CONVEXITY OF POLYHEDRA

BY
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Introduction

The intuitive feeling persists that the convexity of a polyhedron should be determined by the nature of the vertex set of the polyhedron. A few minutes with paper and pencil, however, convince the newcomer to convexity that the precise relationship between the vertex set and convexity is not an obvious one. This paper investigates that relationship. In Section II we demonstrate that a finite geometric simplicial n-complex (polyhedron) is convex if and only if it fulfills a certain vertex condition similar to convexity, is contained in an n-hyperplane and is either starlike or $(n - 1)$-connected. In Section III these results are applied to the problem of subdividing a complex in such a way as to make the star at each vertex convex. Some of the complexes in Section III are not simplicial, but, in the absence of explicit comment to the contrary, it should be assumed that all complexes are simplicial.

I. Definitions

If S and T are sets in E^n (real n-space), $S \circ T$ denotes the union of all closed line segments st with $s \in S$ and $t \in T$. It is easy to see that a set X in E^n is convex if and only if $S \subset X$ and $T \subset X$ imply $S \circ T \subset X$. If K is a complex, $| K |$ will denote the point set occupied by K although, where the distinction is not important, we may refer to the complex K itself (as opposed to the set $| K |$) as convex. A set T in E^n is n-connected if and only if every map $f : S^k \rightarrow T$ $(0 \leqq k \leqq n)$ is homotopic to a constant map $g : S^k \rightarrow T$, where S^k denotes the k-sphere. A set T in E^n is starlike if and only if it contains a point v such that $s \in T$ implies the segment vs is contained in T. A polyhedron K is *n-vertex-convex* (n-vc) if and only if the simplex spanned by any $n + 1$ of its vertices is contained in $| K |$. All discussions take place in E^n.

II. Vertex convexity and convexity

If K is the n-complex obtained by joining the barycenter of the n-simplex to each of its vertices, let K' be the complex obtained by removing one of the n-simplexes of K and the $(n - 1)$-face of this simplex which lies on the boundary of the original simplex. The set $| K' |$ is not convex, but it is starlike, $(n - 1)$-connected and lies in an n-hyperplane. The complex K' however is not $(n - 1)$-vc although it is $(n - 2)$-vc. The key to assuring convexity is $(n - 1)$-vertex convexity.

THEOREM 1. *If K is an n-polyhedron, $(n \geqq 2)$, then $| K |$ is convex if and*

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only if K is $(n - 1)$-vc and $|K|$ is $(n - 1)$-connected and contained in an n-hyperplane.

Proof. If $|K|$ is convex, it is contained in an n-hyperplane and is a topological n-cell. Necessity is clear.

Now suppose the three conditions hold and let q_1 and q_2 belong to $|K|$. Suppose $q_1 q_2$ is not contained in $|K|$. Since $|K|$ is compact, $q_1 q_2 - |K|$ contains an interval $p_1 p_2$ with p_1 and $p_2 \in \mathrm{Bd}\ (|K|)$. The segment $p_1 p_2$ has no points of $|K|$ on its interior. If p_1 and p_2 are vertices, we are through, since $(n - 1)$-vertex convexity implies 1-vertex convexity. We now consider two cases.

Case 1. One of the points p_1 and p_2 (say p_1) is a vertex, but the other is not. Let σ^k by the simplex of lowest dimension containing p_2 and $\pi(\sigma^k)$ the unique K-hyperplane containing σ^k. If $p_1 \in \pi(\sigma^k)$, $p_1 p_2$ is in $\pi(\sigma^k)$. Now p_2 must be on the interior of σ^k, so $p_1 p_2$ contains interior points of σ^k and hence points of $|K|$. This is impossible and we may assume that $p_1 \notin \pi(\sigma^k)$. Hence $p_1 \circ \sigma^k$ is a $(k + 1)$-simplex, $k + 1 \leqq n$.

If $k + 1 < n$, $k + 1 \leqq n - 1$, and by $(n - 1)$-vertex convexity $p_1 \circ \sigma^k$ must be contained in $|K|$. This again contradicts the fact that $p_1 p_2$ misses $|K|$ completely.

If $k + 1 = n$, we consider $\mathrm{Bd}\ (p_1 \circ \sigma^k) = (p_1 \circ \mathrm{Bd}\ \sigma^k) \cup \sigma^k$. Each simplex in $p_1 \circ \mathrm{Bd}\ \sigma^k$ is of dimension $\leqq n - 1$ and by $(n - 1)$-vertex convexity is contained in $|K|$. Hence $p_1 \circ \sigma^k$ is an n-cell whose boundary is contained in $|K|$ but which is not itself contained in $|K|$. This contradicts the $(n - 1)$-connectedness.

