QUASI-REGULAR ELEMENTS AND DORROH EXTENSIONS

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0. Introduction

In this paper, all rings are to be associative, and all mappings are to be written (at least in spirit) to the left. We shall consider certain portions of $Q(U)$, the group of quasi-regular (q.r.) elements of a ring U, portions which, in particular, extend the Jacobson radical $J(U)$ of U. The portion under consideration will depend upon the way in which U is regarded as an algebra. Our principal device will be the introduction of a different multiplication on U, an introduction brought about by employing a q.r. operator on U. We shall show (Theorem 1) that if a suitable change of multiplication is introduced into the bimultiplication ring $M(U)$, [5], on U, then this modified $M(U)$ can be injected into the bimultiplication ring of a modified version of U which is obtained from U by making a related change of multiplication theorem. By ememploying a properly chosen commutative subring $S(U)$ of $M(U)$, it is possible to turn U into an $S(U)$-algebra. If U is an algebra over a commutative ring T, and if U has trivial bicenter [5], [3], then the map α which effects the action of T on U can be factored through $S(U)$ (Theorem 2).

Let $\mathfrak{Q}(U, \alpha, T)$ be the set of all q.r. elements of U which are also q.r. with respect to all the changes of multiplication on U which are induced by the members of $Q(T)$. One finds that $\mathfrak{Q} \geq J(U)$. If $Q(U)$ is central in U, and if U is treated as an $S(U)$-algebra, the resulting $\mathfrak{Q}$, here called $\mathfrak{R}(U)$, is (Theorem 3) a subring of U, an algebra over a certain $\mathfrak{Q}$ of $S(U)$. If U is without divisors of zero, is commutative, is not a radical ring, and has its underlying abelian group U^+ irreducible as a U-module, then $\mathfrak{R}(U)$ vanishes (Theorem 4).

If a ring extension is not too formidable, it is possible to obtain information about its Jacobson radical. We select an uncomplicated extension $V(U, \alpha, T)$ of U by T (going back to Dorroh [1]) which happens to be splitting. If T is an integral domain, and if $\mathfrak{Q}(U, \alpha, T)$ has been turned into an appropriate algebra, then the obvious $\mathfrak{Q}$ of V is a related extension of $\mathfrak{Q}(U, \alpha, T)$ by a certain $\mathfrak{Q}$ of T (Theorem 5). If T is commutative and if the members of $J(T)$ operate on U in such a way that $ru = ru^2$ for all $r \in J(T)$ and all $u \in U$, then $J(V)$ is a related extension (Theorem 6) of $J(U)$ by $J(T)$. Finally, if U^+ is an irreducible T-module (T commutative and U a T-algebra), and if the members of $J(T)$ do not act as automorphisms on U^+, then (Theorem 7) $J(V)$ reduces to the algebra direct sum of $J(T)$ and U.

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We shall take for granted the reader's knowledge of the introductory portions of [5]. See also [3]. If σ is a bimultiplication on U, then the endomorphisms σ_L and σ_R are to be the respective left and right mappings on U^+ induced by σ. Similarly, if $u \in U$, then u_L and u_R are to be the respective left and right multiplications on U by u. Such standard objects and concepts as q.r. elements u, their quasi-inverses (q.i.'s) u^*, the circle composition, and the Jacobson radical and its properties are treated as in such easily available sources as [2] and [4]. A subset A of a ring U is said to be *central in* U if A is extended to the center of U. For any subset B of U, $C(B, U)$ is to be the centralizer of B in U. The center of U is, of course, just $C(U, U)$. If ζ is a map with domain B, and if $A \leq B$, then $\zeta \mid A$ is just ζ with its domain cut down to A. The symbol ζ^{-1} will be used whenever a complete inverse image is required whether or not ζ is one-to-one. If A and B are sets, then A^B has its usual meaning of all functions with domain B and range included in A. The symbol $\oplus$ denotes direct sum. A subset A of a ring U is said to have *the left (right) ideal property in* U if $ua(au) \in A$ for each $u \in U$ and each $a \in A$.

1. Symmetric bimultiplications

Let U be a ring with bimultiplication ring $M(U)$. Choose any $\tau \in M(U)$ which is *symmetric* in that $\tau u = u\tau$ for each $u \in U$. Write

$$g_\tau(u_1, u_2) = u_1 u_2 - \tau u_1 u_2$$

for all $u_1, u_2 \in U$. Under $+$ and g_τ, the set U is reconstituted as a ring U_τ where $U_\tau^+ = U^+$. If, for instance, τ is the zero of $M(U)$, then U_τ is the same ring as U, while if τ is the unity of $M(U)$, then U_τ is the zero ring on U^+. Let ν_U be the member of Hom $(U, M(U))$ which carries each $u \in U$ onto that inner bimultiplication $\nu_U(u)$ which consists of the pair of maps u_L and u_R. One calls $\tau \in M(U)$ *permuting* if $(\sigma u)\tau = \sigma(u\tau)$ and if $(\tau u)\sigma = \tau(u\sigma)$ for each $u \in U$ and for each $\sigma \in M(U)$. It is known [5] that the members of Im ν_U are permuting in $M(U)$, although the only symmetric maps among these are the images of the central members of U.

THEOREM 1. *Suppose that $\tau \in M(U)$ is both symmetric and permuting. Then τ is central in $M(U)$, and $\tau' = \nu_{M(U)}(\tau)$ is symmetric in $M(M(U))$. If, further, τ is q.r. in $M(U)$, then $(M(U))_{\tau'}$ can be injected into $M(U_\tau)$ in such a way that the image of τ is central.*

Proof. For each $x \in U$ and each $\sigma \in M(U)$, $x(\tau'\sigma) = x(\tau\sigma) = (x\tau)\sigma = (\tau x)\sigma$, this last since τ is symmetric. Likewise, $x(\sigma\tau') = x(\sigma\tau) = (x\sigma)\tau = \tau(x\sigma) = (\tau x)\sigma$, this last since τ is permuting. Thus $\tau'\sigma$ and $\sigma\tau'$ are equal as right operators (similarly, equal as left operators). Thus, τ' is symmetric as a member of $M(M(U))$, so that $(M(U))_{\tau'}$ exists. Because $\tau'\sigma = \sigma\tau'$ can be rewritten as $\tau\sigma = \sigma\tau$, τ is central in $M(U)$.

