map of a manifold L into a manifold M then ϕ induces, thru its differential ϕ_*, a map ϕ' of $G_p(L) \to G_p(M)$. The object of this calculation is: given the Jacobian of ϕ relative to given coordinate systems of L and M, to find the Jacobian of ϕ' relative to the associated coordinate systems of $G_p(L)$ and $G_p(M)$.

Let L and M be manifolds of dimensions c and d, and ϕ a non-singular $1:1$ map of a neighborhood of $l \in L$ into a neighborhood of $m \in M$. Let $\{v_a\}$ be a coordinate system of L at l and $\{x_b\}$ a coordinate system of M at m. We further assume that the domain of ϕ is the domain of the $\{v_a\}$ and that ϕ maps this domain inside the domain of the $\{x_b\}$. Let J be the Jacobian of ϕ relative to these coordinate systems, i.e.

$$J = (J_{ba}), \qquad J_{ba} = \partial(x_b \circ \phi)/\partial x_a$$

$(1 \leq a \leq c, 1 \leq b \leq d)$. So the J_{ba} are functions whose domain is the domain of ϕ. Because ϕ is assumed C^∞ and non-singular its differential, ϕ_*, induces a map, that we denote by ϕ', of π^{-1} (domain of ϕ) into $G_p(M)$, where π, here and throughout this paper, denotes the projection of a Grassmann bundle onto onto its base manifold. That is, ϕ' is defined by

$$\phi'(l, P) = (\phi l, \phi_* P).$$

Let I be any subset of $\{1, \cdots, c\}$ with p elements and K any subset of $\{1, \cdots, d\}$ with p elements. The $\{v_a\}$ and I give rise to a coordinate system $\{w_a, w_r^i\}$ of $G_p(L)$ whose domain we denote by Q_I; and the $\{x_b\}$ and K give rise to a coordinate system $\{y_b, y_s^j\}$ of $G_p(M)$ whose domain we denote by Q_K. These were defined in [II] and for the case where $I = K = \{1, \cdots, p\}$ were defined in [I] (the definition for general I, K being only trivially different). Here $1 \leq a \leq c$, $i \in I$, $r \in I^c$, $1 \leq b \leq d$, $j \in K$, $s \in K^c$ (I^c being the complement of I in $\{1, \cdots, c\}$ and K^c being the complement of K in $\{1, \cdots, d\}$). Then ϕ' has a Jacobian J' relative to these associated coordinate systems and we wish to calculate J'. $J' = (J'_{\beta\alpha})$ will be a matrix of functions with β running thru the set

$$\{1, \cdots, d\} \cup [(s, j) : j \in K, s \in K^c]$$

and α running thru the set

$$\{1, \cdots, c\} \cup [(r, i) : i \in I, r \in I^c].$$

These functions are defined by

$$\begin{aligned} J_{ba} &= \partial(y_b \circ \phi')/\partial w_a \\ J_{b,(r,i)} &= \partial(y_b \circ \phi')/\partial w_r^i \\ J_{(s,j),a} &= \partial(y_s^j \circ \phi')/\partial w_a \\ J_{(s,j),(r,i)} &= \partial(y_s^j \circ \phi')/\partial w_r^i . \end{aligned}$$

So we first need formulas for the $y_b \circ \phi'$ and $y_s^j \circ \phi'$. These have been derived in Section 1 of [II] and are not difficult to obtain so we assume them here. (To relate this discussion to that in [II] one takes our ϕ_* to be the T of [II] and our J_{ba} for the t_{ba} of [II].) To express the desired formulas we introduce the following auxiliary functions, A_{si}, B_{ij}, C_{ji}.

$$
\begin{aligned}
A_{si} &= J_{si} \circ \pi + \textstyle\sum_r (J_{sr} \circ \pi) w_r^i \\
B_{ji} &= J_{ji} \circ \pi + \textstyle\sum_r (J_{jr} \circ \pi) w_r^i \\
\textstyle\sum_i B_{ki} C_{ij} &= \delta_{kj}
\end{aligned} \tag{1.2}
$$

where $i \in I$, $r \in I^c$, $j \in J$, $k \in K$, $s \in K^c$. One has (e.g. see formulas (1.4) and (1.5) of [II])

$$Q_I \cap (\phi')^{-1}(Q_K) = [(l, P) \in G_p(L) : B \text{ is defined and } C \text{ exists}]. \tag{1.3}$$

Then the desired formulas are

$$y_b \circ \phi' = x_b \circ \phi \circ \pi, \qquad y_s^j \circ \phi' = \textstyle\sum_i A_{si} C_{ij} \tag{1.4}$$

Hence

$$
\begin{aligned}
J'_{ba} &= J_{ba} \circ \pi, \qquad J'_{b,(r,i)} = 0 \\
J'_{(s,j),a} &= \sum_i \frac{\partial A_{si}}{\partial w_a} C_{ij} + \sum_i A_{si} \frac{\partial C_{ij}}{\partial w_a} \\
J'_{(s,j),(r,i)} &= \sum_k \frac{\partial A_{sk}}{\partial w_r^i} C_{kj} + \sum_k A_{sk} \frac{\partial C_{kj}}{\partial w_r^i}
\end{aligned} \tag{1.5}
$$

where $k \in I$. We use the trivial formula for the derivative of the inverse of a matrix of functions, which says that if $BC = I$ then $C' = -CB'C$ (primes denoting differentiation). Using it the above gives

$$J'_{(s,j),a} = \sum_i \frac{\partial A_{si}}{\partial w_a} C_{ij} - \sum_{ikn} A_{si} C_{ik} \frac{\partial B_{kn}}{\partial w_a} C_{nj} \tag{1.5'}$$

where $i \in I$, $k \in K$, $n \in I$. Similarly we can rewrite the last formula in (1.5); from the definition of the A_{si} and B_{ij} we have

$$\partial A_{sn} / \partial w_r^i = (J_{sr} \circ \pi) \delta_{ni}, \qquad \partial B_{jn} / \partial w_r^i = (J_{jr} \circ \pi)\, \delta_{ni}\,.$$

