ALGEBRAS WITH THE SPECTRAL EXPANSION PROPERTY

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Introduction

Assume that A is the algebra of all completely continuous operators on a Hilbert space. If T is a normal operator in A, then T has a spectral expansion in A in the sense that $T = \sum_k \lambda_k E_k$ where the set $\{\lambda_k\}$ is the non-zero spectrum of T and $\{E_k\}$ is a corresponding set of self-adjoint projections (of course these sets are either finite or countably infinite). This is the standard spectral theorem for normal completely continuous operators (see for example, [2, Theorem 4, p. 183, and Theorem 6, p. 186]). In this paper we consider general algebras A with involution in which a spectral theorem of this type holds for every normal element in A. The formal definitions of what this means in an arbitrary algebra are given in Definitions 3.1 and 3.2. In Theorems 3.3 and 3.5 we characterize these algebras as *-subalgebras of the completely continuous operators on a Hilbert space which are modular annihilator algebras. It is a consequence of Theorem 3.3 that every semi-simple normed modular annihilator algebra A with a proper involution has the property that every normal element in A has a spectral expansion in A.

The first version of this paper was concerned only with a proof of this result. We acknowledge a debt to the referee who strengthened the original theorem and simplified its proof. In particular the proof of Lemma 2.6 is due to the referee.

1. Preliminaries

In general we use the definitions in C. Rickart's book, [4]. We assume throughout this paper that A is a complex algebra. For M a subset of A, we denote by $R[M]$ and $L[M]$ the right and the left annihilator of M respectively (that is $R[M] = \{a \in A \mid Ma = 0\}$). When A is semi-simple, A is a modular annihilator algebra if for any maximal modular left ideal M of A, $R[M] \neq 0$; the elementary properties of modular annihilator algebras are given in [1] and [7]. A subset M of A is orthogonal if whenever $u, v \in M$, $u \neq v$, then $uv = 0$.

We shall be concerned with algebras which have an involution $*$. $*$ is a proper involution if whenever $vv^* = 0$, then $v = 0$. If A has an involution $*$ and a norm $\|\cdot\|$ such that $\| vv^* \| = \| v \|^2$ for all $v \in A$, then we say that the norm $\|\cdot\|$ has the B^*-property. $u \in A$ is self-adjoint if $u = u^*$ and normal if $uu^* = u^*u$.

Now assume that A is a semi-simple modular annihilator algebra with a

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proper involution. A has minimal left ideals, and therefore, minimal idempotents (see [4, Lemma (2.1.5), p. 45]). Then since A has a proper involution, A contains self-adjoint minimal idempotents by [4, Lemma (4.10.1), p. 261]. Furthermore it follows from [1, Theorem 4.2, p. 569] that every non-zero left or right ideal of A contains a self-adjoint minimal idempotent, and that every maximal modular left (right) ideal M is of the form $A(1 - h)$ $((1 - h)A)$ where h is a self-adjoint minimal idempotent. We denote the set of all self-adjoint minimal idempotents in A by H_A and the socle of A by S_A.

DEFINITION. Assume that A is semi-simple and that K is a right (left) ideal of A which is the sum of a finite number of minimal right (left) ideals of A. Then we say that K has finite order and define the order of K to be the smallest number of minimal right (left) ideals which have the sum K.

Assume that A is semi-simple. Using a modification of the proof of the lemma on page 573 of [1], we can prove the following:

(1.1) If K is a right (left) ideal of A with finite order n, and

$$\{e_1, e_2, \cdots, e_m\}$$

is an orthogonal set of minimal idempotents of A in K, then $m \leq n$.

Let A be a normed algebra with norm $\|\cdot\|$. The normed algebra A is called a Q-algebra if the set of all quasi-regular elements of A is open in the topology of the norm; in this case the norm is called a Q-norm. If A is a Banach algebra, then A is a Q-algebra by a standard theorem; see [4, Theorem (1.4.20), p. 18]. When A is a semi-simple Q-algebra with dense socle, then A is a modular annihilator algebra by [7, Lemma 3.11, p. 41]. Also if A is a semi-simple modular annihilator algebra, then any norm on A is a Q-norm by [6, Lemma 2.8, p. 376].

For any $v \in A$, we denote the spectrum of v in A as $\sigma_A(v)$, and we define

$$\rho_A(v) = \sup \{|\lambda| \mid \lambda \in \sigma_A(v)\}.$$

When the algebra A is understood from the context, we write simply $\sigma(v)$ and $\rho(v)$. A norm $\|\cdot\|$ on A is a Q-norm if and only if $\rho(v) \leq \|v\|$ for all $v \in A$ by [6, Lemma 2.1, p. 373].