For each $\eta \in (M(U))_{\tau'}$, construct a pair of maps $\lambda_\eta^{[\tau]} = \lambda_\eta$ from U_τ to U_τ

via

$$\lambda_\eta a = \eta(a - \tau_a) \quad \text{and} \quad a\lambda_\eta = (a - \tau a)\eta$$

for each $a \in U_\eta$. For $a, b \in U$ one can establish such relationships as $g_\tau(a, \lambda_\eta b) = g_\tau(a\lambda_\eta, b)$ by appealing to the assumptions on τ and η. In this routine way we can show that $\lambda_\eta \in M(U_\tau)$. We define $\lambda^{[\tau]} = \lambda$ by setting $\lambda(\eta) = \lambda_\eta$. We find that λ preserves both addition and $g_{\tau'}$-multiplication. If, for any η, $\lambda(\eta) = 0$, then $\lambda_\eta a = 0 = a\lambda_\eta$ for all $a \in U$. Upon expanding, we have $\eta = \tau\eta \in M(U)$. Let ι here denote the unity of $M(U)$. We now have $(\iota - \tau)\eta = 0$. But τ was assumed to be q.r. so that $\iota - \tau$ is regular, whence $\eta = 0$, and λ is an injection.

To show that $\lambda(\tau)$ is central in $M(U_\tau)$, we first note that τ^*, the q.i. of τ, is symmetric; for, τ, $\iota - \tau$, and therefore $(\iota - \tau)^{-1} = \iota - \tau^*$, are symmetric. From this it is easy to show that $\tau^* \in M(U_\tau)$. Thus $(U_\tau)_{\tau^*}$ is meaningful; but a brief calculation shows that this last reduces as a ring to U. Now if W is a ring and if $\rho \in M(W)$ is symmetric, then one readily shows that $M(W) \leq M(W_\rho)$. In particular, $M(U_\tau) \leq M((U_\tau)_{\tau^*} = M(U)$. Thus, under the supposition that $\gamma \in M(U_\tau)$,

$$\lambda_\tau \gamma - \gamma\lambda_\tau = \tau\gamma - \gamma\tau - \tau^2\gamma + \gamma\tau^2.$$

Since τ is central in $M(U)$, this last sum is zero, so that each $\gamma \in M(U_\tau)$ commutes with λ_τ, as we wished to show.

One should observe that the quasi-regularity of τ implies that $\lambda^{[\tau]}(\tau^*) = -\tau$ as members of $M(U_\tau)$. For, since $\tau \in M(U) \leq M(U_\tau)$, $\lambda(\tau^*) + \tau \in M(U_\tau)$, and

$$(\lambda_{\tau^*} + \tau)b = \tau^*(b - \tau b) + \tau b = 0$$

for each $b \in U$ (likewise for operators on the right).

2. Factorization of some algebra-producing maps

If U and T are rings, then U is said to be a *T-algebra via* $\alpha \in \mathrm{Hom}\,(T, \mathrm{End}\, U^+)$ if $\mathrm{Im}\,\alpha$ centralizes all left and all right multiplications on U by elements of U: $(\alpha(t)u)v = \alpha(t)(uv) = u(\alpha(t)v)$ for all $t \in T$ and all $u, v \in U$. We shall call such an α a *T-algebra-producing mapping* (or *map*) for U. The set $\mathrm{Sm}\,(U)$ of symmetric bimultiplications on U, though closed under addition and subtraction, cannot be multiplicatively closed if $\mathrm{Sm}(U)$ is non-commutative. Nevertheless, the intersection $S(U)$ of $\mathrm{Sm}\,(U)$ with the centralizer in $M(U)$ of $\mathrm{Sm}\,(U)$ is easily shown to be a subring of $M(U)$. The members of $\mathrm{Sm}\,(U)$, and therefore of $S(U)$, are certain pairs of equal endomorphisms on U^+. Let κ_U be the map which carries each $\gamma \in S(U)$ onto the common endomorphism of its endomorphism pair. Not only is κ_U in $\mathrm{Hom}\,(S(U), \mathrm{End}\, U^+)$, but U is an $S(U)$-algebra via monomorphic κ_U. Let U be any commutative ring. Then U is a U-algebra via $\kappa_U \nu_U$. We shall call this algebra *the multiplication algebra on the commutative ring* U, and we shall have occasion to refer to it in the sequel.

Recall [5] [3] that *the bicenter of* U is defined as $\ker \nu_U$. If U has a unity, or any other left or right non-divisor of zero, then the bicenter is trivial.

THEOREM 2. (a) *If U is a ring with trivial bicenter, then* $\mathrm{Sm}(U) = S(U)$.
(b) *Let U, a ring with a trivial bicenter, be a T-algebra via α where T is commutative. Then there exists $\beta \in \mathrm{Hom}(T, S(U))$ such that $\alpha = \kappa_U \beta$.*

Proof. (a) Let $U^{\#}$ be the ideal in U which is generated by all the composite members of U, that is, by all $u \in U$ where $u = ab$ for some $a, b \in U$. If $\sigma, \delta \in \mathrm{Sm}(U)$ then

$$(ab)\sigma\delta = \delta[a(b\sigma)] = (\delta a)(b\sigma) = [(a\delta)b]\sigma = [a(b\delta)]\sigma = (ab)(\delta\sigma).$$

Likewise, $\sigma\delta(ab) = \delta\sigma(ab)$, so that $\sigma\delta$ and $\delta\sigma$ are equal if their domains are cut down to $U^{\#}$.

Now suppose that one could find an $a \in U$ for which $b = (\sigma\delta - \delta\sigma)a \neq 0$. Since U has a trivial bicenter, there must exist some $c \in U$ such that at least one of bc and cb is non-zero. From $bc \neq 0$, $(\sigma\delta - \delta\sigma)(ac) \neq 0$, contradicting the statement that $(\sigma\delta - \delta\sigma)| U^{\#} \neq 0$. From $cb \neq 0$, $c[(\sigma\delta - \delta\sigma)a] = ca(\sigma\delta - \delta\sigma) \neq 0$, again a contradiction. (In this last step, we first use the assumption that σ lies in $\mathrm{Sm}(U)$, not just in $M(U)$.) We have proved that $\delta\sigma$ and $\sigma\delta$ coincide as left mappings. Similarly, they coincide as right mappings, giving (a).