Case 2. Neither p_1 nor p_2 is a vertex. We pick σ^k and σ^t of minimal dimension containing p_1 and p_2 respectively; $\sigma^k \circ \sigma^t$ is a convex j-cell, and since both are contained in an n-hyperplane, $j \leqq n$.

If $j < n$ we may triangulate $\sigma^k \circ \sigma^t$ without adding vertices. (To see that this is possible, note that it is possible for convex 2-polyhedra. The general case follows by induction, triangulating boundary cells and taking the cone from any vertex of the polyhedron.) By $(n - 1)$-vertex convexity each simplex of the resulting complex is contained in $|K|$. The contradiction is established by noting that $\sigma^k \circ \sigma^t$ is now contained in $|K|$.

If $j = n$, $\mathrm{Bd}\ (\sigma^k \circ \sigma^t)$ consists of $(n - 1)$-cells, each of which, by the above argument, is contained in $|K|$. Hence $\sigma^k \circ \sigma^t$ is an n-cell which is not contained in $|K|$, but whose boundary is contained in $|K|$. This contradiction to the $(n - 1)$-connectedness establishes the theorem.

The topological property, $(n - 1)$-connectedness, can be replaced in the above theorem by a non-topological property, starlikeness, since a starlike polyhedron is a contractible absolute neighborhood retract thus an absolute retract and n-connected. We will, however, avoid Theorem 1 and prove the following theorem in a vector space over an arbitrary ordered field.

THEOREM 2. *If K is an n-polyhedron, then $|K|$ is convex if and only if K is $(n-1)$-vc and $|K|$ is starlike and contained in an n-hyperplane.*

Proof. Again necessity is clear.

The proof of sufficiency is the same as that used in Theorem 1 until we consider the cases in which $p_1 \circ \sigma^k$ is an n-simplex and $\sigma^k \circ \sigma^t$ is an n-cell. It is only in these cases that the $(n-1)$-connectedness is used.

Let v be the point with respect to which K is starlike.

Case 1. Suppose $p_1 \circ \sigma^k$ is an n-simplex. If $q \in \operatorname{Int} p_1 p_2$ and $q \neq v$, $q \in \operatorname{Int}(p_1 \circ \sigma^k)$, since $p_2 \in \operatorname{Int} \sigma^k$. Let L be the line determined by q and v. Now $\operatorname{Bd}(p_1 \circ \sigma^k)$ is contained in $|K|$ by $(n-1)$-vertex convexity and by convexity of $p_1 \circ \sigma^k$, L meets $\operatorname{Bd}(p_1 \circ \sigma^k)$ in exactly two points, at least one of which (call it x) lies on the opposite side of q from v. The starlikeness assures that $vx \subset |K|$. However, q is now on both vx and $p_1 p_2$. This contradiction establishes the theorem in this case.

Case 2. Suppose $\sigma^k \circ \sigma^t$ is an n-cell. By $(n-1)$-vertex convexity $\operatorname{Bd}(\sigma^k \circ \sigma^t)$ is contained in $|K|$ and the same contradiction as in above case establishes the theorem.

An obvious corollary of these two theorems is that for $(n-1)$ n-vc n-complexes in E^n starlikeness and $(n-1)$-connectedness are equivalent.

III. Convex subdivisions

The principal tool of this section is Theorem 2. All the results hold in a vector space over an arbitrary ordered field.

If v is a vertex of a complex K, the star at v, denoted by $\operatorname{St}(v)$, is the complex consisting of those simplexes of K which contain v, and all faces of such simplexes. A subdivision $\operatorname{Sd}(K)$ of a complex K is called convex if and only if $|\operatorname{St}(v)|$ is convex for each vertex v of $\operatorname{Sd}(K)$. The obvious advantages of convex complexes lead us to ask whether an arbitrary convex polyhedral n-cell can be triangulated in a convex fashion. If we allow subdivision by polyhedral cells, rather than insisting upon simplexes, the following theorem gives an affirmative answer.

THEOREM 3. *If K is a convex n-complex, there exists a subdivision of K into n-cells in which $|\operatorname{St}(v)|$ is convex for each vertex v of $\operatorname{Sd}(K)$.*

Proof. The subdivision is that given by the $(n-1)$-hyperplanes determined by the $(n-1)$-simplexes of K. Of course $\operatorname{St}(v)$ is a cell-complex, and each cell is convex. We prove the convexity of $|\operatorname{St}(v)|$ by induction on n.

We first triangulate $\operatorname{St}(v)$ by triangulating each of its n-cells without adding vertices and so that v is a vertex of each simplex in $\operatorname{Sd}(\operatorname{St}(v))$. Since $|\operatorname{St}(v)|$ is starlike and contained in an n-hyperplane, we need only show $(n-1)$-vertex convexity.

