ON MIXED GROUPS OF TORSION-FREE RANK ONE

BY
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1. Introduction

All groups considered in this paper are assumed to be additively written abelian groups. We follow for the most part the notation and terminology of [2]. We mention the following notations: if G is an abelian group, then G_t denotes the maximal torsion subgroup of G, G_p denotes the primary component of G_t for the prime p, $G^1 = \bigcap_{n=1}^{\infty} nG$, and $f_G(\alpha)$ denotes the α-th Ulm invariant of G if G is primary. The additive group of the rationals will be denoted by Q and the subgroup of Q consisting of the integers by Z. All modules are assumed to be unital.

The purpose of this paper is to study mixed groups of torsion-free rank one; that is, mixed groups G with the property that if x and y are elements of G having infinite order, then there exist nonzero integers m and n such that $mx = ny$ In Section 2, we give an invariant that, along with the Ulm invariants, determines the isomorphism classes of such groups in the countable case. The invariant is just that given by Rotman [9] for a more restricted class of groups and indeed is the obvious generalization of the invariant introduced by Kaplansky and Mackey in [6] for countably generated reduced R-modules of torsion-free rank one where R is a complete discrete valuation ring. Rotman in [8] generalized the techniques of [6] to certain modules over a not necessarily complete discrete valuation ring and in [9] to certain countable mixed groups having torsion-free rank one. To be specific, if G denotes a countable reduced mixed group of torsion-free rank one, then Rotman has given satisfactory invariants in the following cases:

(1) G_t is a p-group, $G/G_t \cong Q$ and $\bigcap_{n=1}^{\infty} p^n G = 0$ (see [8]).

(2) $\bigcap_{n=1}^{\infty} p^n G \subseteq \sum_{q \neq p} G_q$ for all primes p (see [9]).

Observe that in both (1) and (2), G_t is necessarily a direct sum of cyclic groups. Case (2) is more restrictive than it may seem, since, for example, it does not include the extensions of $\sum C(p)$ (summation over all primes p) by Q. As in Rotman's work, our approach is to attempt to push the methods of [6] as far as possible. Our success in dealing with elements of transfinite heights depends on the imbedding of a reduced group in its cotorsion completion (see [3]).

In the third section, we solve the existence question for countable groups with given invariants, and in the fourth section, we give some consideration

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to uncountable mixed groups of torsion-free rank one. We show that in some rather special instances the invariant given in the countable case suffices, but that in general it fails to distinguish nonisomorphic mixed groups of torsion-free rank one having the same maximal torsion subgroup. In the final section, some applications of our invariant are considered.

Let G be an abelian group and let p be a prime. We define $p^\alpha G$ inductively for all ordinals α as follows:

$$pG = [x \in G : x = pg \text{ for some } g \in G].$$
$$p^\alpha G = p(p^{\alpha-1}G) \quad \text{if } \alpha - 1 \text{ exists.}$$
$$= \textstyle\bigcap_{\beta<\alpha} p^\beta G \quad \text{if } \alpha \text{ is a limit ordinal.}$$

If λ is the first ordinal such that $p^\lambda G = p^{\lambda+1}G$, set $p^\infty G = p^\lambda G$ and observe that $p(p^\infty G) = p^\infty G$. We define the p-height $h_p(x)$ of the element x in G by

$$h_p(x) = \alpha \quad \text{if } x \in p^\alpha G \text{ but } x \notin p^{\alpha+1}G.$$
$$= \infty \quad \text{if } x \in p^\infty G.$$

In the sequel, we shall need the following lemma and its corollary.