(b) If $\gamma \in \mathrm{End}\, U^{+}$ has the property that it centralizes both the left and the right multiplications by the elements of U on U, define a pair of maps $\gamma^{\#}$ from U to U by $\gamma^{\#}u = \gamma u = u\gamma^{\#}$. Then $\gamma^{\#} \in \mathrm{Sm}(U)$ which last equals $S(U)$, by (a). For each $t \in T$, take $\beta(t) = (\alpha(t))^{\#}$, and observe that $\kappa_U(\alpha(t))^{\#} = \alpha(t)$. It is clear that β preserves addition; and the commutativity of T, used only here, causes β to preserve multiplication, completing the proof.

3. Change of multiplication

Let T be a commutative ring, and let U be a T-algebra via α. For $t \in T$, form $s(t) = (\alpha(t))^{\#} \in S(U)$, as in the proof of Theorem 2. Introduce a new production U, as in Section 1, by setting

$$g_{s(t)}(u, v) = uv - (\alpha(t))^{\#}(uv).$$

It will simplify notation considerably if we write g_t instead of $g_{s(t)}$, if we write $uv - \alpha(t)(uv)$ or $uv - t(uv)$ instead of $uv - (\alpha(t))^{\#}(uv)$, and if we write U_t instead of $U_{s(t)}$. If 0 is the zero of T, we shall write U instead of U_0. To say that $u \in Q(U_t)$ means that there exists (an actually unique) $u^{(t)} \in U_t$ such that $g_t(u, u^{(t)}) = u + u^{(t)} = g_t(u^{(t)}, u)$. One calls $u^{(t)}$ (if it exists) *the t-quasi-inverse* (t-*q.i.*) *of* $u \in U$ and says that u is *t-quasi-regular* (t-*q.r.*) *in* U. Here, $(u^{(t)})^{(t)} = u$, and $u^{(t)} \in Q(U_t)$. If u is q.r. we write the usual u^* instead of $u^{(0)}$ and substitute the standard notations q.i. and q.r. for respective 0-q.i. and 0-q.r.

Let $\mathfrak{Q}(U, \alpha, T) = \bigcap_q Q(U_q)$ where q runs over all of $Q(T)$. It will be convenient (i) to write $\mathfrak{Q}(T)$ instead of $\mathfrak{Q}(T, \kappa_T \nu_T, T)$, T here being considered as the multiplication algebra on T; (ii) to write $\mathfrak{R}(U)$ instead of $\mathfrak{Q}(U, \kappa_U, S(U)$. Just as we fashioned members of $M(U_\tau)$ from elements of $M(U)$ in Section 1, we now construct T_t-algebra-producing maps for U_t from T-algebra-producing maps for U where $t \in T$. That is, if U is a T-algebra (T commutative) via α, let $\alpha_t \in (U^U)^T$ be defined by setting $\alpha_t(s)u = \alpha(s - ts)u$ for all $s \in T$ and all $u \in U$. Then $\alpha_t \in \operatorname{Hom}(T_t, \operatorname{End} U^+)$, and U_t is a T_t-algebra via α_t. The basic computational result is as follows:

LEMMA 1. *Let U be a T-algebra (T commutative).*

(a) *If $u \in U$ and if $q_1, q_2 \in Q(T)$, then $u \in Q(U_{q_1})$ if and only if $u - (q_1 \circ q_2^*)u \in Q(U_{q_2})$, in which case*

$$(u - (q_1 \circ q^*)u)^{(q_2)} = u^{(q_1)} - (q_1 \circ q_2^*)u^{(q_1)}.$$

(b) *For each $t \in Q(T)$, $\mathfrak{Q}(U, \alpha, T) = \mathfrak{Q}(U_t, \alpha_t, T_t)$ as subsets of U.*

Proof. (a) can be verified directly. As for (b), fix $t \in Q(T)$ throughout the discussion. Then $u \in \mathfrak{Q}(U_t, \alpha_t, T_t)$ if and only if $u \in Q((U_t)_r)$ for each $r \in Q(T_t)$. But $u \in Q((U_t)_r)$ if and only if there exists $v = v(r) \in U$ for which

$$\begin{aligned} g_t(u, v) - \alpha_t(r)g_t(u, v) &= uv - (t \circ (r - tr))uv = u + v \\ &= vu - (t \circ (r - tr))vu = g_t(v, u) - \alpha_t(r)g_t(v, u). \end{aligned}$$

By part (a), $r - tr \in Q(T)$, whence $u \in Q((U_t)_r)$ if and only if $u \in Q(U_{d(r;t)})$ where $d(r; t) = t \circ (r - tr) \in Q(T)$. We now have

$$Q(U_t, \alpha_t, T_t) = \textstyle\bigcap_r Q(U_{d(r;t)})$$

as r ranges over $Q(T_t)$. One easily finds that

$$r = (t^* \circ d(r; t)) - t^*(t^* \circ d(r; t)).$$

Now suppose that j is any member of $Q(T)$. Define $e(j; t) \in T$ by

$$e(j; t) = (t^* \circ j) - t^*(t^* \circ j).$$

By part (a), $e(j; t) \in Q(T_t)$. A short calculation gives $d(e(j; t); t) = j$, so that $\bigcap_r Q(U_{d(r;t)}) = \bigcap_j Q(U_j)$ as j ranges over $Q(T)$. But this last intersection is just $\mathfrak{Q}(U, \alpha, T)$, completing the proof. We have, incidentally, established that, for given $t \in Q(T)$, $Q((U_t)_r) = Q(U_{d(r;t)})$ and $Q(U_j) = Q((U_t)_{e(j;t)})$ for all $j \in Q(T)$ and all $r \in Q(T_t)$. These identities will be used below without reference.

THEOREM 3. (a) *Let T be a commutative ring with unity 1_T, and suppose that U is a T-algebra via monomorphic α where $\alpha(1_T)$ is the identity automorphism on U^+. Suppose that $Q(U)$ is central and that, for each $u \in Q(U)$, there is (necessarily precisely) one $s \in Q(T)$ such that $u_L = \alpha(s^*)$. Then $\mathfrak{Q}(U, \alpha, T)$ is a subring of U in such a way that it is a $\mathfrak{Q}(T)$-algebra.*

(b) *If T is a commutative ring and unity 1_T, then $\mathfrak{Q}(T)$ is a subring of T.*

(c) *Let U be a ring for which $Q(U)$ is central. Then $\mathfrak{R}(U)$ is a subring of U in such a way that it is a $\mathfrak{Q}(S(U))$-algebra.*

Proof. (a) Suppose that $v_1, v_2 \in \mathfrak{Q}(U, \alpha, T)$. In particular, $v_2 \in Q(U)$ so that hypothesis provides us with $q \in Q(T)$ such that $(v_2)_L = \alpha(q^*)$. Since $v_1 \in \mathfrak{Q}(U, \alpha, T)$, $v_1 \in Q(U_q)$. There exists, by hypothesis, $p \in Q(T)$ such that $(v^*)_L = \alpha(p)$ Since $v_2 \circ v^* = 0$, $\alpha(q^* \circ p) = (v_2 \circ v_2^*)_L = 0$. But α is a monomorphism by assumption, so that $q^* \circ p = 0$, and $p = q$. Using the centrality of $Q(U)$ and this last we have (I) $v_2^* v_1 = qv_1 = v_1 v_2{}^*$.