Together these give

$$J'_{(s,j),(r,i)} = (J_{sr} \circ \pi) C_{ij} - \textstyle\sum_{kn} A_{sn} C_{nk} (J_{kr} \circ \pi) C_{ij} \tag{1.5''}$$

where again $k \in K$, $n \in I$. These give

$$
\begin{aligned}
\phi'_* \frac{\partial}{\partial w_a} &= \sum_b (J_{ba} \circ \pi) \frac{\partial}{\partial y_b} \\
&\quad + \sum_{sj} \left(\sum_i \frac{\partial A_{si}}{\partial w_a} C_{ij} - \sum_{ikn} A_{si} C_{ik} \frac{\partial B_{kn}}{\partial w_a} C_{nj} \right) \frac{\partial}{\partial y_s^j} \\
\phi'_* \frac{\partial}{\partial w_r^i} &= \sum_{sj} \left((J_{sr} \circ \pi) C_{ij} - \sum_{kn} A_{sn} C_{nk} (J_{kr} \circ \pi) C_{ij} \right) \frac{\partial}{\partial y_s^j} \\
&= \sum_{sj} \left(J_{sr} \circ \pi - \sum_k y_s^k J_{kr} \circ \pi \right) C_{ij} \frac{\partial}{\partial y_s^j}
\end{aligned} \tag{1.6}
$$

where, as above, $1 \leq a \leq c$, $1 \leq b \leq d$, $i \in I$, $n \in I$, $r \in I^c$, $j \in K$, $k \in K$, $s \in K^c$.

We now use these formulas to show: if λ is any lift form of $G_p(M)$ and ϕ is as above then $\phi'^*\lambda = \lambda \circ \phi'$ is a lift form of $G_p(L)$. One could prove this directly from the definitions but we prefer to prove it by a coordinate calculation because we wish later to use the coordinate formula that we obtain. So we now prove the following, in which we continue with the notation used above: if

$$\lambda_s = \sum_j y_s^j \, dy_j - dy_s$$

then

$$\phi'^*\lambda_s = \sum_r K_{sr}\Big(\sum_i w_r^i \, dw_i - dw_r\Big) \tag{1.7}$$

where

$$K_{sr} = J_{sr} \circ \pi - \sum_j (y_s^j \circ \phi')(J_{jr} \circ \pi).$$

Proof. Using (1.2), (1.4) and (1.5),

$$\begin{aligned}
\phi'^*\lambda &= \sum_j (y_s^j \circ \phi')\phi'^* \, dy_j - \phi'^* \, dy_s \\
&= \sum_j \Big(\sum_i A_{si} C_{ij}\Big) \sum_a J'_{ja} \, dw_a - \sum_a J'_{sa} \, dw_a \\
&= \sum_a \Big(\sum_{ji} A_{si} C_{ij} J'_{ja} - J'_{sa}\Big) \, dw_a \\
&= \sum_k \Big(\sum_{ji} A_{si} C_{ij} J'_{jk} - J'_{sk}\Big) \, dw_k + \sum_r \Big(\sum_{ji} A_{si} C_{ij} J'_{jr} - J'_{sr}\Big) \, dw_r \\
&= \sum_k \Big(\sum_{ji} A_{si} C_{ij}\Big(B_{jk} - \sum_r (J_{jr} \circ \pi) w_r^k\Big) - J'_{sk}\Big) \, dw_k \\
&\quad + \sum_r \Big(\sum_{ji} A_{si} C_{ij} J_{jr} \circ \pi\Big) - J_{sr} \circ \pi\Big) \, dw_r \\
&= \sum_k \Big(A_{sk} - \sum_{ijr} A_{si} C_{ij}(J_{jr} \circ \pi) w_r^k - J_{sk} \circ \pi\Big) \, dw_r \\
&\quad + \sum_r \Big(\sum_{ji} A_{si} C_{ij}(J_{jr} \circ \pi) - J_{sr} \circ \pi\Big) \, dw_r \\
&= \sum_k \Big(\sum_r (J_{sr} \circ \pi) w_r^k - \sum_{ijr} A_{si} C_{ij}(J_{jr} \circ \pi) w_r^k\Big) \, dw_k \\
&\quad + \sum_r \Big(\sum_{ji} A_{si} C_{ij}(J_{jr} \circ \pi) - J_{sr} \circ \pi\Big) \, dw_r \\
&= \sum_r \Big(J_{sr} \circ \pi - \sum_{ij} A_{si} C_{ij}(J_{jr} \circ \pi)\Big(\sum_k w_r^k \, dw_k - dw_r\Big)
\end{aligned}$$

and this proves (1.7). We note that using K_{sr} one part of (1.6) becomes

$$\phi'^* \frac{\partial}{\partial w_r^i} = \sum_{sj} K_{sr} C_{ij} \frac{\partial}{\partial y_s^j}. \tag{1.6$'$}$$

We now describe the notion of "second order tangency" for elements of $G_q(G_p(M))$. Let ϕ be a non-singular $1:1$ C^∞ map of a neighborhood U of 0 in R^q into M. Then, as above, ϕ gives rise to a map ϕ' of $G_p(U) \to G_p(M)$, defined by

$$\phi'(x, P) = (\phi(x), \phi_* P).$$

For each $t \in U$ we define $P_p(t)$, a subset of R_t^q (R_t^q denotes the tangent space to R^q at t), by

$$P_p(t) = \mathrm{sp}\left\{\frac{\partial}{\partial u_1}(t), \cdots, \frac{\partial}{\partial u_p}(t)\right\}$$

where sp denotes "the span of" and the u_i are the usual coordinate functions on R^q. Then we define

$$\tilde{\phi} : U \to G_p(M) : t \to \phi'(t, P_p(t))$$

i.e. if ψ is the map of $U \to G_p(R^q)$ defined by

$$\psi(t) = \left(t, \mathrm{sp}\left\{\frac{\partial}{\partial u_1}(t), \cdots, \frac{\partial}{\partial u_p}(t)\right\}\right)$$

then

$$\tilde{\phi} = \phi' \circ \psi.$$

Finally we define $\tilde{\phi}' : U \to G_q(G_p(M))$ by

$$\tilde{\phi}'(t) = (\tilde{\phi}(t), \tilde{\phi}_* R_t^q).$$

Definition. Let $(m, P, Q) \in G_q(G_p(M))$. Then (m, P, Q) is *second order tangent* to a q-dimensional submanifold of M iff there exists a non-singular $1:1$ C^∞ map ϕ of a neighborhood of 0 in R^q into M such that $\tilde{\phi}'(0) = (m, P, Q)$.