Lemma 1. *Let p be a prime and α an ordinal. If G is a neat subgroup of K such that $K[p] \subseteq \{G[p], p^\beta K\}$ for all $\beta < \alpha$, then $p^\beta K \cap G = p^\beta G$ for all $\beta \leq \alpha$.*

Proof. Since G is a neat subgroup of K, $pK \cap G = pG$. Suppose $\beta \leq \alpha$ and that $p^\gamma K \cap G = p^\gamma G$ for all $\gamma < \beta$. If β is a limit ordinal, we obtain $p^\beta K \cap G = p^\beta G$ by taking intersections. We may assume that $\beta - 1$ exists. Let $x \in p^\beta K \cap G$ and write $x = py$ with $y \in p^{\beta-1}K$. Since G is neat, there is a $g \in G$ such that $x = pg$. Then $y - g \in K[p]$, and we can write $y - g = g_1 + z$ with $g_1 \in G[p]$ and $z \in p^{\beta-1}K$. Therefore

$$y - z = g + g_1 \in G \cap p^{\beta-1}K = p^{\beta-1}G \quad \text{and} \quad x = pg = p(g + g_1) \in p^\beta G.$$

Corollary. *If K/G is torsion-free, then $p^\alpha K \cap G = p^\alpha G$ for all primes p and all ordinals α.*

For each prime p, let J_p be a copy of the p-adic integers and let 1_p denote the identity of J_p. For technical convenience, we assume that the J_p are disjoint. We then form the ring $J^* = \sum^* J_p$, the complete direct sum of the J_p. Clearly J^* has an identity 1 whose J_p-component is 1_p for each prime p, and we may consider the ring J of integers as being imbedded as a subring of J^* by identifying the integer n with the element $n1$ of J^*. We also consider the J_p to be, in the obvious manner, ideals in J^*. Since $pJ_q = J_q$ for $q \neq p$, we have $1 = 1_p + pq^*$ for a unique $q^* \in J^*$. From this equation, it follows that $h_p(x) = h_p(1_p x)$ whenever x is an element of a J^*-module M. (Indeed, we have a direct decomposition $M = 1_p M + K$ where $K = pK$ and $1_p K = 0$, and therefore $p^\alpha M = p^\alpha(1_p M) + K$ for all ordinals α.) More generally, $h_p(x) = h_p(\pi x)$ whenever π is a unit in J_p and $h_p(x) = h_p(nx)$ whenever n is an integer prime to p.

Certain groups are J^*-modules in a natural fashion. For example, if T is a torsion group, then $T = \sum T_p$ and, since each T_p is a J_p-module, T becomes a J^*-module in an obvious manner. If G is a complete Hausdorff space in the topology obtained by taking the subgroups nG as neighborhoods of 0, then G is also a J^*-module (see [1]). The key to this observation is the fact that J^* may be viewed as the completion of the ring J of integers in the topology on J obtained by taking the ideals $nJ = (n)$ as neighborhoods of 0. The groups G that are complete Hausdorff in the above topology are precisely the reduced algebraically compact groups. More generally, we shall show that cotorsion groups are J^*-modules. Recall that G is cotorsion if and only if G is reduced and isomorphic to Ext $(Q/Z, G)$. Now

$$\text{Ext}\,(Q/Z, G) \cong \text{Ext}\,(Q/Z, G_t) + \text{Ext}\,(Q/Z, G/G_t)$$

and Ext $(Q/Z, G/G_t)$ is algebraically compact (see [3]). It then suffices to show that $T^* = \text{Ext}\,(Q/Z, T)$ is a J^*-module whenever T is a reduced torsion group. Now T can be viewed as the maximal torsion subgroup of T^*, and we therefore have the exact sequence

$$0 = \text{Hom}\,(T^*/T, T^*) \to \text{Hom}\,(T^*, T^*) \to \text{Hom}\,(T, T^*) \to \text{Ext}\,(T^*/T, T^*) = 0.$$

It follows then that every endomorphism of T extends uniquely to an endomorphism of T^*. Therefore there is a natural and unique manner in which to make T^* a J^*-module containing T as a submodule. We now see that every reduced group G is a subgroup of a J^*-module M with $p^\alpha M \cap G = p^\alpha G$ for all primes p and all ordinals α. Indeed, let $M = \text{Ext}\,(Q/Z, G)$. Then G can be viewed as a subgroup of M with M/G torsion-free and the corollary to Lemma 1 applies.