From $v_1 + v_1^{(q)} = (1_T - q)v_1^{(q)}v_1$, we have

$$v_2 v_1 + v_2 v_1^{(q)} = q(1_T - q)v_1^{(q)}v_1,$$

while

$$v_1 v_2 + v_1^{(q)}v_2 = (1_T - q)v_1^{(q)}v_1 v_2 = (1_T - q)v_1^{(q)}v_2 v_1$$
$$= (1_T - q)v_1^{(q)}(qv_1) = q(1_T - q)v_1^{(q)}v_1,$$

again by the central position of $Q(U)$. Thus we have (II) $v_2 v_1^{(q)} = v_1^{(q)}v_2$. It is now quite simple, employing (I) and (II), to show that $(v_1 + v_2)^*$exists and equals $v_2^* + (1_T - q)^2 v_1^{(q)}$.

Suppose $v \in Q(U_t)$ where $t \in Q(T)$. Then, by Lemma 1 (a), $w = v - tv \in Q(U)$ so that $v = w - t^*w$. If u is any member of U, then $g_t(v, u) = (v - tv)u = wu = s^*u$ for some $s \in Q(T)$, as provided by hypothesis. By Lemma 1 (a), $r = s - t^*s \in Q(T_t)$, and $r^{(t)} = s^* - t^*s^*$, from which $s^* = r^{(t)} - tr^{(t)}$. That is, $g_t(v, u) = (r^{(t)} - tr^{(t)})u = \alpha_t(r^{(t)})u$; thus, (III) each $v \in Q(U_t)$ can be realized as a left multiplication under g_t-composition by $\alpha_t(r^{(t)})$ for some $r \in Q(T_t)$. The centrality of w in U as a member of central $Q(U)$ allows us to assert that $uv = uw - t^*uw = wu - t^*wu = vu$, so that v is central under ordinary, therefore under g_t-, multiplication. We now have that (IV) the center of U_t extends $Q(U_t)$.

One can readily check that (V) T_t is commutative with unity $1_{T_t} = 1_T - t^*$. Since $\alpha_t(1_T - t^*)u = u$, (VI) $\alpha_t(1_{T_t})$ is the identity automorphism on U^+. If $\alpha_t(b) = 0$ for any $b \in T_t$, then $\alpha(b - tb) = 0$. But α is a monomorphism, so that $b(1_T - t) = 0$. Since $1_T - t$ is regular, $b = 0$. Thus (VII) α_t is a monomorphism. By (III)–(VII), U_t, T_t, α_t, and 1_{T_t} can replace their respective counterparts without t in the hypothesis of (a). Recall, from Lemma 1 (b), that, as sets, $\mathfrak{Q}(U_t, \alpha_t, T_t) = \mathfrak{Q}(U, \alpha, T)$. If we change from ordinary to g_t-multiplication, the steps of the argument can now be repeated to put $v_1 + v_2$ in $Q(U_t)$. It follows that $v_1 + v_2 \in \mathfrak{Q}(U, \alpha, T)$, whence this set is closed under the addition of U.

If $v \in \mathfrak{Q}(U, \alpha, T)$, then $v \in Q(U_q)$ for each $q \in Q(T)$, from which

$$(-v) + (-v^{(q)}) = (q - 1_T)(-v)(-v^{(q)}) = (q - 1_T)(-v^{(q)})(-v).$$

Since $q - 1_T$ is a unit of T (with inverse $q^* - 1_T$), $-v \in Q(U_{2_T - q})$ where

$2_T = 2(1_T)$, and $(-v)^{(2_T - q)} = -v^{(q)}$. If r is any member of $Q(T)$, then r may be written in the form $2_T - q$ where $q \in Q(T)$ with $q^* = 2_T - r^*$. That is, $2_T - q$ is as general a member of $Q(T)$ as is q itself, whence $-v \in \mathfrak{Q}(U, \alpha, T)$, and this latter set is now closed under both addition and subtraction.

If $v \in \mathfrak{Q}(U, \alpha, T)$, and if $q, t \in Q(T)$, then

$$v - tv = v - ((t \circ q) \circ q^*)v.$$

Since $v \in \mathfrak{Q}(U_{t \circ q})$, Lemma 1 (a) places $v - tv$ in $Q(U_q)$. Allowing q to run over $Q(T)$, we have $v - tv \in \mathfrak{Q}(U, \alpha, T)$, a set which was just shown to be closed under subtraction. Thus, $tv \in \mathfrak{Q}(U, \alpha, T)$. If $v_1, v_2 \in \mathfrak{Q}(U, \alpha, T)$, then the hypothesis provides us with $r \in Q(T)$ such that $v_1 v_2 = r^* v_2$. But we have just proved that all elements like $r^* v_2$ lie in $\mathfrak{Q}(U, \alpha, T)$; that is, the set $\mathfrak{Q}(U, \alpha, T)$ is a subring of U. If $r \in \mathfrak{Q}(T)$, and if $u \in \mathfrak{Q}(U, \alpha, T)$, then $r \in Q(T)$, so that $ru \in \mathfrak{Q}(U, \alpha, T)$. Since U is a T-algebra, and since $\mathfrak{Q}(T)$ operates on $\mathfrak{Q}(U, \alpha, T)$, this last must be a $\mathfrak{Q}(T)$-algebra, establishing all of (a). Since T as the multiplication algebra satisfies the conditions in (a), $\mathfrak{Q}(T)$ is a subring of T, and we have (b).