Remark. This notion of second order tangency is relative to q and $p < q$ and is really second order in some directions and first order in others. This imprecise statement can be made precise in, among other ways, terms of the coordinate systems introduced below for $I_{q,p}(M)$. There is another (and different) notion of second order tangency in terms of $G_p(G_q(M))$ but we shall not consider that.

We now wish to do the following things: (1) show that ϕ'_* at $(0, P_p(0))$ is determined by $\tilde{\phi}_*$ at 0; (2) express integrability in terms of coordinate systems and use that to show $I_{q,p}(M)$ is a submanifold of $G_q(G_p(M))$; (3) prove this kind of second order tangency equivalent to integrability. We now turn to the first of these.

Let ϕ and ψ be as above, with $\tilde{\phi} = \phi' \circ \psi$. Let $\{w_a, w_r^i\}$ be the coordinate system of $G_p(R^q)$ associated with the usual coordinate system $u_1, \cdots, u_q$ of R^q, using the set $I = \{1, \cdots, p\}$ as a subset of $\{1, \cdots, q\}$; thus $1 \le a \le q$, $1 \le i \le p$, $p + 1 \le r \le q$. Trivially,

$$\psi_* \frac{\partial}{\partial u_a} = \frac{\partial}{\partial w_a}$$

so to prove ϕ'_* at $(0, P_p(0))$ is determined by $\tilde{\phi}_*$ at 0 it is sufficient to prove the

$$\phi'_*\left(\frac{\partial}{\partial w_a}(0, P_p(0))\right)$$

determine the

$$\phi'_*\left(\frac{\partial}{\partial w_r^i}(0, P_p(0))\right).$$

For this let $\{x_b\}$ be any coordinate system of M at $m = \phi(0)$ such that $\phi'(0)$

lies in the domain of the associated coordinate system $\{y_b, y_s^j\}$. We are now taking $J = \{1, \cdots, p\}$ as a subset of $\{1, \cdots, d\}$, so $1 \leq b \leq d$, $1 \leq j \leq p$, $p + 1 \leq s \leq d$. Using (1.2) and (1.6) and the fact that the $w_r^i(0, P_p(0)) = 0$ we find, at $\phi'(0, P_p(0))$,

$$\phi'_* \frac{\partial}{\partial w_a}(0, P_p(0)) = \sum_b J_{ba}(0) \frac{\partial}{\partial y_b}$$

$$(1.6'') \qquad + \sum_{sj} \left(\sum_i \frac{\partial J_{si}}{\partial u_a}(0) C_{ij} - \sum_{ikn} J_{si}(0) C_{ik} \frac{\partial J_{km}}{\partial u_a}(0) C_{mj} \right) \frac{\partial}{\partial y_s^j}$$

$$\phi'_* \frac{\partial}{\partial w_r^i}(0, P_p(0)) = \sum_{sj} (J_{sr}(0) C_{ij} - \sum_{kn} J_{sn}(0) C_{nk} J_{kr}(0) C_{ij}) \frac{\partial}{\partial y_s^j}$$

where now $1 \leq a \leq q$, $1 \leq b \leq d$, $1 \leq i, j, k, n, \leq p$, $p + 1 \leq r \leq q$, $p + 1 \leq s \leq d$. From these it is clear that $\phi'_*(0)$ is determined by the

$$\phi'_* \left(\frac{\partial}{\partial w_a}(0, P_p(0)) \right).$$

We shall need below certain coordinate systems for $G_q(G_p(M))$ and we now introduce them. Let $\{x_b\}$ be any coordinate system of M, and let $\{y_b, y_s^j\}$ be the associated coordinate system of $G_p(M)$ (formed using the subset $I = \{1, \cdots, p\}$ of $\{1, \cdots, d\}$) with $1 \leq b \leq d$, $1 \leq j \leq p$, $p + 1 \leq s \leq d$. In the rest of this section these indices will always have these ranges. The $\{y_b, y_s^j\}$ now generate, by the same process, a coordinate system for $G_q(G_p(M))$ whose functions we denote by z_b, z_s^j, $\bar{z}_t^a$, z_s^{ja} where $1 \leq a \leq q$, $q + 1 \leq t \leq d$; this notation will also be kept fixed in the rest of this section. These functions are hence defined by

$$z_b = y_b \circ \pi', \qquad z_s^j = y_s^j \circ \pi'$$

$$(1.8) \qquad \frac{\partial}{\partial y_a}(m, P) + \sum_t \bar{z}_t^a(m, P, Q) \frac{\partial}{\partial y_t}(m, P) + \sum_{s,j} z_s^{ja}(m, P, Q) \frac{\partial}{\partial y_s^j}(m, P) \in Q$$

We shall refer to these $\{y_b, y_s^j\}$ and $\{z_b, z_s^j, \bar{z}_t^a, z_s^{ja}\}$ as the y and z coordinate systems associated with the x-coordinate system $\{x_b\}$.