For $n = 2$, we must show 1-vertex convexity. If v_1 and v_2 are vertices of

$\mathrm{St}(v)$ which lie in the same cell, $v_1 v_2$ is in the cell by convexity and hence in $|\mathrm{St}(v)|$. If v_1 and v_2 are not in the same cell and p is a point of $v_1 v_2 - |\mathrm{St}(v)|$, pv intersects $\mathrm{Bd}\,|\mathrm{St}(v)|$ in a point q of the triangle $vv_1 v_2$. If $q = v$, p fails to belong to K, contradicting the convexity of K. Since it lies on $\mathrm{Bd}\,|\mathrm{St}(v)|$, if q is on a 1-cell containing v, it is an endpoint of the cell. In this case q must also lie on a 1-simplex of $\mathrm{Bd}(\mathrm{St}(v))$. Hence, in any case, q is on a 1-simplex of $\mathrm{Bd}(\mathrm{St}(v))$, which does not contain v. But this simplex must have been extended in the subdivision process and hence must cut vv_1 (or vv_2) on its interior, implying that v_1 (or v_2) is not a vertex of $\mathrm{St}(v)$.

Now assume the theorem for n and let K be an $(n + 1)$-complex and $v_1, v_2, \cdots, v_{n+1}$ vertices of $\mathrm{St}(v)$. We will show that if the simplex $v_1 v_2 \cdots v_{n+1}$ is not contained in $|\mathrm{St}(v)|$ at least one of the v_i's is not a vertex of $\mathrm{St}(v)$.

Suppose $p \in v_1 v_2 \cdots v_{n+1} - |\mathrm{St}(v)|$, and let $v_1 v_2 \cdots \hat{v}_i \cdots v_{n+1}$ be the $(n - 1)$-face of $v_1 v_2 \cdots v_{n+1}$ which does not contain v_i and π_i the n-hyperplane determined by $v, v_1, v_2, \cdots, v_{i-1}, v_{i+1}, \cdots, v_{n+1}$. Suppose first that $p \in v_1 v_2 \cdots \hat{v}_i \cdots v_{n+1}$. Since the subdivision of K automatically subdivides $|K \cap \pi_i|$ in a similar fashion we may consider $K \cap \pi_i$ as an n-complex. The star in $K \cap \pi_i$ at v is $\mathrm{St}(v) \cap \pi_i$ and by the induction hypothesis $|\mathrm{St}(v) \cap \pi_i|$ is convex. However, $p \notin |\mathrm{St}(v)| \cap \pi_i$ and each of

$$(v, v_1, v_2, \cdots, v_{i-1}, v_{i+1}, \cdots, v_{n+1})$$

does. Hence $p \notin v_1 v_2 \cdots \hat{v}_i \cdots v_{n+1}$.

Now letting $q \in pv \cap |\mathrm{Bd}(\mathrm{St}(v))| \cap (vv_1 v_2 \cdots v_{n+1})$, q is on a bounding hyperplane π of $\mathrm{St}(v)$ which does not contain v. Since, by the previous paragraph

$$v_1 v_2 \cdots \hat{v}_i \cdots v_{n+1} \subset |\mathrm{St}(v)|,$$

the same is true of $vv_1 v_2 \cdots \hat{v}_i \cdots v_{n+1}$ by starlikeness. Hence

$$q \in \mathrm{Int}\,(vv_1 v_2 \cdots v_{n+1}) \quad \text{or} \quad q \in v_1 v_2 \cdots v_{n+1}.$$

The line segment pv intersects $\mathrm{Bd}\,(vv_1 v_2 \cdots v_{n+1})$ in p and v only. Otherwise the entire segment would be on a face of $vv_1 v_2 \cdots v_{n+1}$ other than $v_1 v_2 \cdots v_{n+1}$, and all these faces are contained in $|\mathrm{St}(v)|$. Hence if $q \in v_1 v_2 \cdots v_{n+1}$, $q = p$, but $p \notin \mathrm{St}(v)$ so $q \neq p$.

Suppose now that $q \in \mathrm{Int}(vv_1 v_2 \cdots v_{n+1})$. If the plane π contained $n + 1$ vertices of $vv_1 v_2 \cdots v_{n+1}$, it would intersect this simplex in an n-face, but we have just established that q is not on any n-face of $vv_1 v_2 \cdots v_{n+1}$. Thus π contains at most n vertices of $vv_1 v_2 \cdots v_{n+1}$ and at least one v_i is not in π. If all the vertices $v, v_1, v_2, \cdots, v_{n+1}$ were on the same side of π as v, π could contain only boundary points of $vv_1 v_2 \cdots v_{n+1}$ and $q \notin \pi$. Therefore at least one vertex v_i is separated from v by π. This vertex cannot be a vertex of $\mathrm{St}(v)$.