(c) Since members of $Q(U)$ are central, $w \in Q(U)$ implies that $\nu_U(w) = (f(w))^*$ for some $f(w) \in Q(S(U))$. It follows that $w_L = \kappa_U((f(w))^*)$. Recall that $S(U)$ has a unity and that κ_U is a monomorphism which carries this unity onto the identity automorphism of U^+. Since the conditions of (a) hold, we have (c).

COROLLARY. *Let U, a ring with trivial bicenter, be a T-algebra (T commutative with unity 1_T) via monomorphic α, where $\alpha(1_T)$ is the identity automorphism on U^+, in such a way that, for each $u \in Q(U)$, $u_L = \alpha(s^*)$ and $u_R = \alpha(t^*)$ for some $s, t \in Q(T)$. Then $\mathfrak{Q}(U, \alpha, T)$ is a $\mathfrak{Q}(T)$-algebra.*

Proof. If $u \in Q(U)$ and if $y, w \in U$, then $(uy)w = u(yw) = s^*(yw) = y(s^*w) = y(uw) = (yu)w$, so that $(uy - yu)w = 0$; and $w(uy) = (wu)y = (t^*w)y = w(t^*y) = w(yu)$, so that $w(uy - yu) = 0$. If $uy - yu \neq 0$, the assumption that U has trivial bicenter provides us with at least one non-zero $w \in U$ such that at least one of $(uy - yu)w$ and $w(uy - yu)$ is non-zero. The resulting contradiction shows that each $u \in Q(U)$ is central. Now apply (a).

4. The Jacobson radical

Let T be a commutative ring, U be a T-algebra, and X be a T-subalgebra of U. Recall [2] that $(X:U) = [s;\ s \in T \text{ and } su \in X \text{ for all } u \in U]$ is an ideal ($= T$-subalgebra) of T. Let Epen U^+ be the set of ependomorphisms on U^+.

LEMMA 2. *Let T be a commutative ring, and let U be a T-algebra via α. Then*

(a) *$J(U)$ is a T-subalgebra of U via some $\alpha_J \in \mathrm{Hom}\,(T, \mathrm{End}\,(J(U))^+)$;*
(b) *if $(J(U):U) \cap \alpha^{-1}(\mathrm{Epen}\ U^+)$ is non-empty, then U is a radical ring;*
(c) *as sets, $J(U_q) = J(U)$ for each $q \in Q(T)$;*
(d) *$J(U) \leq \mathfrak{Q}(U, \alpha, T), \mathfrak{R}(U)$;*

(e) *if* $\mathfrak{Q}(U, \alpha, T)$ *has the (left, right) ideal property in* U, *then* $\mathfrak{Q}(U, \alpha, T) = J(U)$; *and*

(f) $\mathfrak{Q}(J(U), \alpha_J, T) = J(U)$.

Proof. (a) Suppose that $s \in T$ and that $u \in J(U)$. Then, for each $v \in U$, $(su)v = u(sv)$, q.r. since right multiples of radical elemenes u are q.r., so that su has q.r. right multiples exclusively and is thus itself a radical element.

(b) If $r \in (J(U):U) \cap \alpha^{-1}(\text{Epen } U^+)$, $\alpha(r) \in \text{Epen } U^+$, so that, to each $u \in U$, there corresponds at least one $u' \in U$ with $\alpha(r)u' = ru' = u$. Since $r \in (J(U):U)$, $u = ru' \in J(U)$ from which $U = J(U)$.

(c) $w \in J(U_q)$ if and only if $v = v(a) = g_q(w, a) \in Q(U_q)$ for all $a \in U$. Equivalently,

$$v - qv = (w - 2qw + q^2w)a = w'a \in Q(U)$$

where $w' = w - 2qw + q^2w$. But $w'a \in Q(U)$ for each $a \in U$ if and only if $w' \in J(U)$. Now suppose that $w \in J(U)$. Since $J(U)$ is a T-subalgebra, $w' \in J(U)$. By what we have just shown, $w \in J(U_q)$, giving $J(U) \leq J(U_q)$. By an exchange of roles, $J(U_q) \leq J(U)$, and we have (c). From (c), (d) is immediate.

(e) If $u \in \mathfrak{Q}(U, \alpha, T)$, and if $a \in U$, then $ua \in \mathfrak{Q}(U, \alpha, T)$ should this set have the right ideal property. Since $\mathfrak{Q}(U, \alpha, T) \leq Q(U)$, $ua \in Q(U)$ so that u is a radical element. That is, $\mathfrak{Q}(U, \alpha, T) \leq J(U)$. Combining this last with (d), we have the right case of (e). The left case is similar. As for (f),

$$J(U) = J(J(U)) \leq \mathfrak{Q}(J(U), \alpha_J, T) \leq J(U).$$

THEOREM 4. *Let U be a non-trivial commutative ring without divisors of zero. Suppose, further, that U is not a radical ring and that U^+ as a U-module is irreducible. Then* $\mathfrak{R}(U) = 0$.

Proof. Since U^+ is irreducible, and since $J(U) \neq U$, we must have $J(U) = 0$. Now $u \in \mathfrak{R}(U)$ if and only if $u \in Q(U_\gamma)$ for each $\gamma \in Q(S(U))$. If we let $w_\gamma = u - \gamma u$, we can solve for u to obtain

$$u = w_\gamma - \gamma^* w_\gamma = (\iota - \gamma_L^*)w_\gamma$$

where, here, ι is the identity automorphism on U^+.

If v is a non-zero member of U, then v_L is a monomorphism on U^+ since U has no divisors of zero. Since U^+ is U-irreducible, U is strictly cyclic on v [2, Prop. 1, p. 6], so that v_L is an ependomorphism and, therefore, an automorphism on U^+. Since U is commutative, it is possible to construct $\tau = (\iota - v_L)^\# \in \text{Sm}(U)$ where $\tau_L = \iota - v_L$. See the proof of Theorem 2 (b). We shall show that $\tau \in Q(S(U))$.

Suppose that $\eta \in \text{Sm}(U)$. For each $w \in U$,

$$(\tau\eta)w = (\iota - v_L)\eta w = \eta w - v(\eta w) = \eta w - (v\eta)w = \eta w - (\eta v)w$$
$$= \eta(w - vw) = \eta(\iota - v_L)w = \eta\tau w,$$

making $\tau\eta$ and $\eta\tau$ equal as left operators (similarly, as right operators). Thus $\tau \in S(U)$.