2. Structure theorem

We now initiate a series of lemmas that lead to the proof of our main theorem on countable mixed groups of torsion-free rank one. We first establish a crucial generalization of Lemma 2 in [6].

LEMMA 2. *Let M be a reduced J_p-module and suppose that $S = \{y\} + V$ is a subgroup of M such that V is finite and $0(y) = \infty$. If x is an element of M such that $px \in S$, then there is an $x' \in M$ such that $\{x', S\} = \{x, S\}$ and the coset $x' + S$ contains an element of maximal height.*

Proof. Since M is a J_p-module, $h_q(m) = \infty$ for all $m \in M$ and all primes $q \neq p$. Hence by height we mean p-height and we write $h(m)$ for $h_p(m)$. Since $px \in S$, we may write $px = t_0 y + v_0$ where t_0 is an integer and $v_0 \in V$. We consider various cases.

Case 1. $(t_0, p) = 1$. Write $1 = \lambda t_0 + \mu p$ and let $x' = \lambda x + \mu y$. Then $\{x, S\} = \{x', S\}$ and $px' = y'$ where $y' = y + \lambda v_0$. Observe that $S = \{y'\} + V$. Suppose the coset $x' + S$ contains no element of maximal height. Then, since V is finite, there will exist a $v \in V$ and a sequence t_n of integers such that

$$h(x' + t_{n+1} y' + v) > h(x' + t_n y' + v)$$

for all n. But note that $x' + t_n y' + v = (1 + t_n p)x' + v$, and therefore there is a unit $\sigma_n \in J_p$ such that

$$\sigma_n(x' + t_n y' + v) = x' + \sigma_n v.$$

Since v has finite order, the $\sigma_n v$ can assume at most finitely many values Therefore we have for some integer t the contradiction $h(x' + tv) > h(x' + tv)$.

Case 2. $p^j x = 0$ for some j. Let $S^* = J_p y + V$. Then by Lemma 2 in [6], there is a $\pi \in J_p$ and a $v \in V$ such that $x + \pi y + v$ has maximal height in $x + S^*$. We may assume that $\pi \neq 0$ (otherwise $x + v$ has maximal height in $x + S$) and we write $\pi = p^i \sigma$ where σ is a unit in J_p. Then there is an integer k prime to p such that

$$\sigma^{-1}(x + \pi y + v) = kx + p^i y + kv.$$

Let λ and μ be integers such that $1 = \lambda k + \mu p^j$. Then

$$\lambda(kx + p^i y + kv) = x + \lambda p^i y + \lambda kv \in x + S.$$

Since $\lambda\sigma^{-1}$ is a unit in J_p, $x + \lambda p^i y + \lambda kv$ has maximal height in $x + S \subseteq x + S^*$.

Case 3. $t_0 = pt_1$ for some integer t_1. Then set $x' = x - t_1 y$. Since $px' = v_0$, x' has finite order. By Case 2, the coset $x' + S = x + S$ contains an element of maximal height.

We next generalize Lemma 2 to the situation with which we are concerned.

LEMMA 3. *Let G be a reduced group and let S be a finitely generated subgroup of G having torsion-free rank one. If x is an element of G such that $px \in S$, then there is an $x' \in G$ such that $px' \in S$, $\{x', S\} = \{x, S\}$ and the coset $x' + S$ contains an element of maximal p-height.*

Proof. We may write $S = \{y\} + V$ where $0(y) = \infty$ and V is finite. We may assume that G is imbedded in a reduced J^*-module M such that the elements of G have the same p-height in M as in G. Then $1_p S = \{1_p y\} + 1_p V$ is a subgroup of the reduced J_p-module $1_p M$. By Lemma 2, there is a $z \in 1_p M$ such that $\{z, 1_p S\} = \{1_p x, 1_p S\}$ and the coset $z + 1_p S$ has an element of maximal p-height. Moreover, by the proof of Lemma 2, we may assume that $z = 1_p(\lambda x + s_0)$ where $s_0 \in S$ and $(\lambda, p) = 1$. Set $x' = \lambda x + s_0$ and observe that $\{x', S\} = \{x, S\}$. Choose $v \in V$ and an integer t such that

$z + t1_p y + 1_p v = 1_p(x' + ty + v)$ has maximal p-height in $z + 1_p S$. Then $x' + ty + v$ has maximal p-height in $x' + S$.