We now use (1.6) and (1.8) to express ϕ'_* in terms of these z-coordinates in the special case where

$$(\mathrm{A}) \qquad x_a \circ \phi = u_a \quad \text{if } 1 \leq a \leq q.$$

Note that (A) implies

$$J_{ba} = \delta_{ba} \quad \text{at all points in the domain of } \phi \text{, if} \quad 1 \leq a, b \leq q.$$

Let $f_t = x_t \circ \phi$ $(q + 1 \leq t \leq d)$ and we now express (1.6) in terms of these

f_t. First, using (A), (1.2) gives

$$\begin{aligned} A_{si} &= w_s^i && \text{if } p+1 \le s \le q \\ &= \frac{\partial f_s}{\partial u_i} \circ \pi + \sum_r \left(\frac{\partial f_s}{\partial u_r} \circ \pi \right) w_r^i && \text{if } q+1 \le s \le d, \\ B_{ji} &= C_{ij} = \delta_{ij} \end{aligned}$$

and (1.4) gives (because of (A))

$$y_s^j \circ \phi' = A_{sj}\,.$$

So (1.6) becomes

$$\begin{aligned} \phi'_* \frac{\partial}{\partial w_a} &= \frac{\partial}{\partial y_a} + \sum_t \left(\frac{\partial f_t}{\partial u_a} \circ \pi \right) \frac{\partial}{\partial y_t} \\ &\quad + \sum_{t,j} \left(\frac{\partial^2 f_t}{\partial u_j\, \partial u_a} \circ \pi + \sum_r \left(\frac{\partial^2 f_t}{\partial u_r\, \partial u_a} \circ \pi \right) y_r^j \right) \frac{\partial}{\partial y} \\ \phi'_* \frac{\partial}{\partial w_r^i} &= \frac{\partial}{\partial y_r^i} + \sum_t \left(\frac{\partial f_t}{\partial u_r} \circ \pi \right) \frac{\partial}{\partial y_t^i}\,. \end{aligned} \tag{1.6A}$$

With (1.8) this shows, under assumption (A), that at $(m, P, Q) = \tilde{\phi}'(u)$,

$$\begin{aligned} z_t^a(m, P, Q) &= \frac{\partial f_t}{\partial u_a}(u) \\ z_s^{ja}(m, P, Q) &= 0 && \text{if } p+1 \le s \le q \\ &= \frac{\partial^2 f_s}{\partial u_j\, \partial u_a}(u) && \text{if } q+1 \le s \le d. \end{aligned} \tag{1.9}$$

LEMMA 1.2. *Let $\{x_b\}$ be any coordinate system of M. Let*

$$\{z_b\,,\, z_s^j\,,\, \bar{z}_t^a\,,\, z_s^{ja}\}$$

be associated z-coordinate system of $G_q(G_p(M))$ and let

$$\{w_b\,,\, w_s^j\,,\, \bar{w}_s^i\,,\, w_s^{ij}\}$$

be the coordinate system of $G_p(G_p(M))$ associated to the $\{x_b\}$. Let $(m, P, Q) \in G_q(G_p(M))$ be in the domain of this z-coordinate system and $(m, P, Q_0) \in G_p(G_p(M))$ be in the domain of this w-coordinate system. Then $Q_0 \subseteq Q$ iff both the following hold:

(a) $\bar{w}_t^i(m, P, Q_0) - \bar{z}_t^i(m, P, Q) = \sum_l \bar{w}_l^i(m, P, Q_0) \bar{z}_t^l(m, P, Q)$

(b) $w_s^{ji}(m, P, Q_0) - z_s^{ji}(m, P, Q) = \sum_l \bar{w}_l^i(m, P, Q_0) z_s^{jl}(m, P, Q)$

where $1 \le i, j \le p$, $q+1 \le t \le d$, $p+1 \le l \le q$, $p+1 \le s \le d$.

Proof. The functions of the w-coordinate system are w_a, w_s^i, $\bar{w}_s^i$, w_s^{ij} ($1 \le i, j \le p$, $p+1 \le s \le d$) and are defined (via the y-coordinates of $G_p(M)$

obtained from the x-coordinates of M) by

$$w_a = y_a \circ \pi'', \qquad w_s^i = y_s^i \circ \pi''$$

$$\frac{\partial}{\partial y_i}(m, P) + \sum_s \bar{w}_s^i(m, P, Q_0) \frac{\partial}{\partial y_s}(m, P) + \sum_{s,j} w_s^{ji}(m, P, Q_0) \frac{\partial}{\partial y_s^j}(m, P) \in Q_0$$

where π'' is the projection of $G_p(G_p(M)) \to G_p(M)$.

Consider now any (m, P, Q) in the domain of this z-coordinate system and any (m, P, Q_0) (with the same (m, P)) in the domain of this w-coordinate system. Let $f_1, \cdots, f_q$ be the elements of Q that project to

$$\frac{\partial}{\partial y_1}(m, P), \cdots, \frac{\partial}{\partial y_q}(m, P)$$

under the usual projection of the tangent space at (m, P) onto the span of those $\frac{\partial}{\partial y_a}(m, P)$ and let $e_1, \cdots, e_p$ be the elements of Q_0 that project to

$$\frac{\partial}{\partial y_1}(m, P), \cdots, \frac{\partial}{\partial y_p}(m, P)$$

under the corresponding projection onto these. Clearly $Q_0 \subseteq Q$ is equivalent to $e_i - f_i \in Q$, for $1 \leq i \leq p$, so we now consider the meaning, in terms of our coordinates, of the statement that these $e_i - f_i \in Q$. The statement that $e_i - f_i \in Q$ is equivalent to

$$e_i - f_i = \textstyle\sum_a c_i^a f_a, \qquad 1 \leq i \leq p, 1 \leq a \leq q$$

for some real numbers c_i^a. In terms of coordinates this says, at (m, P, Q),

$$\begin{aligned}\sum_s \bar{w}_s^i(m, P, Q_0) \frac{\partial}{\partial y_s}(m, P) &+ \sum_{sj} w_s^{ji}(m, P, Q_0) \frac{\partial}{\partial y_s^j}(m, P) \\ &- \sum_t \bar{z}_t^a(m, P, Q) \frac{\partial}{\partial y_t}(m, P) - \sum_{js} z_s^{ji}(m, P, Q) \frac{\partial}{\partial y_s^j}(m, P) \\ &= \sum_a c_i^a \left(\frac{\partial}{\partial y_j}(m, P) + \sum_t \bar{z}_t^a(m, P, Q) \frac{\partial}{\partial y_t}(m, P) \right. \\ &\qquad \left. + \sum_{sj} z_s^{ja}(m, P, Q) \frac{\partial}{\partial y_s^j}(m, P) \right)\end{aligned}$$