Let δ be the automorphism on U^+ which is inverse to v_L. With η and w as above, $v(w\eta) = (vw)\eta$. Replacing w by δx where $x = vw$ and operating on both sides of the resulting identity by δ, we have $(\delta x)\eta = \delta(x\eta)$. Since x is as general a member of U as w is, we have established the permutation property $\eta_R\,\delta = \delta\eta_R$ (similarly, $\eta_L\,\delta = \delta\eta_L$) which will be of use presently.

In this context only, let us write $w' = \delta w$ for all $w \in U$. If $a, b, \in U$, then $(\iota - \delta)(ab) = ab - \delta v_L(a'b) = ab - a'b = ab - (\delta a)b = [(\iota - \delta)a]b$. Similarly, $(\iota - \delta)(ab) = ab - ab' = a[(\iota - \delta)b]$. Moreover,

$$[(\iota - \delta)a]b = ab - a'b = ab - va'b' = a[(\iota - \delta)b].$$

Thus, it is possible to construct $\sigma = (\iota - \delta)^{\#} \in \mathrm{Sm}\,(U)$ where $\sigma_L = \iota - \delta$. The permutation property of δ, above, can now be used to show that $\sigma\eta = \eta\sigma$ for each $\eta \in \mathrm{Sm}\,(U)$. At once, $\sigma \in S\,(U)$, and it is readily verified that σ and τ are q.i. to each other, placing both in $Q(S(U))$.

For $u \in \mathfrak{R}(U)$ and $v \in U$,

$$vu = v_L\,u = (\iota - \tau_L)u = (\iota - \tau_L)(\iota - {\gamma^*}_L)w_\gamma$$

for each $\gamma \in Q(S(U))$. That is, $vu = (\iota - (\tau \circ \gamma^*)_L)w_\gamma$. But $\rho = \gamma \circ \tau^*$ is as general a member of $Q(S(U))$ as is γ itself. That is, $vu = (\iota - \rho_L^*)y_\rho$ where $y_\rho = w_{\tau\circ\rho} \in Q(U)$. Equivalently, $vu - \rho vu \in Q(U)$ for each $\rho \in Q(S(U))$ since the expression in question reduces to y_ρ. By the initial remarks in the proof, $vu \in \mathfrak{R}(U)$ giving this last the left ideal property. We can now apply Lemma 2 (e) to show that $\mathfrak{R}(U) = J(U)$. But $J(U)$ has already been shown to be 0, so that the proof is complete.

5. A splitting extension

Let U be a T-algebra via α where, throughout this section, T is a commutative ring. Then there is a standard way, which goes back to Dorroh [1], of extending the T-algebra U to a splitting extension by the T-algebra T; let $V(U, \alpha, T)$ be the set of all (s, u), where $s \in T$ and $u \in U$, under direct-sum addition, with multiplication given by

$$(s, u)(t, w) = (st, sw + tu + uw) \qquad (s, t \in T \quad \text{and} \quad u, w \in U)$$

and with V turned into a T-algebra via the $\alpha_{\#}$ in Hom $(T, \mathrm{End}\,(T^+ \oplus U^+))$ which is defined by setting $\alpha_{\#}(t)(s, u) = t(s, u) = (ts, tu)$. This extension of U by T is not the most general splitting extension [3], but it does have enough inherent commutativity to make questions concerning the radical accessible.

One readily checks that $Q(V)$ is the set of all $(q^*, x) \in V$ where $q \in Q(T)$ and $x \in Q(U_q)$; here,

$$(q^*, x)^* = (q, x^{(q)} - (q \circ q(x^{(q)}).$$

If ϕ_Q is the map which carries each such (q^*, x) onto q^*, then the sequence of groups (each under circle composition)

$$0 \to Q(U) \to Q(V(U, \alpha, T)) \xrightarrow{\phi_Q} Q(T) \to 0$$

is exact. Note that, as a map from a set to a set, $\phi_Q = \phi \mid Q(V)$ where ϕ is the map given by $\phi(s, u) = s$, this ϕ making

$$0 \to U \to V \xrightarrow{\phi} T \to 0$$

exact as a sequence of T-algebras. Likewise, let $\phi_J = \phi \mid J(V)$, a T-algebra homomorphism which makes

$$0 \to J(U) \to J(V) \xrightarrow{\phi_J} J(T)$$

exact. In particular, if $(r, u) \in J(V)$, then $r \in J(T)$, while the Lie product $[u, x] = ux - xu$ lies in $J(U)$ for every $x \in U$.

Similarly, if $u \in J(U)$ and if $r \in J(T) \cap (J(U):U)$, then $(r, u) \in J(V)$. For, taking any $(t, y) \in V$, $rt \in J(T) \leq Q(T)$ so that $rt = q^*$ for some $q \in Q(T)$, and $(r, u)(t, y) = (q^*, z)$ where $z = ry + tu + uy$. Since $J(U)$ is both an ideal and a T-subalgebra in U, $tu + uy \in J(U)$. But $r \in (J(U):U)$ so that all of z lies in $J(U)$. It follows that, for each $s \in T$, $z - sz \in J(U) \leq Q(U)$. Now take $s = q$, so that Lemma 1 (a) gives $z \in Q(U_q)$. By our remarks on the nature of the elements of $Q(V)$, (q^*, z) is in this last so that $(r, u) \in J(V)$, as we wished to show.

The case where T is a field is discussed from a somewhat different point of view in [6]. We shall say something below about the case where T is an integral domain. Let $\alpha_{\mathfrak{Q}} = \alpha \mid \mathfrak{Q}(T)$, a ring map whenever $\mathfrak{Q}(T)$ is a subring of T. If $\mathfrak{Q}(U, \alpha, T)$ is a T-algebra it is also a $\mathfrak{Q}(T)$-algebra via $\alpha_{\mathfrak{Q}}$ whenever $\mathfrak{Q}(T)$ is a ring. In this case

$$B = B(U, \alpha, T) = V(\mathfrak{Q}(U, \alpha, T), \alpha_{\mathfrak{Q}}, \mathfrak{Q}(T))$$

is a subring of $V(U, \alpha, T)$. It is also a $\mathfrak{Q}(T)$-algebra: if $s \in \mathfrak{Q}(T)$ and if $(p, x) \in B$, then $s(p, x) = (sp, sx) \in B$, and the operators from T commute with the left and right multiplications on B. Let $A = A(U, \alpha, T)$ denote $\mathfrak{Q}(V(U, \alpha, T), \alpha_{\#}, T)$, and let $\phi_{\mathfrak{Q}} = \phi \mid B$.