With but a simple argument to handle the case $h_p(x) = \infty$, the proof of Lemma 5 in [9] can now, with the aid of our Lemma 3, be reproduced to yield

LEMMA 4. *Suppose that G and H are reduced groups having the same Ulm invariants for the prime p. Let S and T be finitely generated subgroups of G and H having torsion-free rank one, σ a height preserving isomorphism of S onto T and x an element of G such that $x \notin S$ but $px \in S$. Then σ can be extended to a height preserving isomorphism of $\{x, S\}$ onto a suitable subgroup of H containing T.*

Our main result will follow rather readily now. But first we must introduce the invariant we have promised. Let x be an element of G and let p be a prime. Then we associate with x the p-Ulm sequence

$$U_G^p(x) = (\alpha_0, \cdots, \alpha_i, \cdots)$$

of x in G where α_i is the p-height of $p^i x$ in G. We say that any two sequences $(\alpha_0, \cdots, \alpha_i, \cdots)$ and $(\beta_0, \cdots, \beta_i, \cdots)$ are equivalent if there exist integers n and m such that $\alpha_{n+j} = \beta_{m+j}$ for $j = 0, 1, \cdots$. Let $p_1, p_2, \cdots, p_i, \cdots$ be the increasing sequence of primes. Then with each $x \in G$ we associate an infinite matrix $U_G(x) = (\alpha_{ij})$, which we shall call the Ulm matrix of x, where the i-th row is just the p_i-Ulm sequence of x. We now define an equivalence relation $\sim$ between such infinite matrixes, the motivation for which will be evident shortly. We write $(\alpha_{ij}) \sim (\beta_{ij})$ if and only if for almost all i the i-th rows of the two matrices are identical and for the remaining i's the i-th rows are equivalent as sequences. It is easy to check that we have indeed defined an equivalence relation. If x and y are elements of G such that $nx = my$ for nonzero integers n and m, then it is immediately seen that $U_G(x) \sim U_G(y)$. Therefore, if G has torsion-free rank one, we associate with G the equivalence class $U(G)$ of matrices determined by $U_G(x)$ where x is any element of infinite order in G. If G and H have torsion-free rank one and if $U(G) = U(H)$, then G and H contain elements x and y such that $U_G(x) = U_H(y)$. Indeed, if x' and y' are elements of G and H having infinite order, then there exist nonzero integers n and m such that $U_G(nx') = U_H(my')$.

THEOREM 1. *Let G and H be countable mixed groups of torsion-free rank one. Then $G \cong H$ if and only if $G_t \cong H_t$ and $U(G) = U(H)$.*

Proof. The conditions are clearly necessary, so let us assume that they are satisfied. Also we may assume that the maximal torsion subgroups are reduced. Then the only possibility that G not be reduced is that $G \cong G_t + Q$. But it is obvious that this happens if and only if $U(G)$ contains the matrix all of whose entries are ∞'s. Therefore we may assume that the groups G and H are reduced. Since $U(G) = U(H)$, we can find elements $x \in G$ and $y \in H$ having infinite order and such that $U_G(x) = U_H(y)$. The mapping $x \to y$ then yields a height preserving isomorphism of $\{x\}$ onto $\{y\}$. Since our groups are countable and have the same Ulm invariants for all primes p, we can with the aid of Lemma 4 extend this mapping in the usual step-by-step back-and-forth method (see the proof of Ulm's theorem in [2], [5], or [6]) to an isomorphism between G and H.