where $1 \leq i, j \leq p$, $1 \leq a \leq q$, $p + 1 \leq s \leq d$, $q + 1 \leq t \leq d$. If $1 \leq a \leq p$ there is no $\partial/\partial y_a$ on the left side, hence

$$c_i^a = 0 \quad \text{if } 1 \leq i, a \leq p.$$

If $p + 1 \leq a \leq q$ then we have, by equating coefficients,

$$c_i^a = \bar{w}_a^i(m, P, Q_0) \quad \text{if } 1 \leq i \leq p, p + 1 \leq a \leq q.$$

Hence the preceding relation becomes

$$\sum_s \bar{w}_s^i(m, P, Q_0) \frac{\partial}{\partial y_s}(m, P) + \sum_{sj} w_s^{ji}(m, P, Q_0) \frac{\partial}{\partial y_s^j}(m, P)$$

$$- \sum_t \bar{z}_t^i(m, P, Q) \frac{\partial}{\partial y_t}(m, P) - \sum_{sj} z_s^{ji}(m, P, Q) \frac{\partial}{\partial y_s^j}(m, P)$$

$$= \sum_l \bar{w}_l^i(m, P, Q_0) \Big(\frac{\partial}{\partial y_l}(m, P) + \sum_t \bar{z}_t^l(m, P, Q) \frac{\partial}{\partial y_t}(m, P)$$

$$+ \sum_{sj} z_s^{jl}(m, P, Q) \frac{\partial}{\partial y_s^j}(m, P)\Big)$$

where $1 \leq i, j \leq p, p + 1 \leq s \leq d, q + 1 \leq t \leq d, p + 1 \leq l \leq q$. Equating coefficients now gives (a) and (b). Reversing the argument gives the converse.

As a corollary of this lemma we can give a coordinate characterization of the integrable elements of $G_q(G_p(M))$ (for those elements that lie in the domain of the coordinate system considered).

COROLLARY. *The integrable elements of $G_q(G_p(M))$ which lie in the domain of such a z-coordinate system are exactly those which satisfy the following set of relations:*

(a′) $z_t^i - \bar{z}_t^i = \sum_l z_l^i \bar{z}_t^l$

(b′) $z_s^{ij} - z_s^{ji} = \sum_l z_l^i z_s^{jl} - \sum_l z_l^j z_s^{il}$

where $1 \leq i, j \leq p, q + 1 \leq t \leq d, p + 1 \leq l \leq q, p + 1 \leq s \leq d$.

Proof. First suppose (m, P, Q) is integrable and lies in the domain of this coordinate system. Let $(m, P, Q_0) \in I_{p,p}(M)$ with $Q_0 \subset Q$, and let the w-coordinates be as in the lemma. Because $\pi_* Q_0 = P$ it is clear that (m, P, Q_0) is in the domain of this coordinate system. By [I] we know that integrability of (m, P, Q_0) implies

$$\bar{w}_s^i(m, P, Q_0) = w_s^i(m, P, Q_0) \quad \text{and} \quad w_s^{ij}(m, P, Q_0) = w_s^{ji}(m, P, Q_0).$$

By definition we have

$$w_s^i(m, P, Q_0) = y_s^i(m, P) = z_s^i(m, P, Q)$$

and these, with (a) and (b) of the lemma, give (a′) and (b′) at (m, P, Q).

Now suppose (a′) and (b′) hold at (m, P, Q). Then $\pi_* Q$ has dimension q because (m, P, Q) is in the domain of this coordinate system. We define a point $(m, P, Q_0) \in I_{p,p}(M)$ by

$$w_a(m, P, Q_0) = z_a(m, P, Q),$$

$$w_s^i(m, P, Q_0) = z_s^i(m, P, Q),$$

$$w_s^{ij}(m, P, Q_0) = z_s^{ij}(m, P, Q) + \textstyle\sum_l z_l^j(m, P, Q) z_s^{il}(m, P, Q).$$

By (b′) we have $w_s^{ij}(m, P, Q_0) = w_s^{ji}(m, P, Q_0)$ so, by [I], these define a unique $(m, P, Q) \in I_{p,p}(M)$. By (a′) and (b′) it now follows that (a) and (b) of the lemma hold.

COROLLARY. *$I_{q,p}(M)$ is a submanifold of $G_q(G_p(M))$.*

Proof. By (a′) and (b′) we can take the z_b, z_l^i, $\bar{z}_t^i$, z_t^l, z_s^{ij} with $i \leq j$, and z_s^{il} for a coordinate system. It is only tedious to show that these overlap properly.

THEOREM. *Let $(m, P, Q) \in G_q(G_p(M))$. Then (m, P, Q) is integrable iff it is second order tangent to a submanifold of M.*

Proof. First suppose (m, P, Q) is second order tangent to a submanifold. Let ϕ be a non-singular $1:1$ C^∞ map of a neighborhood of 0 in R^q into M such that $\tilde{\phi}'(0) = (m, P, Q)$. Then $\pi_* Q = \phi_*(0)$ so $\pi_* Q$ has dimension q. Let A be the map of

$$R^p \to R^q : (t_1, \cdots, t_p) \to (t_1, \cdots, t_p, 0, \cdots, 0)$$

and define $\bar{\phi} = \phi \circ A$. Clearly $\bar{\phi}_* R_0^p = P$ and we define Q_0 by $\bar{\phi}''(0) = (m, P, Q_0)$, thus $(m, P, Q_0) \in I_{p,p}(M)$. It is trivial that $Q_0 \subset Q$, hence (m, P, Q) is integrable.