We close this section with a technical lemma needed in Section 3.

LEMMA 1.2. *Let A be a *-algebra with a norm $\|\cdot\|$ which satisfies the B^*-property. If for every self-adjoint $u \in A$, $\|u\| \geq \rho(u)$, then $\|v\| \geq \rho(v)$ for all $v \in A$. Thus $\|\cdot\|$ is a Q-norm on A.*

Proof. Assume that $\lambda \in \sigma(v)$, $\lambda \neq 0$. Let $w = v/\lambda$. Then $1 \in \sigma(w)$, and we may assume that $1 \in \sigma(w + w^* - ww^*)$. Thus by hypothesis,

$$\| w + w^* - ww^* \| \geq 1.$$

Therefore

$$2\| w \| + \| w \|^2 \geq 1 \quad \text{and} \quad (1 + \| w \|)^2 \geq 2.$$

Finally $\| w \| \geq (\sqrt{2} - 1)$. Let $\alpha = (\sqrt{2} - 1) > 0$. We have shown that for any $\lambda \in \sigma(v)$, $\| v \| \geq \alpha |\lambda|$. Thus for any $v \in A$, $\| v \| \geq \alpha\rho(v)$. Then

$$\| v \| \geq \| v^n \|^{1/n} \geq (\alpha\rho(v^n))^{1/n} = \alpha^{1/n}\rho(v).$$

Taking the limit as $n \to \infty$, we have that $\| v \| \geq \rho(v)$, as was to be shown.

2. Modular annihilator algebras with involution

Throughout this section we assume that A is a semi-simple, normed, modular annihilator algebra with a proper involution $*$. The results in this section are the basis of the proofs of the main theorems of this paper, Theorems 3.3 and 3.5.

LEMMA 2.1. *Assume that $u \in A$ is normal. If $h \in H_A$ has the property that $hu = uh$, then there exists $\lambda \in \sigma(u)$ such that $(\lambda - u)h = 0$. Conversely if $\lambda \in \sigma(u)$ and $\lambda \neq 0$, then there exists $h \in H_A$ such that $uh = \lambda h$. Finally whenever $uh = \lambda h$, $h \in H_A$, then $uh = hu$.*

Proof. First assume that $h \in H_A$ and $hu = uh$. Then there is a scalar λ such that $\lambda h = huh = uh$ (hAh is a complex normed division ring). Clearly $\lambda \in \sigma(u)$.

Conversely assume that $\lambda \in \sigma(u)$ and $\lambda \neq 0$. Then either $A(\lambda - u) \neq A$ or $(\lambda - u)A \neq A$. We may assume that $A(\lambda - u) \neq A$; then $A(\lambda - u)$ is contained in some maximal modular left ideal M of A. Since A is a modular annihilator algebra, M has the form $A(1 - h)$ for some $h \in H_A$ (see Section 1). Therefore $(\lambda - u)h = 0$. Then $h(\lambda - u)(\bar{\lambda} - u^*)h = 0$ since u is normal. But $*$ is a proper involution, and hence $h(\lambda - u) = 0$. Thus $uh = hu = \lambda h$.

If $\lambda \in \sigma(u)$ and there exists $h \in H_A$ such that $uh = \lambda h$, then we call λ an eigenvalue of u. By Lemma 2.1, when $u \in A$ is normal, then all non-zero elements of the spectrum of u are eigenvalues. Also we have the following interesting fact:

(2.2) Let λ_1 and λ_2 be distinct eigenvalues of a normal element $u \in A$. Assume that h_1 and $h_2 \in H_A$ are such that $uh_1 = \lambda_1 h_1$ and $uh_2 = \lambda_2 h_2$. Then h_1 and h_2 are orthogonal.

The proof of (2.2) is easy: Note that $\lambda_2 h_2 h_1 = uh_2 h_1 = h_2 uh_1$ (by Lemma 2.1), and $h_2 uh_1 = \lambda_1 h_2 h_1$. Then $h_2 h_1 = 0$, and taking the involution of this equation, $h_1 h_2 = 0$.

LEMMA 2.3. *Assume that K is a right ideal of A of finite order. Then there exists a unique self-adjoint idempotent $e \in S_A$ such that $K = eA$.*

Proof. We may assume that $K \neq 0$. Let M be a maximal orthogonal set of self-adjoint minimal idempotents in K (note that M is non-empty).