3. Existence theorem

We now turn to the question of the existence of countable mixed groups with given invariants. First we must look at our invariant more closely. By an increasing sequence of ordinals and symbols ∞, we mean a sequence $(\alpha_0, \cdots, \alpha_n, \cdots)$ where each α_n is either an ordinal or a symbol ∞ and where $\alpha_{n+1} > \alpha_n$ if α_n is an ordinal and $\alpha_{n+1} = \infty$ if $\alpha_n = \infty$. An increasing sequence $(\alpha_0, \cdots, \alpha_n, \cdots)$ of ordinals and symbols ∞ is said to be p-compatible with the reduced torsion group T if

(i) $\alpha_{n+1} > \alpha_n + 1$ only if the α_n-th Ulm invariant of the p-primary component of T does not vanish; and

(ii) if $\alpha_n \neq \infty$, then $\alpha_n < \lambda + \omega$ where λ is the length of the p-primary component of T.

The restriction (i) is, of course, familiar (see Lemma 22 in [5]) and the restriction (ii) is also easily seen to be essential to our discussion. Indeed, suppose G is a mixed group of torsion-free rank one having T as its maximal torsion subgroup and that $p^\alpha G \neq p^\infty G$ contains an element of infinite order for some $\alpha \geq \lambda + \omega$. Note that if $T = T_p + S$, then $p^\lambda G/S$ is torsion-free of rank one. It follows that $p^\lambda G/p^\alpha G$ is torsion and consequently isomorphic to $C(p^\infty)$. This, however, implies that $p^\lambda G = p^{\lambda+1} G = p^\infty G$.

Now let (α_{ij}) $(i = 1, 2, \cdots; j = 0, 1, \cdots)$ be an infinite matrix whose rows are increasing sequences of ordinals and symbols ∞. Then (α_{ij}) will be called an Ulm matrix for the reduced torsion group T if for each i the i-th row is p_i-compatible with T. Our existence theorem can now be stated.

THEOREM 2. *Let (α_{ij}) be an Ulm matrix for the countable reduced torsion group T. Then there exists a mixed group G of torsion-free rank one with $G_t \cong T$ and $(\alpha_{ij}) \in U(G)$.*

We first reduce the proof of Theorem 2 to the case of primary groups. For each positive integer i, let T_i be the p_i-primary component of T, and suppose that we have constructed a mixed group G_i of torsion-free rank one having T_i as its maximal torsion subgroup and such that $U(G_i)$ contains a matrix whose i-th row agrees with that of (α_{ij}) and that has ∞'s elsewhere. Then set $K = \sum_{i=1}^{\infty *} G_i$ and observe that T can be considered to be the maximal torsion subgroup of K. Observe that, for each i and each ordinal α, we have

$$p_i^\alpha K = p_i^\alpha G_i + \textstyle\sum_{j \neq i} G_j .$$

The equation is also valid for $\alpha = \infty$. Now let x_i be an element of G_i having infinite order and such that $(\alpha_{i0}, \cdots, \alpha_{in}, \cdots)$ is its p_i-Ulm sequence in G_i.

Then choose x to be that element of K whose G_i-component is x_i for each i. Finally we let G to be that unique pure subgroup of K containing $\{x, T\}$ and having torsion-free rank one. Since K/G is torsion-free, heights computed in G are the same as computed in K. Therefore, to show that $U_G(x) = (\alpha_{ij})$, it suffices to observe that $U_K(x) = (\alpha_{ij})$. This latter equation is readily checked. Indeed, from the decomposition of $p_i^\alpha K$ noted above, we see that $U_K^{p_i}(x) = U_K^{p_i}(x_i) = U_{G_i}^{p_i}(x_i)$.

Given a countable reduced p-primary group T and a p-compatible sequence $(\alpha_0, \alpha_1, \cdots)$, we now wish to construct a mixed group G of torsion-free rank one such that $G_t \cong T$, $U_G^p(x) = (\alpha_0, \alpha_1, \cdots)$ for some $x \in G$ having infinite order and $nG = G$ whenever n is prime to p. This latter condition guarantees enough ∞'s as entries in the matrices in $U(G)$. By an existence theorem of Rotman and Yen [10][2], there is a J_p-module M such that $M_t = T$ and M contains an element x having zero order ideal with $U_M^p(x) = (\alpha_0, \alpha_1, \cdots)$. It suffices to take G to be the minimal pure subgroup of M containing T and x.