Now suppose (m, P, Q) is integrable. Choose any coordinate system $\{x_b\}$ of M at m such that all $x_b(m) = 0$ and

$$P = \mathrm{sp}\left\{\frac{\partial}{\partial x_1}(m), \cdots, \frac{\partial}{\partial x_p}(m)\right\}$$

$$\pi_* Q = \mathrm{sp}\left\{\frac{\partial}{\partial x_1}(m), \cdots, \frac{\partial}{\partial x_q}(m)\right\} \tag{1}$$

We again let our indices have the ranges $1 \leq i, j \leq p$, $p + 1 \leq s \leq d$, $q + 1 \leq t \leq d$, $1 \leq a \leq q$, $1 \leq b \leq d$. We now consider the associated y and z coordinate systems. By (1) we have

$$z_s^j(m, P, Q) = y_s^j(m, P) = 0 \qquad \text{for all } s, j,$$

$$z_t^{ja}(m, P, Q) = 0 \qquad \text{for all } t, j. \tag{2}$$

Because (m, P, Q) is integrable this, with (a′) and (b′) of the first corollary above, gives

$$\bar{z}_t^i(m, P, Q) = z_t^i(m, P, Q) = 0, \qquad z_s^{ij}(m, P, Q) = z_s^{ji}(m, P, Q). \tag{3}$$

Now let ϕ be any C^∞ map of a neighborhood U of 0 in R^q into the domain of the $\{x_b\}$ and suppose ϕ so chosen that assumption (A) holds, i.e.

$$x_a \circ \phi = u_a \quad \text{if } 1 \leq a \leq q, \tag{4}$$

hence the Jacobian (J_{ba}) of ϕ (relative to the usual u_a and the x_b) satisfies:

$$J_{ba} = \delta_{ba} \quad \text{if } 1 \leq a, b \leq q$$

at all points of U. Define f_t by

$$f_t = x_t \circ \phi \quad \text{if } q + 1 \leq t \leq d.$$

Because of (1), and the fact that all $x_b(m) = 0$, we have

$$\bar{z}_b(\tilde{\phi}'(0)) = 0, \qquad z_s^j(\tilde{\phi}'(0)) = 0$$

and by (1.9),

$$\bar{z}_t^a(\tilde{\phi}'(0)) = \frac{\partial f_t}{\partial u_a}(0)$$

$$(5) \qquad z_s^{ja}(\tilde{\phi}'(0)) = 0 \qquad \text{if } p + 1 \leq s \leq p$$

$$= \frac{\partial^2 f_t}{\partial u_j \, \partial u_a}(0) \quad \text{if } q + 1 \leq s \leq d$$

Hence if we choose the f_t so that

$$\frac{\partial f_t}{\partial u_a}(0) = 0 \qquad \text{for all } a, t$$

$$\frac{\partial^2 f_t}{\partial u_j \, \partial u_a}(0) = z_t^{ja}(m, P, Q)$$

we see from (1)–(5) that all coordinates of (m, P, Q) are the same as for $\tilde{\phi}'(0)$, so $\tilde{\phi}'(0) = (m, P, Q)$ and the theorem is proved.

We now wish to associate with each $(m, P, Q) \in I_{q,p}(M)$ a subspace $\tilde{Q}$ of $G_p(M)_{(m,P)}$ in such a way that if ϕ is a non-singular C^∞ 1 : 1 map of a neighborhood of 0 in R^q into M with $\tilde{\phi}'(0) = (m, P, Q)$ then

$$\phi'_*(G_p(R^q)_{(0,P_p(0))}) = \tilde{Q}.$$

For this purpose we introduce the following notation: if W is any finite-dimensional linear space over R of dimension $q > p$ then $g_p(W)$ is the Grassmann manifold of all p-dimensional linear subspaces of W. So dim $g_p(W)$ is $p(q - p)$. If P is a q-dimensional subspace of M_m then $g_p(P)$ is a submanifold of $G_p(M)$ and we identify its tangent space at $P \in g_p(P)$ with a subspace of $G_p(M)_{(m,P)}$. Now given $(m, P, Q) \in I_{q,p}(M)$ we define $\tilde{Q}$, a subspace of $G_p(M)_{(m,P)}$, by

$$(1.10) \qquad \tilde{Q} = Q + g_p(\pi_* Q)_P.$$

LEMMA 1.3. *If ϕ is a non-singular 1 : 1 C^∞ map of a neighborhood of 0 in R^q into M and $\tilde{\phi}'(0) = (m, P, Q)$ then*

$$\phi'_*(G_p(R^q)_{(0,P_p(0))}) = \tilde{Q}.$$

Proof. It will be sufficient to show that the spaces on the two sides of this equality have the same dimension and that the left side contains the right. Clearly the dimension of $\tilde{Q}$ is $q + p(q - p)$, as is the dimension of $G_p(R^q)_{(0,P_p(0))}$, so the dimension of $\phi'_*(G_p(R^q)_{(0,P_p(0))})$ is $\leq q + p(q - p)$. Hence if we prove the left side contains the right it will follow that these spaces have the same dimension and the lemma will be proved. To prove the inclusion we prove separately that

(α) $\phi'_*(G_p(R^q)_{(0,P_p(0))}) \supseteq Q$
(β) $\phi'_*(G_p(R^q)_{(0,P_p(0))}) \supseteq g_p(\pi_* Q)_P$.

Proof of (α). If ψ is the map of $R^p \to G_p(R^q)$ used previously in the definition of $\tilde{\phi}$ then

$$Q = \tilde{\phi}_* R_0^q = \phi'_* \psi_* R_0^q .$$

Since $\psi_* R_0^q$ is a subspace of $G_p(R^q)_{(0,P_p(0))}$ this proves (α).

Proof of (β). We have $\pi_* Q = \pi_* \tilde{\phi}_* R_0^q = \pi_* \phi'_* \psi_* R_0^q = \phi_* R_0^q$, hence ϕ' carries the fiber of $G_p(R^q)$ which lies over 0 onto $g_p(\pi_* Q)$. From this (β) follows.