By (1.1) M must be a finite set, and we write

$$M = \{h_1, h_2, \cdots, h_n\}.$$

Assume that $v \in K$, and let $w = v - \sum_{k=1}^{n} h_k v$. Clearly $h_k w = 0$ when $1 \leq k \leq n$. If $w \neq 0$, then there exists a self-adjoint minimal idempotent $h \in wA \subset K$. Then $h_k h = 0$ for $1 \leq k \leq n$. But also taking the involution of both sides of this equation, we have that $hh_k = 0$ for $1 \leq k \leq n$. This contradicts the maximality of M. Therefore $w = 0$, and it follows that for any $v \in K$, $v = (\sum_{k=1}^{n} h_k)v$. Let $e = h_1 + \cdots + h_n$. Then $K = eA$. The uniqueness of e is easy to verify.

Now assume that u is a normal element of A, and that $\lambda \in \sigma(u)$, $\lambda \neq 0$. Since A/S_A is a radical algebra u/λ must be quasi-regular modulo S_A. In particular there must exist $v \in A$ and $s \in S_A$ such that $(1 - v)(1 - u/\lambda) = (1 - s)$. Then whenever $(1 - u/\lambda)x = 0$ for some $x \in A$, then $(1 - s)x = 0$. It follows that $R[A(\lambda - u)] \subset sA$. Now sA is of finite order. Thus $R[A(\lambda - u)]$ is of finite order; similarly, $L[(\lambda - u)A]$ is of finite order. Applying Lemma 2.1 and Lemma 2.3, we have the following result:

PROSOPITION 2.4. *Assume that u is a normal element of A. Assume that λ is a non-zero scalar in $\sigma(u)$. Then there is a unique self-adjoint idempotent $e \in S_A$ such that*

$$R[A(\lambda - u)] = eA \quad \textit{and} \quad L[(\lambda - u)A] = Ae.$$

Clearly e has the property that $ue = eu = \lambda e$. We call e the spectral projection in A corresponding to the eigenvalue $\lambda \in \sigma_A(u)$.

LEMMA 2.5. *Assume that B is a semi-simple, modular annihilator *-subalgebra of A. Assume that u is a normal element in B, and $\lambda \in \sigma_B(u)$, $\lambda \neq 0$. Then the spectral projection in B corresponding to λ is the same as the spectral projection in A corresponding to λ.*

Proof. Let f be the spectral projection in B corresponding to the non-zero eigenvalue λ of u. Let $w = u - \lambda f$. w is a normal element of B. Suppose $\lambda \in \sigma_B(w)$. Then by Lemma 2.1, there exists $g \in H_B$ such that $gw = wg = \lambda g$. Now $fw = wf = 0$, and thus $0 = fwg = \lambda fg$; it follows that $fg = gf = 0$. Therefore $ug = gu = \lambda g$, and by the definition of f it follows that $g = fg$. This is a contradiction. Then $\lambda \notin \sigma_B(w)$, and since B is a subalgebra of A, $\lambda \notin \sigma_A(w)$.

Now assume that e is the spectral projection in A corresponding to the eigenvalue λ of u. Note that $f = ef$. But then $(e - f)w = (e - f)(u - \lambda f) = \lambda e - \lambda ef = \lambda(e - f)$. Since $\lambda \notin \sigma_A(w)$, $(e - f) = 0$. This completes the proof of the Lemma.

The last lemma of this section plays an important role in the proof of Theorem 3.3.

LEMMA 2.6. *Assume that A has a norm $\|\cdot\|$ which has the B^*-property. Let*

B be the completion of A in this norm. Then S_B is dense in B with respect to $\|\cdot\|$.

Proof. Let I be the closure of S_A in B. Let π be the natural projection of B onto the quotient algebra B/I. Now since A is a modular annihilator algebra, by [7, Theorem 2.4, p. 38], whenever $v \in A$, then $\pi(v)$ is quasi-regular in B/I. Now assume that u is an arbitrary self-adjoint element in B. There exists a sequence of self-adjoint elements $\{u_n\} \subset A$ such that $\| u_n - u \| \to 0$. Now $\pi(u_n)$ has zero spectral radius and is self-adjoint in B/I. By [4, Theorem (4.9.2), p. 249], B/I is a B^*-algebra. Therefore $\pi(u_n) = 0$ for all n, and it follows that $u \in I$. Therefore $B = I$. It is easy to verify that $S_A \subset S_B$, and therefore the closure of S_B is B.