The proof of Theorem 2 is now complete.

4. Uncountable groups

Although Theorem 1 does not hold in the case of uncountable groups, it can be established for uncountable groups with certain types of maximal torsion subgroups. But before considering such extensions of the theorem, we give an example that demonstrates the inadequacy of the invariant $U(G)$ for uncountable G.

Example. Let p be a prime and let $\bar{B}$ be the torsion subgroup of $\sum_{i=1}^{\infty *} \{b_i\}$, where $0(b_i) = p^i$ for each i. Set $B = \sum_{i=1}^{\infty} \{b_i\}$ and let H be a pure subgroup of $\bar{B}$ such that $B \subseteq H$ and $\bar{B}/H \cong C(p^\infty)$. Clearly H is not a closed p-group and in particular $H \not\cong \bar{B}$. We shall need the following observation: If H is a pure subgroup of a group C with $C^1 = 0$ and $C/H \cong C(p^\infty)$, then $C \cong \bar{B}$. Indeed, B is a basic subgroup of any such group C and there is a canonical monomorphism ψ of C into $\bar{B}$ that leaves the elements of B fixed (see [2, pp. 111–112]). In fact, ψ leaves the elements of H fixed. The mapping of $C/H \cong C(p^\infty)$ into $\bar{B}/H \cong C(p^\infty)$ induced by ψ is necessarily onto and thus ψ is an isomorphism of C onto $\bar{B}$.

Let M be a reduced group containing B as maximal torsion subgroup and such that $M/B \cong Q$ and $M^1 \neq 0$. Since H is not a closed p-group, Pext $(C(p^\infty), H) \neq 0$, and therefore there is a subgroup N of Ext $(C(p^\infty), H)$ such that $N_t = H$, $N/H \cong Q$ and $N^1 \neq 0$. It is evident that $U(M) = U(N)$ since each class contains a matrix having $(\omega, \omega + 1, \omega + 2, \cdots)$ in the row corresponding to p and ∞'s elsewhere. It is easily seen that the mapping $b \to b + M^1$ takes B into a pure subgroup of M/M^1 and has cokernel iso-

[2] The author wishes to thank the referee for calling his attention to the paper [10] and thereby materially shortening his original proof of Theorem 2.

morphic to $C(p^\infty)$. It follows that M/M^1 is a direct sum of cyclic p-groups and indeed $M/M^1 \cong B$ (see Theorem 33.4 in [2]). Similarly, the mapping $h \to h + N^1$ takes H into a pure subgroup of N/N^1 and has cokernel isomorphic to $C(p^\infty)$. It follows then from the observation cited in the preceding paragraph that $N/N^1 \cong \bar{B}$.

Now define groups $G = B + N$ and $K = M + H$. Then $G_t \cong B + H \cong K_t$ and $U(G) = U(N) = U(M) = U(K)$. However $G \not\cong K$ since $G/G^1 \cong B + \bar{B} \not\cong B + H \cong K/K^1$.

When the maximal torsion subgroups are direct sums of cyclic groups, our invariant does suffice.

THEOREM 3. *Let G and K be mixed groups of torsion-free rank one such that $G_t \cong K_t$ and $U(G) = U(K)$. If G_t is a direct sum of cyclic groups, then $G \cong K$.*

Proof. Let S be a countable pure subgroup of G containing an element of infinite order. Then, it is easily seen that $G = \{S, G_t\}$. Since G_t is a direct sum of cyclic groups, we can find a direct summand U of G_t having a countable complement and such that $S \cap U = 0$. It is evident that $G = A + U$, where A is a countable subgroup of G containing S. Similarly, we have a direct decomposition $K = B + V$ with B countable and V torsion. Since any two direct decompositions of a direct sum of cyclic groups have isomorphic refinements, we may assume that $A_t \cong B_t$ and $U \cong V$. Since $U(A) = U(G) = U(K) = U(B)$, we conclude from Theorem 1 that $A \cong B$ and, consequently, that $G \cong K$.