COROLLARY. *If ϕ is a non-singular $1:1$ C^∞ map of $L \to M$, where L and M are manifolds of dimension $\geq p$, then ϕ' is a non-singular map of $G_p(L) \to G_p(M)$.*

Proof. The above formula shows ϕ'_* preserves the dimension of the tangent space to $G_p(L)$ at each point.

2. The characteristic vector field theorem

Our object now is to prove the following

THEOREM. *Let M be a manifold of dimension d and p an integer with $0 < p < d$. Let U be an open set in $G_p(M)$ and $\tilde{U} = \pi^{-1}(U)$ where this π is the projection of $I_{p+1,p}(M) \to G_p(M)$. Let λ be a map which assigns to each $(m, P, Q) \in \tilde{U}$ an element λ_Q of $L(Q)$, with $\lambda_Q \neq 0$ for all $(m, P, Q) \in \tilde{U}$, and such that λ is C^∞ in the sense explained below. Let F be a real-valued C^∞ function defined on U. Then there exists a unique map V which assigns to each $(m, P, Q) \in U$ a tangent vector in $\tilde{Q}$, that we denote by $V(m, P, Q)$, such that both*
(a) *$\lambda(V) = 0$ on U,*
(b) *for each $(m, P, Q) \in U$ there exists a number $G(m, P, Q)$ such that*

$$d\lambda_Q(t, V(m, P, Q)) = dF(t) + G(m, P, Q)\lambda_Q(t) \quad \text{for all} \quad t \in \tilde{Q}$$

where $d\lambda_Q$ is explained in the remark below. This V will be C^∞ as a map of $\tilde{U}$ into the tangent bundle to $G_p(M)$ and will have the following property. If ϕ is any non-singular $1:1$ map of a neighborhood of 0 in R^{p+1} into M such that $\tilde{\phi}'(0) = (m, P, Q) \in \tilde{U}$ and if W is the characteristic vector field of $F \circ \phi'$ (in the classical sense) relative to $\mu = \lambda \circ \phi'$ then $\phi'_* W(0, P_p(0)) = V(m, P, Q)$.

Remarks. (1) If $(m, P, Q) \epsilon I_{p+1,p}(M)$ then $\pi_* Q/P$ is 1-dimensional so we can choose $T(m, P, Q) \epsilon Q$ such that $\pi_* T(m, P, Q) \notin P$. Furthermore, this $T(m, P, Q)$ can be chosen locally to depend in a C^∞ fashion on (m, P, Q). To prove this within the domain of a z-coordinate system obtained as usual from an x-coordinate system of M we choose

$$T(m, P, Q) = \sum_i z^i_{p+1}(m, P, Q)\left(\frac{\partial}{\partial y_i}(m, P) + \sum_t z^i_t(m, P, Q)\frac{\partial}{\partial y_t}(m, P) + \sum_{sj} z^i_{sj}(m, P, Q)\frac{\partial}{\partial y^j_s}(m, P)\right) - \left(\frac{\partial}{\partial y_{p+1}}(m, P) + \sum_t z^{p+1}_t(m, P, Q)\frac{\partial}{\partial y_t}(m, P) + \sum_{sj} z^{p+1}_{sj}(m, P, Q)\frac{\partial}{\partial y^j_s}(m, P)\right)$$

where, as usual, $1 \leq i, j \leq p$, $p + 1 \leq s \leq d$, $p + 2 \leq t \leq d$ (since now $q = p + 2$). This expression shows $T(m, P, Q) \epsilon Q$. Letting

$$\lambda_s = \sum\nolimits_i y^i_s\, dy_i - dy_s$$

we have

$$(2.1)\qquad \begin{aligned} \lambda_{p+1}(T(m, P, Q)) &= \sum\nolimits_i y^i_{p+1}(m, P)^2 + 1 = \sum\nolimits_i z^i_{p+1}(m, P, Q)^{2'} + 1 \\ \lambda_t(T(m, P, Q)) &= z^{p+1}_t(m, P, Q) \end{aligned}$$

Because the first of these is non-zero we see that $\pi_* T(m, P, Q) \notin P$. We have written the second only for later use. We now define the sense in which the above λ is C^∞: for every such $C^\infty T$, mapping an open set in $I_{p+1,p}(M)$ into $G_p(M)$, (including that $T(m, P, Q) \epsilon Q - P$) the function $\lambda(T)$ is in C^∞.

(2) We need to explain the $d\lambda_Q$ used above because λ_Q fails in two ways to be a 1-form: it depends on Q and it is only defined for vectors in Q. But we now show that if λ_1, λ_2 are C^∞ 1-forms of $G_p(M)$ such that $\lambda_1 \mid Q = \lambda_2 \mid Q$ then

$$d\lambda_1 \mid (\tilde{Q} \times \tilde{Q}) = d\lambda_2 \mid (\tilde{Q} \times \tilde{Q}).$$

Proof. Choose ϕ as usual with $\tilde{\phi}'(0) = (m, P, Q)$. Then $\phi'^* d\lambda_1 = \phi'^* d\lambda_2$ on the tangent spaces to $G_p(R^q)$. Using Lemma 1.3 we have the desired fact. We then define $d\lambda_Q$ to be equal to these, but defined only on $\tilde{Q} \times \tilde{Q}$. Since the vectors in (b) lie in $\tilde{Q}$ this explains (b).

(3) We introduced, in the previous section, certain functions K_{sr}, depending on two coordinate systems. Because we now have only one value for r, namely $r = p + 1$, we shall here denote those functions, obtained from our coordinate system $\{y_b, y^j_s\}$ of $G_p(M)$ and $\{w_a, w^i_{p+1}\}$ of $G_p(R^{p+1})$ (these being obtained from an x-coordinate system we assume given on M and the usual coordinate system of R^{p+1}) by K_s, i.e. $K_s = K_{s,p+1}$. From the previous section we have

$$\phi'^*\lambda_s = K_s\left(\sum\nolimits_i w^i_{p+1}\, dw_i - dw_{p+1}\right).$$

Using that dim $L(Q) = 1$ and (2.1) we have

$$(2.2) \qquad K_t/K_{p+1} = z_t^{p+1}/(\textstyle\sum_i (z_{p+1}^i)^2 + 1)$$

This will be used below.