3. Algebras with the spectral expansion property

DEFINITION 3.1. Assume that A is a *-algebra and that A has a norm $\|\cdot\|$ with the B^*-property. Then $u \in A$ has a spectral expansion in A if
(1) either (i) the non-zero spectrum of u in A is a sequence $\{\lambda_k\}$ or (ii) the non-zero spectrum of u in A is a finite set, $\{\lambda_1, \cdots, \lambda_n\}$.
(2) In case (i), there exists an orthogonal sequence of self-adjoint idempotents $\{h_k\} \subset S_A$ such that $u = \sum_{k=1}^{\infty} \lambda_k h_k$ (convergence in the norm $\|\cdot\|$). In case (ii) there exists a finite orthogonal set of self-adjoint idempotents $\{h_1, \cdots, h_n\} \subset S_A$ such that $u = \lambda_1 h_1 + \cdots + \lambda_n h_n$.

For convenience when $u \in A$ has a spectral expansion in A, we shall not distinguish between cases (i) and (ii) in the definition. We write simply $u = \sum \lambda_k h_k$, leaving the summation without limits.

DEFINITION 3.2. An algebra A has the spectral expansion property if A has an involution $*$ and a norm with the B^*-property, and every normal element of A has a spectral expansion in A.

Let $\mathcal{H}$ be a Hilbert space. We denote by $\mathfrak{F}[\mathcal{H}]$, the algebra of bounded operators on $\mathcal{H}$ which have finite-dimensional range. $\mathcal{C}[\mathcal{H}]$ denotes the algebra of completely continuous operators on $\mathcal{H}$. When we say that a *-subalgebra A of $\mathcal{C}[\mathcal{H}]$ has the spectral expansion property, it is to be understood that the involution and the norm in Definition 3.2 are those induced by the unique involution and B^*-norm on $\mathcal{C}[\mathcal{H}]$.

In the next theorem we characterize those algebras which have the spectral expansion property.

THEOREM 3.3. *Assume that A is a semi-simple algebra with a proper involution* *. *Then the following are equivalent:*

(1) *There exists a Hilbert space $\mathcal{H}$ such that A is $*$-isomorphic to a *-subalgebra of $\mathcal{C}[\mathcal{H}]$ which has the spectral expansion property.*

(2) *A has the spectral expansion property.*

(3) *A has dense socle in some Q-norm.*

(4) *A is a normed modular annihilator algebra.*

Proof. Assume (1). Then certainly A has a norm with the B^*-property. The image of A under the given *-isomorphism has the spectral expansion property, and therefore A does also.

If (2) holds, A has a norm with the B^*-property, $\|\cdot\|$. If u is any self-adjoint element of A, u has a spectral expansion $\sum \lambda_k h_k$ in A. Then

$$|\lambda_k| \, \| h_k \| = \| u h_k \| \leq \| u \| \, \| h_k \|.$$

Thus $|\lambda_k| \leq \| u \|$. Then $\rho(u) \leq \| u \|$ and by Lemma 1.2, $\rho(v) \leq \| v \|$ for any $v \in A$. Thus $\|\cdot\|$ is a Q-norm on A. Clearly S_A is dense in A in the norm $\|\cdot\|$.

(3) implies (4) by [7, Lemma 3.11, p. 41].

Now assume that (4) holds. As a consequence of [5, Theorem 5.2, p. 318], A has a faithful *-representation into the bounded operators on a Hilbert space. In particular A has a norm with the B^*-property. Let B be the completion of A in this norm. By Lemma 2.6, S_B is dense in B. Then by a result of I. Kaplansky, [3, Theorem 2.1], there exists a $*$-isomorphism γ of B into the completely continuous operators on some Hilbert space $\mathfrak{H}$. It remains to be shown that $\gamma(A)$ has the spectral expansion property as a subalgebra of $\mathfrak{C}[\mathfrak{H}]$. The norm and involution on $\gamma(A)$ are those induced by the unique involution and B^*-norm on $\mathfrak{C}[\mathfrak{H}]$. Assume that $u \in \gamma(A)$ is normal. Then by the standard spectral theorem for normal completely continuous operators, u has a spectral expansion $\sum \lambda_k h_k$ in $\mathfrak{C}[\mathfrak{H}]$. Now $\mathfrak{C}[\mathfrak{H}]$ is a modular annihilator algebra, and it is easy to verify that h_k is the spectral projection in $\mathfrak{C}[\mathfrak{H}]$ corresponding to the eigenvalue λ_k of u in the sense of Proposition 2.4. Now u has the same non-zero spectrum in $\gamma(A)$ as in $\mathfrak{C}[\mathfrak{H}]$. Then by Lemma 2.5, h_k is the spectral projection in $\gamma(A)$ corresponding to λ_k. Therefore u has the spectral expansion $\sum \lambda_k h_k$ in $\gamma(A)$.