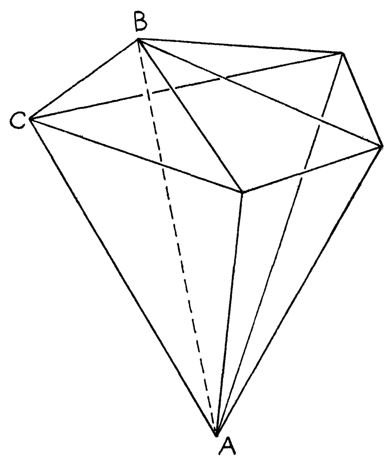
The subdivision into cells rather than simplexes leaves something to be desired, but, at least in low dimensions, convex simplicial subdivisions are possible. The situation in higher dimensions is unsettled.

THEOREM 4. *If K is a convex 2-complex there exists a convex simplicial subdivision of K.*

Proof. We first subdivide the complex in the fashion described in the previous theorem. We now triangulate each of the convex 2-cells formed in this process by joining one vertex to each of the others in the cell. We will let $\mathrm{St}^*(v)$ denote the star at v in the cell subdivision and $\mathrm{St}(v)$ denote the ordinary star. We need only show 1-vertex convexity of $\mathrm{St}(v)$ to establish the theorem.

If v_1 and v_2 are vertices of $\mathrm{St}(v)$, vv_1 and vv_2 are 1-simplexes in the triangulation. Suppose there is a point $p \in v_1 v_2 - |\mathrm{St}(v)|$, and let $q = pv \cap \mathrm{Bd}\,\mathrm{St}(v)$. Now $q \neq p$, and since $v \in \mathrm{Int}\,\mathrm{St}(v)$, $q \neq v$. Now q is on a 1-face of $\mathrm{Bd}\,(\mathrm{St}(v))$ which does not contain v. If this face cuts vv_1 it does so in a vertex of $\mathrm{St}(v)$ which is also a vertex of $\mathrm{St}^*(v)$. Since there is no vertex of $\mathrm{St}^*(v)$ between v and v_1, the 1-face in question cuts neither vv_1 nor vv_2. Therefore it must have a vertex w in the triangle $vv_1 v_2$. Hence w is a vertex of $\mathrm{St}^*(v)$ and the ray from v containing w intersects $v_1 v_2$ in a point which cannot be in $|\mathrm{St}^*(v)|$. This contradicts the convexity of $|\mathrm{St}^*(v)|$.

This proof is possible because the subdivision of convex 2-cells without adding vertices has the property that all stars are convex. This is not the case even for a three cell as can be seen from the diagram. Although K is convex, after the addition of line AB, $\mathrm{St}(C)$ is not convex.



We might ask whether it is possible to triangulate an arbitrary convex polyhedron in a convex fashion. This is possible in low dimensions, but vertices are added in the triangulation and the method of the previous theorem can not be extended to dimension three. The situation in higher dimensions is unsettled.

THEOREM 5. *If K is a convex polyhedral n-cell ($n = 1, 2, 3$) there is a triangulation $T(K)$ in which $|\,\mathrm{St}(v)\,|$ is convex for every vertex v of $T(K)$.*

Proof. The statement is trivial for $n = 1$. For $n = 2$ it is sufficient to join one fixed vertex v to each of the remaining vertices. Clearly $|\,\mathrm{St}(v)\,| = K$ and is convex. We number the vertices in a counterclockwise direction $v = v_0, v_1, v_2, \cdots, v_k$. $\mathrm{St}(v)$ and $\mathrm{St}(v_k)$ are single simplexes. For $1 < i < k$ we consider the two half planes containing v_i and bounded by vv_{i-1} and vv_{i+1}. The intersection of these half planes and K is $\mathrm{St}(v_i)$ and is convex.

For $n = 3$, we pick a vertex v^* and a plane π^* intersecting $|\,K\,|$ in v^* alone. Let π be a plane parallel to π^* which misses $|\,K\,|$ and is so situated that $|\,K\,|$ lies between π and π^*. Let $\phi : K - v^* \to \pi$ be the obvious projection of $K - v^*$ on π (i.e. $\phi(x)$ is the intersection of the line through v^* and x and the plane π). Letting V be the union of those 2-cells on $\mathrm{Bd}(K)$ which do not contain v^*, $\phi(V) = \phi(K)$ is a convex 2-complex in π which can be subdivided in a convex fashion by Theorem 4. We perform the subdivision and then triangulate V by considering $\phi^{-1}\,\mathrm{Sd}\,\phi(V)$. We triangulate K by taking the cone over the triangulation of V from v^*.

Now $|\,\mathrm{St}(v^*)\,| = K$ and is convex. If $v' \in T(V)$, note that $|\,\mathrm{St}\ \phi(v')\,|$ is convex. But

$$|\,\mathrm{St}\ (v')\,| = (v^* \circ \mathrm{St}\ \phi(v')) \cap K,$$

and hence is convex.

REFERENCES

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