A similar argument yields the following generalization of the preceding theorem.

THEOREM 4. *Let G and K be mixed groups of torsion-free rank one such that $G_t \cong K_t$ and $U(G) = U(K)$. If G_t is a direct sum of countable groups, then $G \cong K$.*

The following lemma may sometimes be useful in reducing considerations to the countable case.

LEMMA 5. *Let G and K be mixed groups of torsion-free rank one. Suppose A is a pure mixed subgroup of G and that ϕ is an isomorphism of A onto a pure subgroup of K. If $\phi \mid A_t$ extends to an isomorphism of G_t onto K_t, then ϕ extends to an isomorphism of G onto K.*

Proof. If B is the image under ϕ of A, then $G = \{A, G_t\}$ and $K = \{B, K_t\}$. Define a mapping $\bar{\phi}$ of G onto K by $\bar{\phi}(t + a) = \psi(t) + \phi(a)$ whenever $t \in G_t$ and $a \in A$. It is routine to verify that $\bar{\phi}$ is a well-defined isomorphism of G onto K such that $\bar{\phi} \mid A = \phi$.

We call a torsion group T closed if T_p is a closed p-group for each prime p. Recall that closed p-groups have been characterized by the property that any isomorphism between basic subgroups can be extended to an automorphism (see [7]).

THEOREM 5. *Let G and K be mixed groups of torsion-free rank one such that $G_t \cong K_t$ and $U(G) = U(K)$. If G_t is closed, then $G \cong K$.*

Proof. Let S be a countable mixed pure subgroup of G such that $S \cap p^\omega G \neq 0$ whenever $p^\omega G \neq 0$. This latter requirement insures that $U(S) = U(G)$. Since S_t is countable, it is a direct sum of cyclic groups and therefore is contained in a basic subgroup of G_t. We see then that there is a mixed pure subgroup A of G such that $U(A) = U(G)$ and A_t is a basic subgroup of G_t. Similarly, there is a mixed pure subgroup B of K such that $U(B) = U(K)$ and B_t is a basic subgroup of K_t. Since $G_t \cong K_t$, $A_t \cong B_t$ and, by Theorem 3, $A \cong B$. Since G_t and K_t are closed, Lemma 7 can be applied to yield $G \cong K$.

It is perhaps noteworthy that the classes of groups for which our invariant has been shown to be adequate are classes where the maximal torsion subgroups are characterized by their Ulm invariants.

5. Applications

Rotman has applied the invariant $U(G)$ to solve cancellation, square-root and isomorphic refinement problems. We refer the reader to [9] for these applications. We also mention that it is a simple exercise to verify that Kaplansky's first test problem (see [5, p. 12]) has an affirmative answer for countable mixed groups of torsion-free rank one.

Given a mixed group G of torsion-free rank one and given any matrix (α_{ij}) in $U(G)$, we can, of course, recover the factor group G/G_t from the matrix. Indeed, we define a sequence $(k_1, \cdots, k_n, \cdots)$ as follows: $k_n = \infty$ if the n-th row of (α_{ij}) either contains an infinite ordinal or symbol ∞ or has infinitely many gaps, and $k_n = \alpha_{nj} - j$ if the n-th row contains only integers and has no gaps after α_{nj-1}. The sequence $(k_1, \cdots, k_n, \cdots)$ is then just the height sequence of some element of G/G_t and therefore determines the factor group (see [2, pp. 146–149]).

If G is a countable mixed group of torsion-free rank one and if $(\alpha_{ij}) \in U(G)$, then it is easy to give a criterion for the splitting of G. Indeed, G splits into the direct sum of a torsion and a torsion-free group if and only if the following conditions are satisfied:

(1) almost every row in (α_{ij}) is free of gaps;
(2) the rows not free of gaps have at most a finite number of gaps; and
(3) if a row contains entries other than integers, then it contains an ∞.

If π is a collection of primes, then a similar criterion can be given for $\sum_{p \in \pi} G_p$ to be a direct summand of G.