THEOREM 5. *Let T be a commutative ring with unity 1_T, and let U be a T-algebra via α where $\alpha(1_T)$ is the identity automorphism on U^+. Suppose that $\mathfrak{Q}(U, \alpha, T)$ is a $\mathfrak{Q}(T)$-algebra as a subring of U. Then*

(i) $B(U, \alpha, T) \leq A(U, \alpha, T)$;

(ii) $$0 \to \mathfrak{Q}(U, \alpha, T) \to B(U, \alpha, T) \xrightarrow{\phi_{\mathfrak{Q}}} \mathfrak{Q}(T) \to 0$$

is an exact sequence of $\mathfrak{Q}(T)$-algebras; and

(iii) *if $A(U, \alpha, T)$ is closed under the subtraction of $V(U, \alpha, T)$, or if T is an integral domain, then $A(U, \alpha, T) = B(U, \alpha, T)$.*

Proof. First observe that the identity automorphism on the direct-sum group $T^+ \oplus U^+$ is also a ring isomorphism on $V(U_t, \alpha_t, T_t)$ onto

$(V(U, \alpha, T))_t$ for each $t \in Q(T)$. That is, as sets,

$$A = \bigcap_t \mathfrak{Q}((V(U, \alpha, T))_t) = \bigcap_t \mathfrak{Q}(V(U_t, \alpha_t, T_t)),$$

t running over $Q(T)$. Hence $(p, x) \in V(U, \alpha, T)$ lies in A if and only if $p \in \mathfrak{Q}(T)$ and

$$x \in \bigcap_t Q((U_t)_{p^{(t)}}) = \bigcap_t Q(U_{d(p^{(t)};t)}),$$

as we can see by appealing to our earlier results and definitions. Since $d(p^{(t)}; t) \in Q(T)$, $\bigcap_t Q(U_{d(p^{(t)};t)}) \geq \mathfrak{Q}(U, \alpha, T)$. Now suppose that $(p, x) \in B$ so that $p \in \mathfrak{Q}(T)$ and $x \in \mathfrak{Q}(U, \alpha, T)$. By what we have just done, $(p, x) \in A$, whence $B \leq A$, and we now have (i). The exactness of the sequence of (ii) is immediate.

Let us assume that A is closed under the subtraction of $V(U, \alpha, T)$, and let us take $(s, u) \in A$. By our above remarks on A, $s \in \mathfrak{Q}(T)$. Since $0 \in \bigcap_t Q(U_{d(s^{(t)};t)})$, we have $(s, 0) \in A$; therefore $(s, u) - (s, 0) = (0, u) \in A$. That is, $u \in \bigcap_t Q(U_{d(0;t)})$. But $d(0; t) = t$, whence $u \in \bigcap_t Q(U_t) = \mathfrak{Q}(U, \alpha, T)$. It thus appears that $(s, u) \in B$, whence $A \leq B$.

Assume, alternately, that T is an integral domain and thus $(p, x) \in A$ so that $p \in \mathfrak{Q}(T)$ and $x \in \bigcap_t Q(U_{d(p^{(t)};t)})$ as t runs over $Q(T)$. Recall that $r = 2_T - s \in Q(T)$ whenever $s \in Q(T)$. Since $p \in \mathfrak{Q}(T) \leq Q(T_r)$, Lemma 1 (a) provides that

$$(1_T - r)p = -(1_T - s)p \in Q(T).$$

The quantity $1_T + (1_T - s)p$ is thus a unit of T, and one can show that

$$t(s) = 1_T - (1_T - s)(1_T + (1_T - s)p)^{-1} \in Q(T).$$

From this, one has

$$p^{(t(s))} = -(1_T - s)p^2 - p$$

and

$$p^2 s = p^2 + p + p^{(t(s))} = p(p + p^{(t(s))} - t(s)p^{(t(s))}) = p^2\, d(p^{(t(s))}; t(s)).$$

If $p \neq 0$, the integrity of T yields $s = d(p^{(t(s))}; t(s))$. A consequence is that

$$\mathfrak{Q}(U, \alpha, T) = \bigcap_s Q(U_s) = \bigcap_s Q(U_{d(p^{(t(s))};\, t(s))}) \geq \bigcap_t Q(U_{d(p^{(t)};t)})$$

where both t and s range over $Q(T)$. We saw, however, (VIII) that x lies in this last intersection so that $x \in \mathfrak{Q}(U, \alpha, T)$, and (IX) that $p \in \mathfrak{Q}(T)$. From the definition of B, we now must have $(p, x) \in B$.

If $p = 0$, then we have $p^{(t)} = 0$, $d(p^{(t)}; t) = t$, and again $x \in \bigcap_t Q(U_t) = \mathfrak{Q}(U, \alpha, T)$. That is, $(0, x) \in B$. In any event, $A \leq B$, completing the proof.

6. The radical of the extension

Let $\alpha_{J,J} = \alpha_J \mid J(T)$, where, as before, U is a T-algebra via α, and T is, throughout this section, a commutative ring. Observe that $V(J(U), \alpha_{J,J}, J(T))$ is a T-algebra, a subalgebra of $V(U, \alpha, T)$.

THEOREM 6. *Let U be a non-trivial T-algebra (T commutative) via α. Suppose that $r(u - u^2) = 0$ for each $r \in J(T)$ and for each $u \in U$. Then*

$$J(V(U, \alpha, T)) = V(J(U), \alpha_{J,J}, J(T)).$$

Proof. For $(r, u) \in V(U, \alpha, T)$, $(r, u) \in J(V(U, \alpha, T))$ if and only if $(r, u)(s, x) = (rs, rx + su + ux)$ is q.r. for each $(s, x) \in V$. That is, equivalently, rs is q.r. and $rx + su + ux \in Q(U_{(rs)^*})$. Equivalently, again, $r \in J(T)$, and $rx + su + ux - (rs)^*(rx + su + ux) \in Q(U)$.

If $t \in J(T)$, and if $v \in U$, then

$$tv + t^*v = (t + t^*)v = (tt^*)v = (tt^*)v^2 = (tv)(t^*v),$$

by the special condition in the hypothesis. That is, tv is q.r. with $(tv)^* = t^*v$. For $w \in U$, $(tv)w = t(vw)$ so that, by what we have just done, $(tv)w$ is q.r. for each $w \in U$. But this means that $tv \in J(U)$ from which $J(T) \leq (J(U):U)$.