(4) It follows from (2.2) that $\mu = \phi'^*\lambda$ of the theorem will be everywhere $\neq 0$.

Proof of theorem. Fix any $(m, P, Q) \in U \cap$ (domain of a fixed z-coordinate system). Choose ϕ a non-singular $1:1$ C^∞ map of a neighborhood of 0 in R^{p+1} into M usch that $\tilde{\phi}'(0) = (m, P, Q)$, and let $\mu = \phi'^*\lambda$. Let W be the characteristic vector field of $F \circ \phi'$ relative to μ and we now define

$$V(m, P, Q) = \phi'_* W(0, P_p(0)).$$

It is immediate from the properties of a characteristic vector field that $V(m, P, Q)$ satisfies (a) and (b). We shall now show, by a coordinate calculation, that $V(m, P, Q)$ depends only on (m, P, Q). This plus uniqueness of characteristic vector fields (in the classical sense) will show V is unique and our coordinate calculation will also show V is C^∞.

In making the coordinate calculation we first make it using not the characteristic vector field W of $F \circ \phi'$ relative to $\mu = \phi'^*\lambda$ but, instead, the characteristic vector field W_0 relative to

$$\mu_0 = (\textstyle\sum_i w_{p+1}^i \, dw_i - dw_{p+1}).$$

Since $\mu = K_{p+1}\, \mu_0$ we shall then have (see [I]) $W = W_0/K_{p+1}$. By formula (2.1) of [I],

$$\begin{aligned} W_0 &= \sum_i \frac{\partial(F\circ\phi')}{\partial w_{p+1}^i}\frac{\partial}{\partial w_i} - \sum_i \frac{\partial(F\circ\phi')}{\partial w_i}\frac{\partial}{\partial w_{p+1}^i} \\ &= \sum_i \left(\left(\phi'_* \frac{\partial}{\partial w_{p+1}^i}\right) F\right)\phi'_* \frac{\partial}{\partial w_i} - \sum_i \left(\left(\phi'_* \frac{\partial}{\partial w_i}\right) F\right)\phi'_* \frac{\partial}{\partial w_{p+1}^i} \end{aligned}$$

From (1.6) and (1.6′) and the definition of the $\bar{z}_t^a$ and z_s^{ja} we have

$$\begin{aligned} \phi'_* W_0 = \sum_{ais'j'} K_{s'} C_{ij'} \frac{\partial F}{\partial y_{s'}^{j'}} J_{ai} \left(\frac{\partial}{\partial y_a} + \sum_t \bar{z}_t^a \frac{\partial}{\partial y_t} + \sum_{sj} z_s^{ja} \frac{\partial}{\partial y_s^j}\right) \\ - \sum_{aisj} J_{ai}\left(\frac{\partial F}{\partial y_a} + \sum_t \bar{z}_t^a \frac{\partial F}{\partial y_t} + \sum_{s'j'} z_{s'}^{j'a} \frac{\partial F}{\partial y_{s'}^{j'}}\right) K_s C_{ij} \frac{\partial}{\partial y_s^j}. \end{aligned}$$

We now break the sum on a into the two parts, $a \leq p$ and $a = p + 1$, and we consider these formulas only at $\phi'(0, P_p(0))$, where all the w_{p+1}^i are 0. So (1.2) gives, at such a point, $A_{si} = J_{si} \circ \pi$, $B_{ji} = J_{ji} \circ \pi$, $\sum_i J_{ki} C_{ij} = \delta_{kj}$; also (1.4) then gives $y_s^j \circ \phi' = \sum_i J_{si} C_{ij}$. Hence the preceeding formula becomes, at $\phi'_*(0, P_p(0))$,

$$\phi'_* W_0 = \sum_{s'j'} K_{s'} \frac{\partial F}{\partial y^{j'}_{s'}} \left(\frac{\partial}{\partial y_{j'}} + \sum_t \bar{z}^{j'}_t \frac{\partial}{\partial y_t} + \sum_{sj} z^{jj'}_s \frac{\partial}{\partial y^j_s} \right)$$
$$+ \sum_{s'j'} K_{s'} \frac{\partial F}{\partial y^{j'}_{s'}} z^{j'}_{p+1} \left(\left(\frac{\partial}{\partial y_{p+1}} + \sum_t \bar{z}^{p+1}_t \frac{\partial}{\partial y_t} + \sum_{sj} z^{jp+1}_s \frac{\partial}{\partial y^j_s} \right) \right.$$
$$- \sum_{sj} K_s \left(\frac{\partial F}{\partial y_j} + \sum_t \bar{z}^j_t \frac{\partial F}{\partial y_t} + \sum_{s'j'} z^{j'j}_{s'} \frac{\partial F}{\partial y^{j'}_{s'}} \right) \frac{\partial}{\partial y^j_s}$$
$$- \sum_{sj} z^j_{p+1} K_s \left(\frac{\partial F}{\partial y_{p+1}} + \sum_t \bar{z}^{p+1}_t \frac{\partial F}{\partial y_t} + \sum_{s'j'} z^{j'p+1}_{s'} \frac{\partial F}{\partial y^{j'}_{s'}} \right) \frac{\partial}{\partial y^j_s}.$$

Because the functions K_s occurring here may depend on more than just the coordinates of (m, P, Q) this expression may depend on more than just (m, P, Q). But this W_0 was formed using μ_0 rather than μ. Since $\mu = K_{p+1}\, \mu_0$ we know from [I] that $W = W_0/K_{p+1}$ so if W is used instead of W_0 then all these coefficients K_s are divided by K_{p+1}. We then see from (2.1) that we get coefficients that depend only on the (m, P, Q). This proves the theorem. We note that we could get an explicit coordinate expression for V.

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