Now we concern ourselves specifically with *-subalgebras of $\mathfrak{C}[\mathfrak{H}]$. After the following preliminary lemma, we characterize those *-subalgebras which have the spectural expansion property.

LEMMA 3.4. *Assume that* $\mathfrak{H}$ *is a Hilbert space, and that* A *is a* **-subalgebra of* $\mathfrak{C}[\mathfrak{H}]$. *Then* $S_A = \mathfrak{F}[\mathfrak{H}] \cap A$.

Proof. If $E \in H_A$, then since E is a projection in $\mathfrak{C}[\mathfrak{H}]$, $E \in \mathfrak{F}[\mathfrak{H}]$. This implies that $S_A \subset \mathfrak{F}[\mathfrak{H}] \cap A$.

Now assume that $T \neq 0$, $T \in \mathfrak{F}[\mathfrak{H}] \cap A$. Then the *-algebra TAT^* is finite dimensional (in fact $T\mathfrak{C}[\mathfrak{H}]T^*$ is finite-dimensional) and semi-simple. Let F be a minimal self-adjoint idempotent in TAT^*. For some $V \in A, F = TVT^*$. Then

$$FAF = F(TVT^*)A(TVT^*)F \subset F(TAT^*)F \subset FAF.$$

Thus $FAF = F(TAT^*)F$ which is a division ring, and it follows that $F \in H_A$. Also $F \in TA$. We have shown that whenever $T \in \mathfrak{F}[\mathfrak{H}] \cap A$, then TA contains a minimal idempotent of A.

Again assume that $T \in \mathfrak{F}[\mathfrak{H}] \cap A$, $T \neq 0$. Let M be a maximal set of orthogonal self-adjoint minimal idempotents of A in TA. M must be finite since otherwise there would be infinitely many mutually orthogonal projections in $\mathfrak{F}[\mathfrak{H}]$ with ranges contained in the range of $T \in \mathfrak{F}[\mathfrak{H}]$. Proceeding as in the proof of Lemma 2.3 (and using the conclusion of the previous paragraph), we find that $TA = EA$ where E is a self-adjoint idempotent in S_A. Now for any $W \in A$, $TW = ETW$, and therefore $(T - ET)A = 0$. Thus $T = ET$ and $T \in S_A$.

THEOREM 3.5. *Assume that $\mathfrak{H}$ is a Hilbert space and that A is a *-subalgebra of $\mathfrak{C}[\mathfrak{H}]$. Then the following are equivalent:*

(1) *Whenever $T \in A$ is a normal operator and $\sum \lambda_k E_k$ is the spectral expansion of T in $\mathfrak{C}[\mathfrak{H}]$, then $E_k \in A$ for all k and $\sum \lambda_k E_k$ is a spectral expansion for T in A.*

(2) *$\mathfrak{F}[\mathfrak{H}] \cap A$ is dense in A in some Q-norm.*

(3) *A is a modular annihilator algebra.*

Proof. First we note that A must be semi-simple by [4, Theorem (4.1.19), p. 188]. Next by Lemma 3.4, $S_A = \mathfrak{F}[\mathfrak{H}] \cap A$.

Assume that (1) holds. Then A satisfies Theorem 3.3 (1). Then by Theorem 3.3 (3), A has dense socle in some Q-norm. Thus $\mathfrak{F}[\mathfrak{H}] \cap A$ is dense in A in some Q-norm.

(2) implies (3) by [7, Lemma 3.11, p. 41].

Now assume (3) holds. By Theorem 3.3, A has the spectral expansion property. Assume $T \in A$ is a normal operator. Let $\sum \lambda_k E_k$ be the spectral expansion of T in $\mathfrak{C}[\mathfrak{H}]$. E_k is the spectral projection in $\mathfrak{C}[\mathfrak{H}]$ corresponding to the eigenvalue λ_k of T in the sense of Proposition 2.4. Then by Lemma 2.5, $E_k \in A$ for all k, and thus T has spectral expansion $\sum \lambda_k E_k$ in A.

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