In some instances our invariant $U(G)$ can be simplified. For example, suppose G_t is a p-primary group. Then one sees readily that the rows of any matrix in $U(G)$ that correspond to primes different from p are determined by G/G_t. Therefore the structure of G is determined by G_t, G/G_t and the row in some matrix of $U(G)$ corresponding to p. To be more precise, if we let $U^p(G)$ denote the equivalence class of p-Ulm sequences determined by elements in G of infinite order, then we have the following:

THEOREM 6. *Let G and K be countable mixed groups of torsion-free rank one whose maximal torsion subgroups are p-primary. Then $G \cong K$ if and only if $G_t \cong K_t$, $G/G_t \cong K/K_t$ and $U^p(G) = U^p(K)$.*

The countability restriction in the preceding theorem can, of course, be dropped if we assume that the maximal torsion subgroups are direct sums of cyclic p-groups. It is perhaps worth mentioning that a proof, which is wholly independent of our Theorem 1, can be given for Theorem 6 based on the theorem of Kaplansky and Mackey [6].

Another instance when the invariant $U(G)$ can be modified is when G_t is elementary; that is, when each element of G_t has square-free order. If G has torsion-free rank one and G_t is elementary, then, by Theorem 3, G is determined by G_t and $U(G)$. Suppose $(\alpha_{ij}) \in U(G)$. Since for a given prime at most the 0-th Ulm invariant is nonzero, we have $\alpha_{ij} = \alpha_{i1} + j - 1$ whenever $j \geq 1$. It is then evident that G is determined by G/G_t and the first column of any matrix in $U(G)$. Therefore we associate with (α_{ij}) a sequence $(e_1, \cdots, e_i, \cdots)$ such that $e_i = 0$ if $\alpha_{i1} = \alpha_{i0} + 1$ and $e_i = 1$ if $\alpha_{i1} > \alpha_{i0} + 1$. If we define an equivalence relation among such sequences of 0's and 1's by agreeing that two sequences are equivalent if and only if they differ in at most a finite number of terms, then clearly we have assigned to G an equivalence class $e(G)$ of such sequences. In order to see that G is determined by G_t, G/G_t and $e(G)$, it is enough to observe that if $e_i = 1$, then $\alpha_{i0} = 0$. Therefore we have

THEOREM 7. *Let G and K be mixed groups of torsion-free rank one having elementary maximal torsion subgroups. Then $G \cong K$ if and only if $G_t \cong K_t$, $G/G_t \cong K/K_t$ and $e(G) = e(K)$.*

REFERENCES

1. A. L. S. CORNER, *Every countable reduced torsion-free ring is an endomorphism ring*, Proc. London Math. Soc. (3), vol. 13 (1963), pp. 687–710.
2. L. FUCHS, *Abelian groups*, Budapest, Publishing House of the Hungarian Academy of Sciences, 1958.
3. D. K. HARRISON, *Infinite abelian groups and homological methods*, Ann. of Math. (2), vol. 69 (1959), pp. 366–391.
4. ———, *On the structure of Ext*, Topics in abelian groups, Chicago, Scott, Foresman and Company, 1963.

5. I. Kaplansky, *Infinite abelian groups*, Ann Arbor, University of Michigan Press, 1954.
6. I. Kaplansky and G. W. Mackey, *A generalization of Ulm's theorem*, Summa Brasil. Math., vol. 2 (1951), pp. 195–202.
7. H. Leptin, *Zur Theorie der überabzählbaren abelschen p-Gruppen*, Abh. Math. Sem. Univ. Hamburg, vol. 24 (1960), pp. 79–90.
8. J. Rotman, *Mixed modules over valuation rings*, Pacific J. Math., vol. 10 (1960), pp. 607–623.
9. ———, *Torsion-free and mixed abelian groups*, Illinois J. Math., vol. 5 (1961), pp. 131–143.
10. J. Rotman and T. Yen, *Modules over a complete discrete valuation ring*, Trans. Amer. Math. Soc., vol. 98 (1961), pp. 242-254.

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