If $(r, u) \in J(V)$, then $(r, u)(0, x)$ is q.r. for each $x \in U$, which is to say that $rx + ux \in Q(U)$. We know also that $r \in J(T)$. Replace x by xy to obtain $(rx + ux)y \in Q(U)$ for each $x, y \in U$, from which $rx + ux \in J(U)$. But $J(T) \leq (J(U):U)$ gives $rx \in J(U)$. Hence $ux \in J(U) \leq Q(U)$ for each $x \in U$, so that $u \in J(U)$. We have established that $J(V(U, \alpha, T)) \leq V(J(U), \alpha_{J,J}, J(T))$.

Conversely, if (r, u) is in the right-hand set of the preceding inclusion, then (X) $r \in J(T)$, and $u \in J(U)$. We have rx, su, and ux lying in $J(U)$, so that (XI)

$$rx + su + ux - (rs)^*(rx + su + ux) \in J(U) \leq Q(U).$$

But (X) and (XI) are equivalent to $(r, u) \in J(V(U, \alpha, T))$, completing the proof.

We should observe that the condition $r(u - u^2) = 0$ holds for any Boolean ring U which is also a T-algebra. To obtain another example, let p be a prime, T be the ring of p-adic integers, and U be any ring of characteristic p. It is easy to see that U is a T-algebra. Recall [2] that $J(T)$ is the principal ideal generated by the p-adic integer p. Since $pv = 0$ for each $v \in U$, $r(u - u^2) = 0$ whenever $r \in J(V)$.

If we have $r(u - u^2) = 0$ for each $r \in J(T)$ and each $u \in U$, then $ra = 0$ for each $a \in Q(U)$, and $ru = -r^*u$. For, $raa^* = (ra)aa^* = ra^2 + raa^*$ so that $0 = ra^2 = ra$. Also, $(r + r^*)u = r(r^*u) = ra$ where $a = r^*u \in J(U) \leq Q(U)$, since $J(T) \leq (J(U) : U)$. That is, $ra = 0$ from which $ru = -r^*u$. (Cf., $\lambda^{[\tau]}(\tau^*)$ and $-\tau$ in Section 1.)

COROLLARY. *Under the conditions of the theorem, for no $r \in J(T)$ is $\alpha(r)$ an ependomorphism on U^+.*

Proof. By Lemma 2(b), U is a radical ring whenever any member r_0 of $J(T)$ acts as an endomorphism on U^+. But, by the above remarks, $r_0 u = 0$ for every $u \in U$ since $U = Q(U)$ under these circumstances.

THEOREM 7. *Let U be a T-algebra (T commutative) via α where U^+ is an irreducible T-module.*

(a) *Suppose that no member of $J(T)$ is carried by α onto an automorphism of U^+. Then $J(V(U, \alpha, T))$ is, to within an isomorphism, either $J(T)$ or the algebra direct sum of the T-algebras $J(T)$ and U.*

(b) *If α is a monomorphism, then $V(U, \alpha, T)$ either has trivial radical or has its radical essentially U.*

Proof. (a) Since T is commutative, $(\ker \alpha(r))^+$ is a T-submodule of U^+ for each $r \in T$. By the irreducibility of U^+, this submodule would have to vanish if $\alpha(r)$ were to be an ependomorphism. But $\alpha(r)$ would then have to be an automorphism, contrary to assumption if r is taken from $J(T)$. Hence, if $r \in J(T)$, $\operatorname{Im} \alpha(r) < U$. But $(\operatorname{Im} \alpha(r))^+$ is a T-submodule of irreducible U^+. Thus, $\operatorname{Im} \alpha(r)$ reduces to the trivial algebra, and $J(T)$ operates trivially on U^+ via $\alpha \mid J(T)$. Now suppose that $(r, u) \in J(V)$ so that $r \in J(T)$. For all $y \in U$, $(r, u)(0, y) = (0, ry + uy) \in J(V)$. But $ry = 0$ since $J(T)$ operates trivially on U^+, giving $(0, uy) \in J(V)$ and $uy \in Q(U)$. Thus $u \in J(U)$.

Conversely, if $r \in J(T)$, and if $u \in J(U)$, then

$$rx + su + ux - (rs)^*(rs + su + ux)$$

reduces to $su + ux \in J(U) \leq Q(U)$, from which $(r, u) \in J(V)$, as we see from the proof of Theorem 6. Since $J(T)$ acts trivially on U^+, it is readily verified that $(r, u)(s, w) = (rs, uw)$ where $r, s \in J(T)$ and $u, w \in J(U)$. To within an isomorphism, $J(V)$ is just $J(T) \oplus J(U)$. Finally, the irreducibility of U^+ shows that $J(U) = 0$ or U.

(b) Since U^+ is irreducibile via faithful α, T is primitive and thus has zero radical [4]. The members of $J(V)$ are thereby seen to be all $(0, u)$ where $su + ux \in Q(U)$ as s runs over T and as x runs over U. For a special case, take $s = 0$ from which $ux \in Q(U)$ for all $x \in U$, so that $u \in J(U)$ whenever $(0, u) \in J(V)$. As a T-algebra, therefore, $J(V)$ is isomorphic to $J(U) = 0$ or U.

7. Some examples

(a) Let $T = Z_{24}$, the ring of integers modulo 24. Then

$$J(T) = \mathfrak{Q}(T) = (6).$$

(b) Let $T = Z_{24}$, and let $U = (2) \leq T$, so that U is a T-algebra, and each multiplication on U by a member of $Q(U)$ can be realized by an operation from $Q(T)$. Again $J(U) = \mathfrak{Q}(U, \alpha, T) = (6)$, although α is no monomorphism.

(c) Let T be Z_{24}, and let U be the T-algebra of two-by-two matrices over T. Note that $J(U) = (6I)$ where I is the identity matrix. Now

$$\begin{pmatrix} 0 & 7 \\ 3 & 3 \end{pmatrix} \in \mathfrak{Q}(U, \alpha, T) \backslash J(U);$$

likewise for

$$\begin{pmatrix} 1 & 23 \\ 1 & 23 \end{pmatrix}.$$

Nevertheless, their sum

$$\begin{pmatrix} 1 & 6 \\ 4 & 2 \end{pmatrix}$$

is not even q.r. in U. Notice that there are left multiplications on U by elements of $Q(U)$ which cannot be realized by multiplications from $Q(T)$.

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